On the existence of various realizations

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Abstract— **The aim of this paper is to characterize the existence of polynomial, rational and Nash realizations with respect to one another. The existence of realizations within various classes of systems is the main topic of realization theory. In this paper it is shown that if there exists a polynomial realization of a response map then there exists also its rational realization. To disprove the converse implication we provide an example of a response map which is realizable by a rational system but not by a polynomial system. Further, it is shown that the existence of a rational realization implies the existence of a Nash realization of the same response map. The equivalence is proven for response maps defined on piecewise-constant inputs the values of which are of a finite set. However, the question whether the existence of a Nash realization implies the existence of a rational realization generally is still open. Additionally, we discuss the observability properties of polynomial, rational and Nash systems.**

I. Introduction

This paper deals with the existence of polynomial, rational and Nash realizations of response maps. Realizations of a given response map are the systems of ordinary differential equations with inputs and outputs such that their input-output behavior coincides with the response map. By polynomial and rational systems we refer to continuous-time dynamical systems with irreducible varieties as state-spaces and with dynamics and output functions defined by polynomial and rational functions on those varieties. Nash systems are continuous-time dynamical systems with Nash manifolds (smooth manifolds in $\mathbb{R}^n$ defined by polynomial equalities and inequalities) as state-spaces and with dynamics and output functions defined by analytic functions satisfying an algebraic equation.

Polynomial, rational and Nash systems arise in various fields such as systems biology and engineering. Depending on a chosen modeling framework within systems biology it may be possible to model biological phenomena by a system of one of these classes. For example, considering mass-action or Michaelis-Menten kinetics leads to polynomial or rational systems, respectively ([15]). Power-law modeling framework gives rise to Nash systems ([17]). Therefore, being able to find a polynomial, rational or Nash system realizing data characterized by a response map might simplify modeling techniques of systems biology or any other field where the underlying algebraic structure of the realizations is of any importance.

Realization theory concerns the existence of realizations within a specified class of systems, the system properties of realizations (such as their observability, controllability and minimality), and the algorithms for the construction of such realizations. It is studied for the class of polynomial, rational and Nash systems in [2], [13], [12], respectively. In those papers necessary and sufficient conditions for the existence of polynomial, rational and Nash realizations of a given response map are determined. The relations between the existence of these realizations are treated in this paper. Except their theoretical relevance for establishing the links between the results of realization theory for the classes of polynomial, rational, and Nash systems, the obtained relations allow one to check the existence of a realization within a relevant class of systems more efficiently. The usefulness of the study of system properties of realizations such as their observability lies in the possible applications of realization theory and in the ability to derive the most efficient algorithms. This paper also deals with the observability properties of polynomial, rational and Nash systems. It is proven that in the framework of linear systems the corresponding observability concepts (algebraic, rational and semi-algebraic observability) are equivalent. This leads for example to the conjecture that the existence of a rational realization which is rationally observable implies the existence of a Nash realization which is semi-algebraically observable.

The outline of the paper is as follows. Section II recalls the basic terminology of commutative algebra and algebraic geometry used within the paper. The classes of polynomial, rational and Nash systems are introduced in Section III. The main results are presented in Section IV and Section V. Section IV deals with the existence of polynomial, rational and Nash realizations and their relations. Section V is dedicated to the study of their observability properties. The paper is concluded by Section VI.

II. Preliminaries

The notation and terminology on commutative algebra and real-algebraic geometry is adopted from [19], [16], [4]. We refer to [19], [16] for the definitions of polynomial, algebra, integral domain and transcendence degree of a field. Recall that if A is an integral domain (over $\mathbb{R}$) then the *transcendence degree* of A over $\mathbb{R}$, denoted by trdeg A, is well-defined and it equals the transcendence degree of the field $F = \mathcal{Q}(A)$ of quotients of A over $\mathbb{R}$, i.e. it is the greatest number of algebraically independent elements of F over $\mathbb{R}$. Recall that $\varphi_1, \ldots, \varphi_s \in F$ are *algebraically independent* over $\mathbb{R}$ if there does not exist a non-zero polynomial $p \in \mathbb{R}[X_1, \ldots, X_s]$ such that $p(\varphi_1, \ldots, \varphi_s) = 0$ in F.

