Frobenius Method for Computing Power Series Solutions of Linear Higher-Order Differential Systems

Moulay Barkatou   Thomas Cluzeau   Carole El Bacha

University of Limoges
CNRS ; XLIM UMR 6172 ; DMI
123 avenue Albert Thomas
87060 Limoges cedex, France
{moulay.barkatou,thomas.cluzeau,carole.el-bacha}@xlim.fr

Abstract— **We consider the problem of computing regular formal solutions of systems of linear differential equations with analytic coefficients. The classical approach consists in reducing the system to an equivalent scalar linear differential equation and to apply the well-known Frobenius method. This transformation to a scalar equation is not necessarily relevant so we propose a generalization of the Frobenius method to handle directly square linear differential systems. Finally, we investigate the case of rectangular systems and show how their regular formal solutions can be obtained by computing those of an auxiliary square system.**

INTRODUCTION

We consider higher-order linear differential systems of the form

$$x^{\ell} B_{\ell}(x)\, y^{(\ell)}(x) + x^{\ell-1} B_{\ell-1}(x)\, y^{(\ell-1)}(x) + \cdots + B_0(x)\, y(x) = 0 \tag{1}$$

where x is a complex variable, $B_i(x)$, for $i = 0, \ldots, \ell$, are $m \times n$ matrices of analytic functions, y(x) is an unknown n-dimensional vector and $y^{(i)}(x) = \frac{d^i y}{dx^i}(x)$. We are interested in the local analysis of such systems at the point $x = 0$ and therefore, without loss of generality, we suppose that the entries of the matrices $B_i(x)$ are formal power series, *i.e.*, $B_i(x) \in \mathbb{C}[[x]]^{m\times n}$. Such systems appear in many fields of mathematics and many applications in mathematical physics, mechanics and control theory (see [2], [16], [20] and references therein). Computing power series solutions of such systems around singularities can help in understanding of the underlying problem. For example, the existence of logarithm terms in the formal local solutions of the variational equation of a Hamiltonian system can prove its non-integrability. In this paper, we are merely interested in computing *regular formal solutions* of (1), *i.e.,* solutions of the form $y(x) = x^{\lambda_0} z(x)$ where $\lambda_0 \in \mathbb{C}$ and $z(x) \in \mathbb{C}[\log(x)][[x]]^n$.

Definition 1: A solution of (1) of the form

$$y(x) = x^{\lambda_0} \sum_{i\geq 0} U_i x^i \tag{2}$$

where $U_i \in \mathbb{C}[\log(x)]^n$ and $U_0 \neq 0$ is called a *regular formal solution associated to* $\lambda_0 \in \mathbb{C}$.

Usually, when one is confronted to systems of the form (1), the approach commonly used consists in converting it into an equivalent scalar equation: this allows to apply the well-known Frobenius method (see [12], [11]) that computes the regular formal solutions of a scalar equation. The transformation performed to obtain an equivalent scalar equation can use cyclic vectors techniques (see [8]), Jacobson form or hand calculations using differentiation and elimination (see for example [16]). In some cases this transformation can be tedious and can provide a scalar equation with huge coefficients. The point of view of this paper is to handle directly the system. To our knowledge, there exist only few works on that direction: see [19], [15], [18] and [1] in the restrictive case where the entries of the matrices $B_i(x)$ are polynomials. To achieve this, we propose to generalize the Frobenius method to handle directly systems of the form (1).

In the sequel, for ease of presentation, we shall use the Euler derivation $\vartheta = x\frac{d}{dx}$ instead of the standard derivation $\frac{d}{dx}$: these derivations are related by the following formula:

$$\forall\, i \geq 1, \quad x^i \frac{d^i}{dx^i} = \vartheta(\vartheta - 1)\cdots(\vartheta - i + 1).$$

System (1) can then written as $\mathscr{L}(y) = 0$ with

$$\mathscr{L}(y) = A_{\ell}(x)\,\vartheta^{\ell}(y(x)) + A_{\ell-1}(x)\,\vartheta^{\ell-1}(y(x)) + \cdots + A_0(x) y(x), \tag{3}$$

where $A_0(x) = B_0(x), A_{\ell}(x) = B_{\ell}(x)$ and for $i = 1, \ldots, \ell - 1$, $A_i(x)$ is a linear combination with integers coefficients of the $B_j(x)$'s for $j = i, \ldots, \ell$ hence $A_i(x) \in \mathbb{C}[[x]]^{m\times n}$.

The contributions of the present paper are:

- the extension of the Frobenius method for computing regular formal solutions of a square linear differential system (Section II);
- an algorithm that reduces the integration of a rectangular system of linear differential equations to that of a square linear differential system (Section IV).

In the first section, we give a survey on the classical Frobenius method ([12], [11]) in the scalar case, *i.e.*, we consider Equation (3) with $m = n = 1$. In particular, we recall that the exponent λ_0 of any regular formal solution (2) of the linear differential equation must be chosen among the roots

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of the so-called *indicial polynomial* defined by $\sum_{i=0}^{\ell} A_i(0)\lambda^i$. Then, in Section II, we extend the Frobenius method to handle directly square systems ($m = n \geq 2$). Extensions to first-order systems ($\ell = 1$) have been considered in various textbooks (see [11], [14] for example). For second-order systems ($\ell = 2$), another approach based on the calculation of solutions of algebraic matrix equations is developed in [19]. The method proposed in the present paper follows the line of [11], [14]. It has already been sketched in [3] for systems of the form (3) that further satisfy the hypothesis $A_\ell(0)$ invertible. However, linear differential systems appearing in mathematical physics or control theory do not always satisfy this hypothesis. We prove here that the same algorithm can be applied to general square systems of the form (3). To such systems, we associate the matrix polynomial $L_0(\lambda) = \sum_{i=0}^{\ell} A_i(0)\,\lambda^i$. In our forthcoming paper [4], we prove that we can always reduce the problem to the case where $L_0(\lambda)$ is *regular*, *i.e.*, $\det(L_0(\lambda)) \not\equiv 0$. Thus, without loss of generality, in this paper, we suppose that $L_0(\lambda)$ is regular. We then show that the exponent λ_0 of (2) must be a root of the determinant of the matrix polynomial $L_0(\lambda)$ hence $\det(L_0(\lambda))$ plays the same role as the indicial polynomial in the scalar case. Our method has been implemented in Maple. In Section III, we illustrate it on an example taken from [16] and we discuss alternative approaches. Systems appearing in applications such as control theory are generally non-square hence they cannot be handled directly by the method developed in Section II. The aim of Section IV is to show that the integration of a rectangular system of linear differential equations can always be reduced to that of a square system. The approach used is based on the constructive algebraic analysis techniques developed in [23] and the corresponding algorithm has been implemented in Maple using the package OREMODULES ([10]). Note that another method to perform such a reduction in the ordinary differential case is proposed in [5] where the authors present an algorithm for splitting a given rectangular system into two parts: a (square) differential part and an algebraic part. Finally, we add an appendix where we recall some notions concerning matrix polynomials that are used in Section II.