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A set $X \subseteq \mathbb{R}^n$ of zero points of finitely many polynomials of $\mathbb{R}[X_1,\dots,X_n]$ is called a *variety*. Hence, $X = \{(x_1,\dots,x_n) \in \mathbb{R}^n | f_1(x_1,\dots,x_n) = \cdots = f_N(x_1,\dots,x_n) = 0\}$ where $f_1,\dots,f_N \in \mathbb{R}[X_1,\dots,X_n]$ and $N < +\infty$. We say that X is *irreducible* if it cannot be written as a union of two non-empty varieties strictly contained in X. By a *polynomial* on an irreducible variety $X \subseteq \mathbb{R}^n$ we mean a map $p : X \to \mathbb{R}$ for which there exists $q \in \mathbb{R}[X_1,\dots,X_n]$ such that $p = q$ on X. We denote the algebra of all polynomials on X by A. Since X is irreducible, A is an integral domain which allows one to define the field $Q = \mathcal{Q}(A)$ of quotients of A. The elements of Q are called *rational functions* on X.

A subset $S \subseteq \mathbb{R}^n$ is *semi-algebraic* if it is of the form

$$\bigcap_{i=1}^{d} \bigcup_{j=1}^{m_i} \{(x_1,\dots,x_n) \in \mathbb{R}^n | P_{i,j}(x_1,\dots,x_n) \ \epsilon_{i,j} \ 0\},$$

where for each $i = 1,\dots,d$ and $j = 1,\dots,m_i$ the symbol $\epsilon_{i,j} \in \{<,=\}$ and $P_{i,j} \in \mathbb{R}[X_1,\dots,X_n]$. We say that a semi-algebraic subset $S \subseteq \mathbb{R}^n$ is *semi-algebraically connected* if it cannot be written as a union of two disjoint closed semi-algebraic sets in S. Let $S_1 \subseteq \mathbb{R}^n, S_2 \subseteq \mathbb{R}^m$ be two semi-algebraic sets. A mapping $f : S_1 \to S_2$ is called *semi-algebraic* if its graph $\{(x^\top, y^\top)^\top \in \mathbb{R}^{n+m} | x \in S_1, y \in S_2, f(x) = y\}$ is a semi-algebraic set in $\mathbb{R}^{n+m}$. By a *Nash function* on a semi-algebraic subset S of $\mathbb{R}^n$ we mean an analytic function from S to $\mathbb{R}$ which is also semi-algebraic, for more details see for example [5]. We denote the ring of Nash functions on S by $\mathcal{N}(S)$. A *Nash submanifold* X of $\mathbb{R}^n$ is a semi-algebraic set which is also an analytic manifold.

III. Polynomial, Rational and Nash systems

The framework of polynomial and rational systems used in this paper is adopted from [2], [1]. A polynomial or a rational system Σ is given as a quadruple (X, f, h, x_0). Here X denotes a state-space defined as an irreducible variety, f is a family of polynomial or rational vector fields which determine the dynamics of the polynomial or the rational system, respectively. The output function of Σ is specified by h which is componentwise a polynomial or a rational function on X. The point x_0 of X specifies the initial state of Σ. Nash systems considered in this paper are introduced in [12]. They generalize the class of rational systems by allowing semi-algebraic sets being state-spaces and by considering semi-algebraic functions to define dynamics and output functions.

To define polynomial, rational and Nash systems formally we first recall the notion of input functions. The *input-space* U is a subset of $\mathbb{R}^m$ and the *output-space* is $\mathbb{R}^r$. As the *space of input functions* with a given input-space U we consider the set $\mathcal{U}_{pc}$ of piecewise-constant functions $u : [0, T_u] \to U$, where $T_u \in [0, +\infty)$ is the maximal finite time instance for which u is defined. Thus, any $u \in \mathcal{U}_{pc}$ can be identified with a finite sequence $(\alpha_1, t_1) \cdots (\alpha_n, t_n)$ where $\alpha_i \in U$, $t_i \in [0, \infty)$ for $i = 1,\dots,n$. Then, for $t \in [\sum_{j=0}^{i} t_j, \sum_{j=0}^{i+1} t_j)$, $u(t) = \alpha_{i+1}$ for $i = 0, 1, \dots, n-1$, $t_0 = 0$, and $u(\sum_{j=0}^{n} t_j) = \alpha_n$. Every $u = (\alpha_1, t_1) \cdots (\alpha_n, t_n) \in \mathcal{U}_{pc}$ has a time domain $[0, T_u]$ where $T_u = \sum_{j=1}^{n} t_j$.