I. Frobenius Method in the Scalar Case

We start with a short survey of the Frobenius method ([12], [11, chap. 3]) for computing regular formal solutions of scalar linear differential equations. For more details, see [11, chap. 3].

Consider a linear differential equation of the form

$$L(y) = a_\ell(x)\,\vartheta^\ell(y) + a_{\ell-1}(x)\,\vartheta^{\ell-1}(y) + \cdots + a_0(x)\,y = 0, \quad (4)$$

where for $i = 0,\ldots,n$, $a_i(x) = \sum_{j=0}^{\infty} a_{i,j} x^j$ with $a_{i,j} \in \mathbb{C}$. The idea of the Frobenius method consists in considering a logarithm-free formal power series of the form

$$y(x,\lambda) = x^\lambda \sum_{j=0}^{\infty} c_j\, x^j, \quad (5)$$

where $c_0 = 1$ and $c_j = c_j(\lambda)$ are rational functions to be determined. Then, we search for c_j's such that $y(x,\lambda)$ satisfies the non-homogeneous equation

$$L(y(x,\lambda)) = f(\lambda)\, c_0\, x^\lambda, \quad (6)$$

where $f(\lambda) = a_{\ell,0}\lambda^\ell + a_{\ell-1,0}\lambda^{\ell-1} + \cdots + a_{0,0}$ is known as the *indicial polynomial* of (4). Plugging $y(x,\lambda)$ into (6), we find that $y(x,\lambda)$ is a formal solution of (6) if and only if the c_j's satisfy the following algebraic relations:

$$\forall\, j \geq 1, \quad f(\lambda + j)\, c_j = -\sum_{i=0}^{j-1}\sum_{k=0}^{\ell} a_{k,j-i}\,(\lambda + i)^k\, c_i. \quad (7)$$

These algebraic equations can be solved recursively for $c_1, c_2, \ldots$ as rational functions of λ, except at the zeros of $f(\lambda + j)$. Consequently, if we choose λ_1 as a root of f such that $\forall\, j \geq 1$, $f(\lambda_1 + j) \neq 0$ (such a root always exists), then all the functions c_i are well defined at λ_1 and evaluating Equation (6) at $\lambda = \lambda_1$ shows that $y(x,\lambda_1)$ is a regular solution of (4). Thus, equation (4) admits at least one logarithmic-free regular solution of the form (5), namely $y(x,\lambda_1)$. Otherwise, *i.e.*, if we choose λ_1 s.t. there exists $k \in \mathbb{N}$ with $f(\lambda_1 + k) = 0$, nothing ensures that c_k is well defined at λ_1. Then $y(x,\lambda)$ given by (5) may have a pole at λ_1. In this case, one has to proceed in a different way. This illustrates why, in the Frobenius method, not only the roots of f are taken into account but also whether some roots differ or not from each others by integer.

Let λ_1 be a root of f of multiplicity $m \geq 1$. The number of regular solutions associated to λ_1 provided by the Frobenius method is equal to m. Two cases have to be distinguished:

A. *First case*

Suppose that $\forall\, j \geq 1$, $f(\lambda_1 + j) \neq 0$. If $m = 1$ then $y(x,\lambda_1)$ is the only regular solution associated to λ_1. Otherwise, differentiating both sides of (6) with respect to λ and taking into account the commutativity of the operators $\frac{\partial}{\partial\lambda}$ and ϑ, we find

$$\frac{\partial L(y(x,\lambda))}{\partial\lambda} = L\left(\frac{\partial y}{\partial\lambda}(x,\lambda)\right) = \left(f'(\lambda) + f(\lambda)\log(x)\right) c_0\, x^\lambda. \quad (8)$$

Since $m > 1$, then $f(\lambda_1) = f'(\lambda_1) = 0$. Hence, evaluating (8) at $\lambda = \lambda_1$ we get that $\frac{\partial y}{\partial\lambda}(x,\lambda_1) = y(x,\lambda_1)\log(x) + x^{\lambda_1}\sum_{j=0}^{\infty}\frac{\partial c_j}{\partial\lambda}(\lambda_1)x^j$ is also a formal regular solution associated to λ_1 of (4). Moreover, it is linearly independent of $y(x,\lambda_1)$ since it is of degree 1 in $\log(x)$ (we recall that here $c_0 = 1$). Consequently, differentiating Equation (6) $(m-1)$ times with respect to λ yields m linearly independent formal regular solutions of strictly increasing degrees in $\log(x)$.

B. *Second case*

Suppose that there exist exactly $r-1$ roots $\lambda_2,\ldots,\lambda_r$ of f s.t. $\Re(\lambda_2) < \ldots < \Re(\lambda_r)$, where $\Re(z)$ denotes the real part of a complex number $z \in \mathbb{C}$, and $\lambda_i - \lambda_1 = k_i \in \mathbb{N}^*$ for $i = 2,\ldots,r$. Denote by m_i the multiplicity of λ_i and let $\alpha = \sum_{i=2}^{r} m_i$. Instead of taking $c_0 = 1$ in (5) and (6), we set $c_0 =$

$(\lambda-\lambda_1)^{\alpha}$. Consequently, Equation (7) becomes

$$f(\lambda+1)\,c_1 = -(\lambda-\lambda_1)^{\alpha}\sum_{k=0}^{\ell} a_{k,1}\,\lambda^k,$$

$$\text{and } \forall\, j\geq 2, \quad f(\lambda+j)\,c_j = -\sum_{i=1}^{j-1}\sum_{k=0}^{\ell} a_{k,j-i}\,(\lambda+i)^k\,c_i - (\lambda-\lambda_1)^{\alpha}\sum_{k=0}^{\ell} a_{k,j}\lambda^k. \quad (9)$$