Definition III.1 *A* polynomial/rational/Nash system Σ *with an input-space U and an output-space $\mathbb{R}^r$ is a quadruple (X, f, h, x_0) where*

(i) *the state-space X is*
   - *an irreducible variety in $\mathbb{R}^n$ (for polynomial and rational systems)*
   - *a Nash submanifold of $\mathbb{R}^n$ which is semi-algebraically connected (for Nash systems),*

(ii) *the dynamics of the system is given by $\dot{x}(t) = f(x(t), u(t))$ for an input $u \in \mathcal{U}_{pc}$, where $f : X \times U \to \mathbb{R}^n$ is such that for every input value $\alpha \in U$ the components $f_{\alpha,i} : X \to \mathbb{R}$, $i = 1,\dots,n$ of $f(x,\alpha) = (f_{\alpha,1}(x),\dots,f_{\alpha,n}(x))$ are polynomial/rational/Nash functions on X ($f_{\alpha,i}$ is the ith coordinate of the vector field $f_\alpha : X \ni x \mapsto f(x,\alpha) \in \mathbb{R}^n$),*

(iii) *the output of the system is specified by the map $h : X \to \mathbb{R}^r$, the components $h_1,\dots,h_r$ of h are polynomial/rational/Nash functions on X,*

(iv) *$x_0 = x(0) \in X$ is the initial state of Σ.*

The *state trajectory* of a polynomial/rational/Nash system $\Sigma = (X, f, h, x_0)$ corresponding to an input $\mathcal{U}_{pc} \ni u = (\alpha_1, t_1) \cdots (\alpha_k, t_k) : [0, T_u] \to U$ is a continuous piecewise-differentiable function $x_\Sigma(.; x_0, u) : [0, T_u] \to X$ such that $x_\Sigma(0; x_0, u) = x_0$ and $\frac{d}{dt} x_\Sigma(t; x_0, u) = f(x_\Sigma(t; x_0, u), u(t))$ for $t \in (\sum_{j=0}^{i} t_j, \sum_{j=0}^{i+1} t_j)$, $i = 0,\dots,k-1$, $t_0 = 0$. For Σ one considers a system $\mathcal{U}_{pc}(\Sigma)$ of *admissible inputs* which is the subset of piecewise-constant inputs for which the trajectories of Σ are well-defined.

IV. Existence of realizations

Let us first review the main results on the existence of polynomial, rational and Nash realizations of response maps.

We say that a map $p : \mathcal{U}_{pc} \to \mathbb{R}^r$ is a *response map* if the components $p_i : \mathcal{U}_{pc} \to \mathbb{R}$ of p are the functions analytic with respect to switching times of the inputs of $\mathcal{U}_{pc}$. In particular, we assume that for every input $u = (\alpha_1, t_1) \cdots (\alpha_k, t_k) \in \mathcal{U}_{pc}$ the function $p_{i;\alpha_1,\dots,\alpha_k}(t_1,\dots,t_k) = p_i((\alpha_1, t_1) \cdots (\alpha_k, t_k))$ is analytic. One can prove that the set $\mathcal{A}(\mathcal{U}_{pc} \to \mathbb{R})$ of all real functions defined on $\mathcal{U}_{pc}$ which are analytic with respect to switching times of the inputs of $\mathcal{U}_{pc}$ is an integral domain.

We denote by D_α the derivatives of a function from $\mathcal{U}_{pc}$ to $\mathbb{R}$ at the time where an input switches to the value $\alpha \in U$. Hence, $(D_\alpha \varphi)(u) = \frac{d}{dt} \varphi((u)(\alpha, t))|_{t=0+}$ where $\varphi : \mathcal{U}_{pc} \to \mathbb{R}$ and $(u)(\alpha, t) \in \mathcal{U}_{pc}$.

Note that the response maps considered in [2], [13], [12] are not necessarily defined on $\mathcal{U}_{pc}$, but only on subsets S of $\mathcal{U}_{pc}$ which allow taking D_α derivatives and which imply that $\mathcal{A}(S \to \mathbb{R})$ is an integral domain. For simplicity, in this paper we consider only the response maps defined on $\mathcal{U}_{pc}$.

Definition IV.1 *Let $p : \mathcal{U}_{pc} \to \mathbb{R}^r$ be a response map. The* observation algebra *$A_{obs}(p)$ of p is the smallest subalgebra of $\mathcal{A}(\mathcal{U}_{pc} \to \mathbb{R})$ which contains the components $p_i, i = 1,\dots,r$ of p, and which is closed with respect to D_α derivatives for* *all $\alpha \in U$. The* observation field $Q_{obs}(p)$ *of p is the field of quotients of $A_{obs}(p)$.*

The problem of the existence of polynomial/rational/Nash realizations for a given response map consists of determining a polynomial/rational/Nash system $\Sigma = (X, f, h, x_0)$ such that

$\mathcal{U}_{pc} \subseteq \mathcal{U}_{pc}(\Sigma)$ and $p(u) = h(x_\Sigma(T_u; x_0, u))$ for all $u \in \mathcal{U}_{pc}$.