Choosing $c_0 = (\lambda-\lambda_1)^{\alpha}$ implies that all the functions $c_j(\lambda)$ for $j\geq 1$ are well defined at $\lambda=\lambda_1$. Indeed, for $j=1,\ldots,k_2-1$, as λ_1+j is not a root of f, then c_j is uniquely determined as a rational function of λ, well-defined at λ_1 and admits $(\lambda-\lambda_1)^{\alpha}$ as a factor. For $j=k_2$, $f(\lambda_1+k_2)=f(\lambda_2)=0$ hence $f(\lambda+k_2)$ admits $(\lambda-\lambda_1)^{m_2}$ as a factor. Moreover, remark that the right-hand side of (9) also admits $(\lambda-\lambda_1)^{\alpha}$ as a factor. Consequently, performing the formal simplification by $(\lambda-\lambda_1)^{m_2}$ on both sides of Equation (9) with $j=k_2$, we obtain that c_{k_2} is well defined at $\lambda=\lambda_1$ and admits $(\lambda-\lambda_1)^{\alpha-m_2}$ as a factor. Iterating this process, one shows that all the c_j's are well defined at $\lambda=\lambda_1$ and furthermore $c_j(\lambda_1)=0$ for $j=0,\ldots,\alpha-1$. These latter equalities imply that the solution $y(x,\lambda_1)$ found in this way is a multiple of the regular solution associated to the root λ_r computed as explained in the first case. To find a regular solution properly associated to λ_1, one has to consider the αth derivative of Equation (6) with $c_0=(\lambda-\lambda_1)^{\alpha}$. This leads to

$$L\left(\frac{\partial^{\alpha} y}{\partial \lambda^{\alpha}}(x,\lambda)\right) = \alpha!\,f(\lambda)\,x^{\lambda} + (\lambda-\lambda_1)\,\psi(x,\lambda)$$

where $\psi(x,\lambda)$ and its first $m-2$ derivatives vanish for $\lambda=\lambda_1$. Consequently, $\frac{\partial^{\alpha} y}{\partial \lambda^{\alpha}}(x,\lambda_1)$ is regular solution of (4) having $\alpha!x^{\lambda_1}$ as leading term hence it is properly associated to λ_1. If $m>1$, then as in the first case, the first $m-1$ derivatives of $\frac{\partial^{\alpha} y}{\partial \lambda^{\alpha}}(x,\lambda)$ with respect to λ evaluated at $\lambda=\lambda_1$ yield $m-1$ others linearly independent solutions associated to λ_1.

II. Extension to the Matrix Case

We now consider a linear differential system of order ℓ of the form (3) where $A_i(x)=\sum_{j=0}^{\infty} A_{i,j}x^j \in \mathbb{C}[[x]]^{m\times n}$. In this section, we assume that we have a square system, *i.e.*, $m=n$. The rectangular case $m\neq n$ is considered in Section IV.

Inspired by the Frobenius method in the scalar case and the extension of this latter to handle first-order systems which was given in [11, chap. 4] as an exercise, we give an algorithm to compute the regular formal solutions of higher-order linear differential systems of the form (3). Note that in [3], we have already sketched a generalization of the Frobenius method for systems of the form (3) but in the particular case where $A_{\ell}(0)$ is assumed to be invertible.

To system (3), we associate matrix polynomials defined as follows:

$$\forall\, i\geq 0, \quad L_i(\lambda) = A_{\ell,i}\lambda^{\ell} + A_{\ell-1,i}\lambda^{\ell-1} + \cdots + A_{0,i}.$$

Proceeding as in the scalar case, we consider the non-homogeneous system

$$\mathscr{L}(y) = L_0(\lambda)\,g_0(\lambda)\,x^{\lambda}, \quad (10)$$

where $g_0(\lambda)$ is an arbitrary n-dimensional vector depending on λ and we look for logarithm-free regular formal solutions of (10) of the form

$$y(x,\lambda,g_0) = x^{\lambda}\sum_{i\geq 0} g_i(\lambda)\,x^i, \quad (11)$$

where, for $i\geq 1$, $g_i(\lambda)$ is a n-dimensional vector of functions of λ to be determined. Plugging (11) into (10) and identifying the coefficients of the powers of x in both sides, we get

$$\forall\, i\geq 1, \quad L_0(\lambda+i)\,g_i(\lambda) = -\sum_{j=1}^{i} L_j(\lambda+i-j)\,g_{i-j}(\lambda). \quad (12)$$

As we have mentioned in the introduction, we suppose, without loss of generality (see [4]), that $L_0(\lambda)$ is a regular matrix polynomial, *i.e.*, $\det(L_0(\lambda))\not\equiv 0$. Then $g_i(\lambda)$ can be expressed as the product of the inverse of $L_0(\lambda+i)$ by the right-hand side of (12) and hence, it is well defined at the values of λ where $\det(L_0(\lambda))$ does not vanish. Consequently, if λ_1 is a complex number such that $\forall i\geq 1$, $\det(L_0(\lambda_1+i))\neq 0$, then $y(x,\lambda,g_0)$ is well defined at λ_1. To obtain a solution of the homogeneous system associated to (10), we must choose λ_1 so that the right-hand side of (10) vanishes for $\lambda=\lambda_1$, *i.e.*, $L_0(\lambda_1)\,g_0(\lambda_1)=0$. This happens only when λ_1 is an eigenvalue of $L_0(\lambda)$, *i.e.*, $\det(L_0(\lambda_1))=0$ and $g_0(\lambda_1)$ is an eigenvector associated to λ_1 (see [13] or the Appendix for more details on regular matrix polynomial). Thus, we see that the properties of the associated matrix polynomial $L_0(\lambda)$ will constitute the basic tools for generalizing the Frobenius method to the matrix case. Moreover $\det(L_0(\lambda))$ will play the same role as the indicial polynomial f in the scalar case.

In the sequel, we suppose that λ_1 is an eigenvalue of $L_0(\lambda)$, denote by $m_a(\lambda_1)$ its algebraic multiplicity and by $\kappa_1,\ldots,\kappa_{m_g(\lambda_1)}$ its partial multiplicities. We shall compute $m_a(\lambda_1)$ linearly independent solutions associated λ_1. As in Section I, we distinguish two cases:

A. No eigenvalues differing by an integer

We suppose here that $\forall i\geq 1$, λ_1+i is not an eigenvalue of $L_0(\lambda)$, *i.e.*, $L_0(\lambda_1+i)$ is invertible. If $m_a(\lambda_1)=1$, then $y(x,\lambda_1,g_0)$ with $g_0(\lambda_1)$ an eigenvector associated to λ_1 is the only regular solution of (3) associated to λ_1. Otherwise, as in the scalar, the other solutions involving logarithm-terms are obtained by differentiations w.r.t. λ as follows: for each partial multiplicities κ_i of λ_1, we choose $g_{i,0}(\lambda)$ as a root polynomial of maximal order κ_i associated to λ_1. Hence, the first κ_i-1 derivatives of the right-hand side of (10) vanish for $\lambda=\lambda_1$ (see Definition 3 in the Appendix). Consequently, we find that $y(x,\lambda,g_{i,0})$, the logarithm-free solution of (10) computed for $g_0(\lambda)=g_{i,0}(\lambda)$ as explained above, and $\frac{\partial^j y}{\partial \lambda^j}(x,\lambda_0,g_{i,0})$, for $j=1,\ldots,\kappa_i-1$, form κ_i linearly independent regular solutions associated to λ_1. In this way, we construct $m_a(\lambda_1)=\sum_{i=1}^{m_g(\lambda_1)}\kappa_i$ linearly independent solutions of (3) associated to λ_1. Note that the linear independence of these latter is due to their strictly increasing degrees in $\log(x)$ and to the fact that $g_{0,i}(\lambda_1)$ and $g_{0,j}(\lambda_1)$ with $i \neq j$ are linearly independent (see Lemma 2 in the appendix).

B. Eigenvalues differing by integers

We suppose now that $\lambda_2, \ldots, \lambda_r$ are the unique eigenvalues of $L_0(\lambda)$ such that $\Re(\lambda_1) < \cdots < \Re(\lambda_r)$ and $\lambda_i - \lambda_1 \in \mathbb{N}^*$. We set $\alpha = \sum_{i=2}^{r} m_a(\lambda_i)$. Instead of (10), we consider the non-homogeneous system

$$\mathscr{L}(y) = (\lambda - \lambda_1)^{\alpha} L_0(\lambda) g_0(\lambda) x^{\lambda}, \tag{13}$$

where $g_0(\lambda)$ is an arbitrary n-dimensional vector depending on λ and we look for logarithm-free regular formal solutions of (10) of the form

$$y(\lambda, x, g_0) = \left((\lambda - \lambda_1)^{\alpha} g_0(\lambda) + \sum_{k \geq 1} g_k(\lambda) x^k \right) x^{\lambda}, \tag{14}$$

where, for $i \geq 1$, $g_i(\lambda)$ is a n-dimensional vector function of λ to be determined. Plugging (14) into (13) yields

$$\begin{aligned} L_0(\lambda + k)\, g_k(\lambda) = &- \sum_{j=1}^{k-1} L_j(\lambda + k - j) g_{k-j}(\lambda) \\ &- (\lambda - \lambda_i)^{\alpha_i} L_k(\lambda) g_0(\lambda), \quad k \geq 1. \end{aligned} \tag{15}$$

When $\lambda_1 + k$ is not an eigenvalue of $L_0(\lambda)$, $g_k(\lambda)$ which can be written as $L_0(\lambda + k)^{-1}$ multiplied by the right-hand side of (15), is uniquely determined as a vector of rational functions of λ defined at λ_1. Now, if $\lambda_1 + k$ is an eigenvalue of $L_0(\lambda)$, we can also get a solution $g_k(\lambda)$ of (15) well defined at λ_1. Indeed, the presence of the term $(\lambda - \lambda_1)^{\alpha}$ in (13) and (14) implies that the right-hand side of (15) admits $(\lambda - \lambda_1)$ as a factor of multiplicity greater or equal to the multiplicity of $(\lambda - \lambda_1)$ in the determinant of $L_0(\lambda + k)$. So a simplification can be performed and we obtain a $g_k(\lambda)$ defined at λ_1. In this way, we can solve all systems (15) and compute a (logarithm-free) regular solution of (3) by evaluating (14) at $\lambda = \lambda_1$. But this latter is a multiple of a regular solution associated to λ_r and computed as in Subsection II-A. However, since we search for solutions that are linearly independent from those associated to λ_r, we proceed as in subsections I-B and II-A: for $i = 1, \ldots, m_g(\lambda_1)$, we choose $g_{i,0}(\lambda)$ as a root polynomial associated to λ_1 of maximal order κ_i and we compute the $\frac{\partial^k y}{\partial \lambda^k}(x, \lambda_1, g_{i,0})$ for $k = \alpha, \ldots, \alpha + \kappa_i - 1$ where $y(x, \lambda, g_{i,0})$ is given by (14) with $g_0(\lambda) = g_{i,0}(\lambda)$ as explained above. This provides $m_a(\lambda_1) = \sum_{i=1}^{m_g(\lambda_1)} \kappa_i$ linearly independent regular solutions associated to λ_1.

C. Dimension and Implementation

Using the method described above, we compute exactly $\deg(\det(L_0(\lambda)))$ linearly independent regular solutions of system (3). The following theorem shows that this is the dimension of the regular solution space of (3).

Theorem 1 ([4]): Given a linear matrix differential system of the form (3) with $A_i(x) \in \mathbb{C}[[x]]^{n \times n}$ for $i = 0, \ldots, \ell$ and regular matrix polynomial $L_0(\lambda)$, the dimension of its regular solution space is equal to $\deg(\det(L_0(\lambda)))$.

We have implemented the generalization of Frobenius method in Maple using the package LinearAlgebra (resp. MatrixPolynomialAlgebra) for the linear algebra (resp. matrix polynomial) manipulations. It computes a basis of the regular solution space of a system of the form (3) with regular associated matrix polynomial, where the involved power series are truncated up to a fixed order ν.