The following theorem states the main results on the existence of polynomial, rational and Nash realizations presented in [2], [13, Theorem 5.16] and [12, Theorem IV.3], respectively.

Theorem IV.2 *Let p be a response map. Then,*

(i) *there exists a polynomial realization of p if and only if $A_{obs}(p)$ is finitely generated,*
(ii) *there exists a rational realization of p if and only if $Q_{obs}(p)$ is a finite field extension of $\mathbb{R}$,*
(iii) *if there exists a Nash realization of p then* trdeg $A_{obs}(p) < +\infty$.

A. Polynomial and rational realizations

According to Theorem IV.2(i),(ii), the relation between the existence of polynomial realizations and the existence of rational realizations for a given response map p is determined by the relation between the properties of $A_{obs}(p)$ and $Q_{obs}(p)$ being finitely generated. Obviously, if $A_{obs}(p)$ is finitely generated then $Q_{obs}(p)$ is a finite field extension of $\mathbb{R}$. Thus, if a response map p is realizable by a polynomial system then there also exists a rational realization of p. Note that for proving this implication we do not need to use Theorem IV.2. It follows directly from the fact that polynomial systems are a subclass of rational systems. The following example provides a counter-example for the statement: The existence of rational realizations of p implies the existence of polynomial realizations of p.

Example IV.3 *Let us consider a response map p defined as $p(u) = (exp(1 - \int_0^{T_u} u(\tau)d\tau))(1 - \int_0^{T_u} u(\tau)d\tau)^{-1}$ for the inputs $u : [0, T_u] \to U = \mathbb{R}$ such that $\int_0^{T_u} |u(\tau)|d\tau < 1$, where $T_u \in [0, \infty)$ depends on u.*

Let us denote $\varphi_1 = exp(1 - \int_0^{T_u} u(\tau)d\tau)$ and $\varphi_2 = (1 - \int_0^{T_u} u(\tau)d\tau)^{-1}$. Then, $p = \varphi_1\varphi_2$ which implies that $\varphi_1\varphi_2 \in A_{obs}(p)$. For any $\alpha \in \mathbb{R}$ it holds that $D_\alpha p = -\alpha p + \alpha\varphi_1\varphi_2^2$. This implies that $\varphi_1\varphi_2^2 \in A_{obs}(p)$. Further, for any $\alpha \in \mathbb{R}$ it holds that $D_\alpha(\varphi_1\varphi_2^2) = (D_\alpha\varphi_1)\varphi_2^2 + \varphi_1(D_\alpha\varphi_2^2) = -\alpha\varphi_1\varphi_2^2 - \varphi_1 2\varphi_2(-1)(-\alpha)\varphi_2^2 = -\alpha\varphi_1\varphi_2^2 + 2\alpha\varphi_1\varphi_2^3$ which implies that $\varphi_1\varphi_2^3 \in A_{obs}(p)$. Taking further D_α derivatives one derives that

$$A_{obs}(p) = \mathbb{R}[\varphi_1\varphi_2, \varphi_1\varphi_2^2, \varphi_1\varphi_2^3, \ldots].$$

Thus, by Theorem IV.2(i), p does not have a polynomial realization.

But, because $Q_{obs}(p) = \mathcal{Q}(A_{obs}(p)) = \mathbb{R}(\varphi_1, \varphi_2)$, Theorem IV.2(ii) implies that p has a rational realization.

The theorem below summarizes the results of this section.

Theorem IV.4 *If there exists a polynomial realization of a response map p then there exists also a rational realization of p. On the contrary, there exists a response map p realizable by a rational system such that it is not realizable by a polynomial system.*

B. Rational and Nash realizations

Because irreducible varieties of $\mathbb{R}^n$ are not Nash submanifolds of $\mathbb{R}^n$ in general, rational systems are not a subclass of Nash systems. Hence, the existence of Nash realizations for a given response map p does not follow from the existence of rational realizations of p trivially. However, we prove this implication below.

Theorem IV.5 *Let p be a response map. If p is realizable by a rational system then there exists a Nash realization of p.*

Proof: Let p be a response map realizable by a rational system. Then, by Theorem IV.2(ii), $Q_{obs}(p)$ is a finite field extension of $\mathbb{R}$, i.e. there exist $k \in \mathbb{N}$ minimal and $\varphi_1, \ldots, \varphi_k \in \mathcal{A}(\mathcal{U}_{pc} \to \mathbb{R})$ so that $Q_{obs}(p) = \mathbb{R}(\varphi_1, \ldots, \varphi_k)$. By following the steps of the constructive proof of Theorem IV.2(ii) presented in [13] one derives the rational system $\Sigma = (X, f, h, x_0)$ realizing p such that

$$\begin{aligned} X &= \mathbb{R}^k, \\ f &= \{f_\alpha = \sum_{i=1}^k v_i^\alpha \frac{\partial}{\partial x_i} | \alpha \in U\}, \\ h &= (h_1, \ldots, h_r) \text{ with } h_j(x_1, \ldots, x_k) = w_j(x_1, \ldots, x_k), \\ x_0 &= (\varphi_1(e), \ldots, \varphi_k(e)), \end{aligned}$$

where e is the empty input ($T_e = 0$) and $w_j, v_i^\alpha \in \mathbb{R}(X_1, \ldots, X_k)$ are such that $p_j = w_j(\varphi_1, \ldots, \varphi_k)$ and $D_\alpha\varphi_i = v_i^\alpha(\varphi_1, \ldots, \varphi_k)$ for $j = 1, \ldots, r$, $i = 1, \ldots, k$, $\alpha \in U$.

Hence, if there exists a rational realization of p then there exists the rational realization $\Sigma = (X, f, h, x_0)$ of p of the above form. Because $X = \mathbb{R}^k$ is a Nash submanifold of $\mathbb{R}^k$ which is semi-algebraically connected and because $v_i^\alpha, w_j \in \mathcal{N}(\mathbb{R}^k)$ for all $i = 1, \ldots, k$, $j = 1, \ldots, r$, $\alpha \in U$, it follows that Σ is a Nash system realizing p. ■

Since trdeg $A_{obs}(p) =$ trdeg $Q_{obs}(p)$, it follows from Theorem IV.2(ii),(iii) that to prove the implication "if a response map p is realizable by a Nash system then p is also realizable by a rational system" it is sufficient to prove the following: If trdeg $Q_{obs}(p) < +\infty$ then $Q_{obs}(p)$ is a finite field extension of $\mathbb{R}$. Below we show that this statement is valid for response maps defined on inputs whose values vary only within a finite input-space U. In general, finite transcendence degree (over $\mathbb{R}$) of a field extension E of $\mathbb{R}$ does not imply that E is a finite field extension of $\mathbb{R}$. Therefore, the question whether the statement "trdeg $Q_{obs}(p) < +\infty \Rightarrow \exists k \in \mathbb{N}, \varphi_1, \ldots, \varphi_k \in \mathcal{A}(\mathcal{U}_{pc} \to \mathbb{R}) : Q_{obs}(p) = \mathbb{R}(\varphi_1, \ldots, \varphi_k)$"

holds for all response maps $p : \mathcal{U}_{pc} \to \mathbb{R}^r$ is still an open problem.

Theorem IV.6 *Let $p : \mathcal{U}_{pc} \to \mathbb{R}^r$ be a response map, where $\mathcal{U}_{pc}$ is such that $|U| < +\infty$. Then, if p has a Nash realization then there exists also a rational realization of p.*

Proof: As explained above, it is sufficient to show that trdeg $Q_{obs}(p) < +\infty$ implies that $Q_{obs}(p)$ is a finite field extension of $\mathbb{R}$.

Let trdeg $Q_{obs}(p) = k < +\infty$ and let $\varphi_1, \ldots, \varphi_k \in Q_{obs}(p)$ be a transcendence basis of $Q_{obs}(p)$. By the definition of transcendence degree,

$$\forall \varphi \in Q_{obs}(p)\ \exists p \in \mathbb{R}[X_1, \ldots, X_k, X_{k+1}], p \neq 0 :$$
$$p(\varphi_1, \ldots, \varphi_k, \varphi) = 0.$$