III. Example

The following example is taken from [16]. The incipient buoyant thermal convection in vertical cylindrical geometries (see equations (2.17)-(2.19) of [16]) gives rise to the linear differential system $\mathscr{L}(y) = 0$ given by

$$\begin{aligned} \mathscr{L}(y) = &\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vartheta^3(y) + \begin{pmatrix} m & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vartheta^2(y) + \\ &\begin{pmatrix} -2m & -r^2 & 0 \\ m^2 r^2 - 4 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \vartheta(y) + \begin{pmatrix} -m^3 r^2 & 0 & 0 \\ 0 & m r^4 & \mathrm{Ra}\, r^4 \\ 0 & 0 & -m^2 r^2 \end{pmatrix} y. \end{aligned} \tag{16}$$

We keep the notations of [16]: r is the variable, $y(r) = (\psi(r) \quad \bar{p}(r) \quad \bar{T}(r))^T$ is the vector of unknown functions and m, Ra are constant parameters. Performing hand calculations based on differentiations and eliminations techniques, the authors of [16] reduce system (16) to an equivalent scalar linear ordinary differential equation of order 6 in the unknown function ψ of the variable r. Then, they apply the scalar Frobenius method to compute by hand the regular formal solutions of the scalar equations and deduce those of (16).

We shall show how the direct method developed in the previous section can be applied to handle directly the square linear differential system $\mathscr{L}(y) = 0$ given by (16). Here, the matrix polynomial associated to (16) is singular. However, applying the algorithm developed in [4] and implemented in Maple, we find an equivalent differential system given by

$$\begin{aligned} \overline{\mathscr{L}}(z) = &\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vartheta^3(z) + \begin{pmatrix} m & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vartheta^2(z) \\ + &\begin{pmatrix} -2m & -1 & 0 \\ m^2 r^2 - 4 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \vartheta(z) + \begin{pmatrix} -m^3 r^2 & 2 & 0 \\ 0 & m r^2 & \mathrm{Ra}\, r^4 \\ 0 & 0 & -m^2 r^2 \end{pmatrix} z. \end{aligned} \tag{17}$$

It is obtained by performing the change of variable

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} z, \tag{18}$$

and have a regular associated matrix polynomial given by

$$\overline{L}_0(\lambda)=\begin{pmatrix} m\lambda(\lambda-2) & -\lambda+2 & 0\\ -\lambda^3+4\lambda^2-4\lambda & 0 & 0\\ -\lambda & 0 & \lambda^2\end{pmatrix}.$$

This latter matrix polynomial admits two distinct eigenvalues 0 and 2 of algebraic multiplicities $m_a(0)=3$ and $m_a(2)=3$ respectively.

Applying the Maple implementation of the Frobenius method developed in the previous section to System (17), we compute a basis of the regular solution space of (17) given by 6 elements of the form (2) where the series are truncated up to order 2: we get 3 regular solutions z_i^0, $i=1,2,3$ associated to the eigenvalue 0 given by

$$z_1^0=\begin{pmatrix} O(r^2)\\ O(r^2)\\ 60+15m^2r^2\end{pmatrix},$$

$$z_2^0=\begin{pmatrix} 48+24\log(r)\,m^2r^2-12m^2r^2+O(r^2)\\ O(r^2)\\ 12+12\log(r)\,m^2r^2-9m^2r^2+O(r^2)\end{pmatrix}$$

and

$$z_3^0=\begin{pmatrix} O(r^2)\\ O(r^2)\\ 240\log(r)+60\log(r)\,m^2r^2-60m^2r^2\end{pmatrix},$$

and 3 other regular solutions $z_i^@$, $i=1,2,3$ associated to the eigenvalue 2 given by

$$z_1^2=\begin{pmatrix} r^2(18+\frac{9}{4}m^2r^2+\frac{1}{16}mr^2+O(r^2))\\ r^2(1+\frac{1}{4}m^2r^2+O(r^2))\\ r^2(9+\frac{9}{8}m^2r^2+\frac{1}{64}mr^2+O(r^2))\end{pmatrix},$$

$$z_2^2=\begin{pmatrix} r^2(-10-\frac{5}{2}m^2r^2+O(r^2))\\ r^2(-20m-5m^3r^2+O(r^2))\\ r^2(-5-\frac{15}{16}m^2r^2+O(r^2))\end{pmatrix}$$

and

$$z_3^2=\begin{pmatrix} r^2(7-10\log(r)+(\frac{27}{8}-\frac{5}{2}\log(r))m^2r^2+O(r^2))\\ r^2(-20\log(r)\,m+(5-5\log(r))m^3r^2+O(r^2))\\ r^2(6-5\ln(r)+(\frac{49}{32}-\frac{15}{16}\log(r))m^2r^2+O(r^2))\end{pmatrix}.$$

Finally, to obtain a basis of the regular solution space of (16), it suffices to multiply each above solution by the matrix given by (18).

Let us now discuss different alternative strategies that can be also applied to obtain the regular formal solutions of (16). Computing a change of variable is not the only way to obtain a system equivalent to (16) having a regular associated matrix polynomial. An alternative method is also developed in [4]: it consists in performing elementary operations on the equations of the system. Indeed, multiplying system (16) on the left by the invertible matrix operator (see [17])

$$S=\begin{pmatrix} 1 & 0 & 0\\ r^{-2}(2-\vartheta) & -r^{-2}m & 0\\ 0 & 0 & 1\end{pmatrix},$$

we obtain an equivalent linear differential system of second-order with invertible leading matrix coefficient given by

$$\tilde{\mathscr{L}}(y)=\begin{pmatrix} m & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\vartheta^2(y)+\begin{pmatrix} -2m & -r^2 & 0\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}\vartheta(y)$$
$$+\begin{pmatrix} -m^3r^2 & 0 & 0\\ 0 & -m^2r^2 & -\mathrm{Ra}\,m\,r^2\\ 0 & 0 & -m^2r^2\end{pmatrix}y. \quad (19)$$

The corresponding system $\tilde{\mathscr{L}}(y)=0$ has the same regular solution space as (16) and moreover its associated matrix polynomial is regular. Hence, one could also apply the Frobenius method to it to compute a basis of the regular solution space of (16).

In [3], we develop another direct approach for computing regular solutions of higher-order square linear differential systems which we have also implemented in Maple.

Finally, we can remark that the system (19) has an invertible leading coefficient which allows to convert it into a first-order differential system of first kind: it is of the form $\vartheta(Y)=C(x)Y$ where $Y=(y^T \quad \vartheta(y)^T)^T$ and $C(x)\in\mathbb{C}[[x]]^{6\times 6}$. Then, we can apply to it the method developed in [6] and implemented in the package ISOLDE of Maple ([7]).