Let $\varphi \in Q_{obs}(p)$ be arbitrary. Then there exists a polynomial $p = \sum_{a_1,\ldots,a_{k+1}=0}^{N} c_{a_1,\ldots,a_{k+1}} X_1^{a_1} \cdots X_{k+1}^{a_{k+1}}$ such that

$$0 = p(\varphi_1, \ldots, \varphi_k, \varphi) = \sum_{a_1,\ldots,a_{k+1}=0}^{N} c_{a_1,\ldots,a_{k+1}} \varphi_1^{a_1} \cdots \varphi_k^{a_k} \varphi^{a_{k+1}},$$

where $N \in \mathbb{N}$ and not all of $c_{a_1,\ldots,a_{k+1}} \in \mathbb{R}$ equal zero. For any $\alpha \in U$, $0 = D_\alpha p(\varphi_1, \ldots, \varphi_k, \varphi) = \sum_{a_1,\ldots,a_{k+1}=0}^{N} c_{a_1,\ldots,a_{k+1}} D_\alpha(\varphi_1^{a_1} \cdots \varphi_k^{a_k}) \varphi^{a_{k+1}} + \sum_{a_1,\ldots,a_{k+1}=0}^{N} c_{a_1,\ldots,a_{k+1}} \varphi_1^{a_1} \cdots \varphi_k^{a_k} a_{k+1} \varphi^{a_{k+1}-1} D_\alpha \varphi$. If $\sum_{a_1,\ldots,a_{k+1}=0}^{N} c_{a_1,\ldots,a_{k+1}} \varphi_1^{a_1} \cdots \varphi_k^{a_k} a_{k+1} \varphi^{a_{k+1}-1} = 0$ in $Q_{obs}(p)$ then $\varphi_1, \ldots, \varphi_k, \varphi$ would be algebraically independent over $\mathbb{R}$ which would contradict the fact that $\varphi_1, \ldots, \varphi_k$ is a transcendence basis of $Q_{obs}(p)$. Hence, $\sum_{a_1,\ldots,a_{k+1}=0}^{N} c_{a_1,\ldots,a_{k+1}} \varphi_1^{a_1} \cdots \varphi_k^{a_k} a_{k+1} \varphi^{a_{k+1}-1} \neq 0$ and

$$D_\alpha \varphi = \frac{-\sum_{a_1,\ldots,a_{k+1}=0}^{N} c_{a_1,\ldots,a_{k+1}} D_\alpha(\varphi_1^{a_1} \cdots \varphi_k^{a_k}) \varphi^{a_{k+1}}}{\sum_{a_1,\ldots,a_{k+1}=0}^{N} c_{a_1,\ldots,a_{k+1}} \varphi_1^{a_1} \cdots \varphi_k^{a_k} a_{k+1} \varphi^{a_{k+1}-1}}.$$

Therefore,

$$D_\alpha \varphi \in \mathbb{R}(\varphi_1, \ldots, \varphi_k, D_\alpha \varphi_1, \ldots, D_\alpha \varphi_k, \varphi).$$

By taking further D_α derivatives one proves that $D_{\alpha_1} \cdots D_{\alpha_l} \varphi \in \mathbb{R}(\{\varphi_1, \ldots, \varphi_k, D_\alpha \varphi_1, \ldots, D_\alpha \varphi_k, \varphi | \alpha \in U\})$ for all $l \in \mathbb{N}, \alpha_1, \ldots, \alpha_l \in U$. In particular, for all $l \in \mathbb{N}, \alpha_1, \ldots, \alpha_l \in U$ and $i = 1, \ldots, r$,

$$D_{\alpha_1} \cdots D_{\alpha_l} p_i$$
$$\in \mathbb{R}(\{\varphi_1, \ldots, \varphi_k, D_\alpha \varphi_1, \ldots, D_\alpha \varphi_k, p_1, \ldots, p_r | \alpha \in U\}).$$

Consequently, as $Q_{obs}(p) = \mathbb{R}(\{p_1, \ldots, p_r, D_{\alpha_1} \cdots D_{\alpha_l} p_1, \ldots, D_{\alpha_1} \cdots D_{\alpha_l} p_r | l \in \mathbb{N}, \alpha_1, \ldots, \alpha_l \in U\})$, it follows that $Q_{obs}(p) \subseteq \mathbb{R}(\{\varphi_1, \ldots, \varphi_k, D_\alpha \varphi_1, \ldots, D_\alpha \varphi_k, p_1, \ldots, p_r | \alpha \in U\})$. Moreover, since $\{\varphi_1, \ldots, \varphi_k, D_\alpha \varphi_1, \ldots, D_\alpha \varphi_k | \alpha \in U\} \subseteq Q_{obs}(p)$, we get $Q_{obs}(p) = \mathbb{R}(\{\varphi_1, \ldots, \varphi_k, D_\alpha \varphi_1, \ldots, D_\alpha \varphi_k, p_1, \ldots, p_r | \alpha \in U\})$. Because $|U| < +\infty$ and thus $|\{\varphi_1, \ldots, \varphi_k, D_\alpha \varphi_1, \ldots, D_\alpha \varphi_k, p_1, \ldots, p_r | \alpha \in U\}| < +\infty$, the equality above implies that $Q_{obs}(p)$ is a finite field extension of $\mathbb{R}$. ∎

V. Observability properties

Recall that A stands for the algebra of all polynomials on an irreducible variety X and that Q stands for its quotient field, i.e. Q denotes the field of all rational functions on X.

Definition V.1 *Let $\Sigma = (X, f = \{f_\alpha \mid \alpha \in U\}, h)$ be a polynomial or a rational system. The* observation algebra *$A_{obs}(\Sigma)$ of Σ is the smallest subalgebra of the field Q which contains all components $h_i, i = 1, \ldots, r$ of h, and which is closed with respect to the derivatives given by the vector fields $f_\alpha, \alpha \in U$. The* observation field *$Q_{obs}(\Sigma)$ of Σ is the field of quotients of $A_{obs}(\Sigma)$.*

Recall that since X is an irreducible variety, A is an integral domain. Further, as the field of fractions of an integral domain is also an integral domain, Q is an integral domain. Because $A_{obs}(\Sigma)$ is a subalgebra of Q, it is an integral domain, too. Therefore, the observation field $Q_{obs}(\Sigma)$ is well-defined. Note that $Q_{obs}(\Sigma)$ is also closed with respect to the derivatives given by the vector fields f_α for all $\alpha \in U$.

Definition V.2 *Let $\Sigma = (X, f = \{f_\alpha \mid \alpha \in U\}, h)$ be a polynomial or a rational system. The system Σ is called* algebraically observable *if $A_{obs}(\Sigma) = A$ and* rationally observable *if $Q_{obs}(\Sigma) = Q$.*

Algebraic and rational observability of polynomial/rational systems are related in [1]. Obviously, if a polynomial/rational system is algebraically observable then it is also rationally observable. However, there exist polynomial and rational systems which are rationally observable but not algebraically observable, see Example V.3. Therefore, algebraic observability of a polynomial/rational system implies its rational observability but not the other way round.

Example V.3 *Let $\Sigma = (X, f = \{f_\alpha \mid \alpha \in \mathbb{R}\}, h)$ be a polynomial system given as $X = \mathbb{R}, f_\alpha = \alpha x^2 \frac{\partial}{\partial x}$ for $\alpha \in \mathbb{R}, h = x^2$. By simple calculation we derive that*

$$A_{obs}(\Sigma) = \mathbb{R}[X^2, X^3, X^4, \ldots] \subsetneq \mathbb{R}[X] = A.$$

By Definition V.2, Σ is not algebraically observable. On the other hand, for the observation field of Σ it holds that $Q_{obs}(\Sigma) = \mathbb{R}(X) = Q$ and thus the system Σ is rationally observable.

The observation algebra for a Nash system Σ can be defined in the same way as in Definition V.1 for polynomial and rational systems, see [12]. Then $A_{obs}(\Sigma)$ is a subalgebra of $\mathcal{N}(X)$. Let us define the *Nash extension* $A_{obs}^{Nash}(\Sigma)$ of $A_{obs}(\Sigma)$ as follows:

$$A_{obs}^{Nash}(\Sigma) = \{g : X \to \mathbb{R} \mid \exists\, k \in \mathbb{N}\ \exists\, \varphi_1, \ldots, \varphi_k \in A_{obs}(\Sigma)\ \exists\, q \in \mathcal{N}(X) : g = q(\varphi_1, \ldots, \varphi_k)\}.$$

Definition V.4 *We say that a Nash system Σ is* semi-algebraically observable *if $A_{obs}^{Nash}(\Sigma) = \mathcal{N}(X)$.*

A. Relations with other concepts of observability

Let $\Sigma = (X, f, h)$ be a polynomial/rational/Nash system. We say that Σ is *observable* if it has no indistinguishable states. Two states $x_1 \neq x_2 \in X$ are *distinguishable* if there exists an admissible input u such that $h(x_\Sigma(T_u; x_1, u)) \neq h(x_\Sigma(T_u; x_2, u))$.