IV. Rectangular Systems

Systems appearing in applications such as control theory are generally non-square. Consequently, the Frobenius method developed in section II for computing regular formal solutions cannot be applied directly. In this section, we show that the integration of a rectangular system of linear differential equations can be reduced to that of a square linear differential system. The method used is based on the constructive algebraic analysis techniques developed in [23].

Let D be an Ore algebra (*e.g.*, a ring of differential operators), $\mathscr{F}$ a left D-module, $R\in D^{q\times p}$ and consider the linear rectangular system

$$\ker_{\mathscr{F}}(R.)=\{\eta\in\mathscr{F}^p\,|\,R\,\eta=0\}.$$

In [23], the authors provide an algorithm for computing a parametrization of $\ker_{\mathscr{F}}(R.)$. Let us summarize their method. Using algorithms of [9] and [22], we first compute two matrices $Q\in D^{p\times m}$ and $R'\in D^{q'\times p}$ such that

$$\ker_D(R.)=Q\,D^m,\quad \ker_D(.Q)=D^{1\times q'}R'.$$

In particular, we have $RQ=0$ which implies $D^{1\times q}R\subseteq\ker_D(.Q)=D^{1\times q'}R'$ hence there exists $R''\in D^{q\times q'}$ such that

$$R=R''R'.$$

Then, by a syzygy computation, we compute $T\in D^{r'\times q'}$ such that $\ker_D(.R')=D^{1\times r'}T$. If

$$M=D^{1\times p}/(D^{1\times q}R)$$

denotes the left D-module finitely presented by R and

$$t(M)=\{m\in M\,|\,\exists 0\neq P\in D:\,Pm=0\},$$

its torsion submodule, then the matrices $Q \in D^{p\times m}$, $R' \in D^{q'\times p}$, $R'' \in D^{q\times q'}$ and $T \in D^{r'\times q'}$ defined above satisfy

$$t(M) = (D^{1\times q'} R')/(D^{1\times q} R) \cong D^{1\times q'}/(D^{1\times q} R'' + D^{1\times r'} T),$$

and

$$M/t(M) = D^{1\times p}/\ker_D(.Q) = D^{1\times p}/(D^{1\times q'} R').$$

Furthermore, as $\ker_D(.R') = D^{1\times r'} T$, we have $R' \mathscr{F}^p \subseteq \ker_{\mathscr{F}}(T.)$ and thus

$$R\,\eta = 0 \Longleftrightarrow R''\,R'\,\eta = 0 \Longleftrightarrow \left\{ \begin{array}{rcl} R''\,\zeta &=& 0, \\ T\,\zeta &=& 0, \\ R'\,\eta &=& \zeta. \end{array} \right.$$

We now consider the system

$$R'\,\eta = \zeta, \quad \zeta \in \ker_{\mathscr{F}}\left(\left(\begin{array}{c} R'' \\ T \end{array} \right) . \right).$$

If $\mathscr{F}$ is an injective left D-module (see [25]), then the matrix $Q \in D^{p\times m}$ such that $\ker_D(.Q) = D^{1\times q'} R'$ further satisfies $\ker_{\mathscr{F}}(R'.) = Q\,\mathscr{F}^m$. The general solution of the homogeneous system $R'\,\eta = 0$ is thus given by $Q\,\mu$, for all $\mu \in \mathscr{F}^m$. Let $\zeta \in \ker_{\mathscr{F}}\left(\left(\begin{array}{c} R'' \\ T \end{array} \right) . \right)$ and consider the problem of exhibiting a particular solution of $R'\,\eta = \zeta$. In [23], the authors prove that such a particular solution can be given by $S\,\zeta$ where $S \in D^{p\times q'}$ satisfies that there exists $V \in D^{q'\times q}$ such that $R' - R'\,S\,R' = V\,R$. Indeed, we then have

$$R = R''\,R' \Rightarrow R' - R'\,S\,R' = V\,R''\,R' \Rightarrow (I_{q'} - R'\,S - V\,R'')\,R' = 0.$$

Consequently,

$$D^{1\times q'}\,(I_{q'} - R'\,S - V\,R'') \subseteq \ker_D(.R') = D^{1\times r'}\,T,$$

and there exists $W \in D^{q'\times r'}$ such that

$$I_{q'} - R'\,S - V\,R'' = W\,T.$$

Then we get

$$R'\,S + (V \quad W) \left(\begin{array}{c} R'' \\ T \end{array} \right) = I_{q'},$$

which proves that $R'\,(S\,\zeta) = \zeta$. Constructive algorithms for checking the existence of such matrices S and V and computing them when they exist have been developed in [22] and implemented in the OREMODULES package ([10]). Finally, we have obtained the following parametrization of $\ker_{\mathscr{F}}(R.)$:

$$\ker_{\mathscr{F}}(R.) = \{S\,\zeta + Q\,\mu \mid \left(\begin{array}{c} R'' \\ T \end{array} \right) \zeta = 0, \quad \mu \in \mathscr{F}^m\}.$$

An algorithm for computing such a parametrization has been implemented in the OREMODULES package ([10]). A direct consequence of this parametrization is that $\ker_{\mathscr{F}}(R.)$ can be obtained by computing $\ker_{\mathscr{F}}\left(\left(\begin{array}{c} R'' \\ T \end{array} \right) . \right)$. Note that in the case of a torsion-free module $M = D^{1\times p}/(D^{1\times q} R)$, we get the parametrization $\ker_{\mathscr{F}}(R.) = \{Q\,\mu \mid \mu \in \mathscr{F}^m\}$ which will not be an interested case for our problem.

We shall now prove the main theorem of this section which is a specialization of the previous results for $D = \mathbb{C}[x][\frac{d}{dx}]$. In particular, one can note that the solution space $\mathscr{F}$ considered (namely, regular formal solutions) is not an injective left D-module but thanks to the strong properties of $D = \mathbb{C}[x][\frac{d}{dx}]$ which is an hereditary ring (see [25]), we manage to generalize the previous procedure.

Theorem 2: Let $D = \mathbb{C}[x][\frac{d}{dx}]$ be the first Weyl algebra over the field $\mathbb{C}$ of complex numbers and consider a rectangular system of linear differential equations given by a matrix $R \in D^{q\times p}$ of full row-rank. Assume that $q \geq 2$ and let $\mathscr{F}$ be the space of functions $y(x)$ of the form $y(x) = x^{\lambda_0}\,z(x)$ where $\lambda_0 \in \mathbb{C}$ and $z(x) \in \mathbb{C}[\log(x)][[x]]^p$.

Then, there exist four matrices $Q \in D^{p\times m}$, $R' \in D^{q\times p}$ of full row-rank, $R'' \in D^{q\times q}$ (square) and $S \in D^{p\times q}$ satisfying

$$R = R''\,R', \; \ker_{\mathscr{F}}(R'.) = Q\,\mathscr{F}^m, \; R'\,S = I_q.$$

Moreover, we have

$$\ker_{\mathscr{F}}(R.) = \{S\,\zeta + Q\,\mu \mid R''\,\zeta = 0, \quad \mu \in \mathscr{F}^m\}.$$

Proof: Let $Q \in D^{p\times m}$ be such that $\ker_D(R.) = Q\,D^m$. We have the short exact sequence

$$0 \to \ker_D(.Q) \to D^{1\times p} \to M/t(M) \to 0.$$

By definition, $M/t(M)$ is a torsion-free left D-module so it is projective since $D = \mathbb{C}[x][\frac{d}{dx}]$ is an hereditary ring (see [25]). As a consequence, the previous short exact sequence splits which proves that $\ker_D(.Q)$ is a projective left D-module (see [25]). Moreover, we have

$$\mathrm{rank}(\ker_D(.Q)) = \mathrm{rank}(D^{1\times p}) - \mathrm{rank}(M/t(M)).$$

By definition, $\mathrm{rank}(M) = \dim_K(K \otimes_D M)$ where K is the left field of fractions of D (see [25]). Then, from the short exact sequence

$$0 \to t(M) \to M \to M/t(M) \to 0,$$

we get the short exact sequence

$$0 \to K \otimes_D t(M) \to K \otimes_D M \to K \otimes_D M/t(M) \to 0.$$

Now, $K \otimes_D t(M) = 0$ so that $K \otimes_D M = K \otimes_D M/t(M)$ and finally $\mathrm{rank}(M) = \mathrm{rank}(M/(t(M))$. By hypothesis R is of full row-rank so $\mathrm{rank}(M) = p - q$. Finally, we obtain $\mathrm{rank}(\ker_D(.Q)) = p - (p - q) = q$. Now, a result of Stafford (see [26]) asserts that projective modules of rank at least 2 over the Weyl algebras $A_n(k)$ and $B_n(k)$, where k a field of characteristic zero, are free which implies that $\ker_D(.Q)$ is free of rank q. So, computing a basis of this free module we can find $R' \in D^{q\times p}$ of full row-rank such that $\ker_D(.Q) = D^{1\times q} R'$. With the notations of the discussion before the theorem this implies $T = 0$. Moreover the matrix R'' satisfying that $R = R''\,R'$ is then a square matrix of size q. Now, we have

$$\mathrm{coker}_D(Q.) = D^p/(Q\,D^m) \cong R\,D^p \subset D^q.$$

Since $D = \mathbb{C}[x][\frac{d}{dx}]$ is an hereditary ring (see [25]), $\mathrm{coker}_D(Q.)$ is then a projective right D-module of finite type which implies that Q admits a generalized inverse ([21]). As a consequence, the left D-module $\mathrm{coker}_D(.Q)$ is also projective and the long exact sequence

$$0 \to D^{1\times q} \xrightarrow{.R'} D^{1\times p} \xrightarrow{.Q} D^{1\times m} \to \mathrm{coker}_D(.Q) \to 0,$$

splits, *i.e.*, there exist two matrices $X \in D^{p\times q}$ and $Y \in D^{m\times p}$ such that $I_p = X R' + Q Y$ (see [21]). Hence, for every $\nu \in \ker_{\mathscr{F}}(R'.)$, we get $\nu = X R' \nu + Q Y \nu = Q\,(Y \nu)$, *i.e.*, $\nu \in Q\mathscr{F}^m$ which proves that $\ker_{\mathscr{F}}(R'.) = Q\mathscr{F}^m$ since we clearly have $Q\mathscr{F}^m \subseteq \ker_{\mathscr{F}}(R'.)$. To end the proof, we need to show the existence of a right inverse $S \in D^{p\times q}$ of $R' \in D^{q\times p}$. As $M/t(M)$ is a torsion free module over the hereditary ring D, it is projective so that R' admits a generalized inverse (see [21]) $S \in D^{p\times q}$ satisfying $R' - R' S R' = (I_q - R' S) R' = 0$. Now since R' is of full row-rank, this yields $R' S = I_q$. ∎

In the previous theorem, we can replace $D = \mathbb{C}[x][\frac{d}{dx}]$ by $D = \mathbb{C}[[x]][\frac{d}{dx}]$ since, in [24], the authors prove that $\mathbb{C}[[x]][\frac{d}{dx}]$ have the needed algebraic properties. A direct consequence is that computing regular formal solutions of a full row-rank rectangular system of linear differential equations with power series coefficients having at least two equations reduces to computing regular formal solutions of a square linear differential system. An algorithm performing this reduction has been implemented in Maple using the package OreModules ([10]). Note that another approach for computing such a reduction in the ordinary differential case is proposed in [5] where the authors show how the (square) differential part and the algebraic part of a rectangular system can be uncoupled. We illustrate the previous theorem in the following simple example.

Example 1: Let $D = \mathbb{C}[t][d]$ with $d = \frac{d}{dt}$ and consider the rectangular linear differential system given by the matrix

$$R = \begin{pmatrix} 1 & t^3 d^2 - 2t^3 d + t^3 & 0 & 1 \\ t^2 & 2td + t^2 d - 2t - t^2 & 1+t & -t \\ 0 & 0 & -1 & t \end{pmatrix} \in D^{3\times 4}.$$

With the notations of Theorem 2 and using the package OreModules ([10]), we obtain

$$Q = \begin{pmatrix} -1 \\ 0 \\ t \\ 1 \end{pmatrix}, \quad R' = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & t \end{pmatrix},$$

$$R'' = \begin{pmatrix} 1 & t^3 d^2 - 2t^3 d + t^3 & 0 \\ t^2 & 2td + t^2 d - 2t - t^2 & -t-1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Applying the method developed in Section II, we find that the regular formal solutions of the square linear differential system $R''\zeta = 0$ where the series are truncated up to order 4 are given by

$$\zeta(t) = c \begin{pmatrix} O(t^5) \\ 1 + t + 1/2t^2 + 1/6t^3 + 1/24t^4 + O(t^5) \\ O(t^5) \end{pmatrix},$$

where $c \in \mathbb{C}$ is an arbitrary constant. Note that the second entry of $\zeta(t)$ is a constant times the development of the exponential. Theorem 2 then implies that the regular formal solutions of $R\eta = 0$ where series are truncated up to order 4 are given by $\eta(t) = S\zeta(t) + Q\mu(t)$ where $\mu(t)$ is an arbitrary element of the space of functions of the form $t^{\lambda_0} z(t)$ where $\lambda_0 \in \mathbb{C}$, $z(t) \in \mathbb{C}[\log(t)][[t]]$ and the power series are truncated up to order 4. Consequently, we get that the regular formal solutions of $R\eta(t) = 0 + O(t^5)$ are given by

$$\eta(t) = S\zeta(t) + Q\mu(t) = \begin{pmatrix} -\mu(t) \\ 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + O(t^5) \\ t\mu(t) \\ \mu(t) \end{pmatrix},$$

for all $\mu(t)$ defined as above.

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APPENDIX: MATRIX POLYNOMIALS

For the sake of completeness, we recall some properties of regular matrix polynomials that are used in Section II to extend the Frobenius method to the matrix case. Similar recalls appear in [3] and we refer to [13, Ch. 1 & 7] for more details.

Definition 2: Let $L(\lambda) \in \mathbb{C}[\lambda]^{n\times n}$ be a regular matrix polynomial. A complex number λ_0 is called *eigenvalue of* $L(\lambda)$ if $\det(L(\lambda_0)) = 0$: its multiplicity as a root of $\det(L(\lambda))$ is then called *the algebraic multiplicity of* λ_0 and denoted by $m_a(\lambda_0)$. We denote by $\sigma(L(\lambda))$ the set of all eigenvalues of $L(\lambda)$. For $\lambda_0 \in \sigma(L(\lambda))$, a nonzero vector $v \in \mathbb{C}^n$ satisfying $L(\lambda_0)\,v = 0$ is called *eigenvector of* $L(\lambda)$ *associated to the eigenvalue* λ_0: the dimension of the right nullspace of $L(\lambda_0)$ is called *the geometric multiplicity of* λ_0 and denoted by $m_g(\lambda_0)$.

Lemma 1: [13, Th. S1.10] Let $L(\lambda) \in \mathbb{C}[\lambda]^{n\times n}$ be a regular matrix polynomial, $\lambda_0 \in \sigma(L)$ an eigenvalue of $L(\lambda)$ and $m_g = m_g(\lambda_0)$ its geometric multiplicity. Then, there exist two matrix polynomials $E_{\lambda_0}(\lambda)$ and $F_{\lambda_0}(\lambda)$, invertible at $\lambda = \lambda_0$, such that $L(\lambda) = E_{\lambda_0}(\lambda)S_{\lambda_0}(\lambda)F_{\lambda_0}(\lambda)$ where $S_{\lambda_0}(\lambda)$ is a diagonal matrix polynomial which diagonal entries are $1,\ldots,1,(\lambda-\lambda_0)^{\kappa_1},\ldots,(\lambda-\lambda_0)^{\kappa_{m_g}}$. The κ_i are positive integers satisfying $\kappa_1 \leq \ldots \leq \kappa_{m_g}$, moreover, they are unique and called the partial multiplicities associated to λ_0. Moreover, we have $m_a(\lambda_0) = \sum_{i=1}^{m_g} \kappa_i$.

In what follows, for $p \in \mathbb{N}$, $L^{(p)}(\lambda)$ denotes the p-th derivative of $L(\lambda)$ w.r.t. λ.

Definition 3: Let $L(\lambda)$ be a square matrix polynomial and $\lambda_0 \in \sigma(L)$.

- A sequence of vectors $v_0 \neq 0, v_1, \ldots, v_{k-1}$ in $\mathbb{C}^n$ satisfying
$$\sum_{p=0}^{i} \frac{L^{(p)}(\lambda_0)}{p\,!}\, v_{i-p} = 0, \quad \text{for } i = 0,\ldots,k-1,$$
is called a *Jordan chain of length k associated to* λ_0. Note that v_0 is an eigenvector associated to λ_0 and $v_1,\ldots,v_{k-1}$ are called *generalized eigenvectors associated to* λ_0. Its length k is lower or equal to one of the partial multiplicities of λ_0.
- Let $v_0, v_1, \ldots, v_{k-1}$ be a Jordan chain of length k associated to $\lambda_0 \in \sigma(L)$. The polynomial
$$\phi(\lambda) = \sum_{i=0}^{k-1} v_i\,(\lambda-\lambda_0)^i,$$
is called *root polynomial of* $L(\lambda)$ *of order k associated to* λ_0. It satisfies $L(\lambda_0)\,\phi(\lambda_0) = 0$ and the first $k-1$ derivatives of $L(\lambda)\phi(\lambda)$ vanish at $\lambda = \lambda_0$.

Definition 4: Let $L(\lambda) \in \mathbb{C}[\lambda]^{n\times n}$ be a regular matrix polynomial and $\lambda_0 \in \sigma(L)$ an eigenvalue of $L(\lambda)$. An eigenvector v_0 associated to λ_0 is *of rank k* if the maximal order of a root polynomial $\phi(\lambda)$ associated to λ_0 with $\phi(\lambda_0) = v_0$ is k.

The rank of an eigenvector associated to an eigenvalue λ_0 is necessarily one of the partial multiplicities associated to λ_0. In other words, the partial multiplicities associated to a given eigenvalue λ_0 correspond to the maximum lengths of Jordan chains associated to λ_0. In [27, Ch. 3], Zúñiga proposes several algorithms for computing partial multiplicities and Jordan chains of maximal lengths associated to λ_0.

Lemma 2: Let $L(\lambda) \in \mathbb{C}[\lambda]^{n\times n}$ be a regular matrix polynomial and $\lambda_0 \in \sigma(L)$ an eigenvalue of $L(\lambda)$. Let κ_1 and κ_2 be two partial multiplicities associated to λ_0. Then, any two eigenvectors $v_{0,1}$ and $v_{0,2}$ associated to λ_0 of rank respectively κ_1 and κ_2 are linearly independent.