The algebra A of all polynomials on an irreducible variety X is a system of functions on X which distinguishes the points of X (if $a \neq b$ are two different points of X then there exists $p \in A$ such that $p(a) \neq p(b)$). Since algebraic observability of a polynomial system Σ implies that $A_{obs}(\Sigma) = A$, one obtains that $A_{obs}(\Sigma)$ distinguishes the points of X for an algebraically observable polynomial system $\Sigma = (X, f, h)$. By this property of $A_{obs}(\Sigma)$, Z. Bartosiewicz shows in [2, Proposition 3] that algebraically observable polynomial systems are observable.

Consider a rational system $\Sigma = (X, f = \{f_\alpha \mid \alpha \in U\}, h)$ and two points $x_1 \neq x_2 \in X$ such that all components of h and at least one of f_α, $\alpha \in U$ are defined at x_1 and x_2 (we say that $x_1, x_2 \in X_\Sigma$). Let $\mathcal{U}$ denote the set of inputs of $\mathcal{U}_{pc}$ which are admissible for both systems $\Sigma_1 = (X, f, h, x_1)$ and $\Sigma_2 = (X, f, h, x_2)$, i.e. $\mathcal{U} = \mathcal{U}_{pc}(\Sigma_1) \cap \mathcal{U}_{pc}(\Sigma_2)$. We say that x_1 and x_2 are *indistinguishable* if $h(x_\Sigma(T_u; x_1, u)) = h(x_\Sigma(T_u; x_2, u))$ for all $u \in \mathcal{U}$. Then, a rational system Σ is *observable* if it has no indistinguishable states.

Let us assume that a rational system $\Sigma = (X, f, h)$ is such that all initialized rational systems $\Sigma_0 = (X, f, h, x_0)$, where $x_0 \in X_\Sigma$, have the same admissible inputs $\mathcal{U} = \mathcal{U}_{pc}(\Sigma_0)$. Then, in the same way as in [2, Proposition 3], one concludes that algebraically observable rational system Σ is observable. Moreover, because Q distinguishes the points of X, if Σ is rationally observable then it is observable.

In [12, Proposition IV.11] it is proven that if a Nash system Σ is semi-algebraically observable then any two states $x_1 \neq x_2$ of Σ are distinguishable by an element of $A_{obs}(\Sigma)$, i.e. $\exists g \in A_{obs}(\Sigma) : g(x_1) \neq g(x_2)$. Let $\Sigma = (X, f, h)$ be a Nash system such that for all $x \in X$ and for all $u \in \mathcal{U}_{pc}$ the trajectory $x_\Sigma(T_u; x, u)$ is well-defined. Consequently, if Σ is semi-algebraically observable then Σ is observable, see [12, Corollary IV.12].

Finally, the differential geometric conditions for observability of nonlinear systems yield necessary conditions for algebraic, rational, and semi-algebraic observability of polynomial, rational, and Nash systems, respectively.

There are many observability concepts for nonlinear systems, [14], [18], [11], [7], [10] and others. In [3] the relations between several of these concepts are reviewed. Let us point out that algebraic observability in differential-algebraic setting [6], [9], [7], [8] has a different meaning than algebraic observability introduced in Definition V.2. We leave the further comparison of algebraic, rational, and semi-algebraic observability and other nonlinear notions of observability for future research.

B. Linear systems

In this section we show that the notions of observability and of algebraic, rational and semi-algebraic observability are all equivalent in case of linear systems.

Let Σ be a linear system with the state-space $X = \mathbb{R}^n$ given as

$$\begin{aligned} \dot{x} &= Ax + Bu, \;\; x(0) = x_0, \\ y &= Cx + Du, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{r\times n}, D \in \mathbb{R}^{r\times m}$. We assume that the inputs u are piecewise-constant functions with the values in $U \subseteq \mathbb{R}^m$. Because observability of linear systems does not depend on the inputs, to study observability of the system Σ it is sufficient to study observability of the linear system Σ_0 given as

$$\begin{aligned} \dot{x} &= Ax, \;\; x(0) = x_0, \\ y &= Cx. \end{aligned} \tag{2}$$

So, Σ is observable if and only if Σ_0 is observable if and only if the observability rank condition is satisfied.

Theorem V.5 *Consider a linear system Σ of the form (1) and the corresponding linear system Σ_0 determined by (2). The following statements are equivalent:*

(i) *Σ is observable,*
(ii) *Σ_0 is algebraically observable,*
(iii) *Σ_0 is rationally observable,*
(iv) *Σ_0 is semi-algebraically observable.*

Proof: Σ_0 is a linear system defined on the state-space $X = \mathbb{R}^n$. The dynamics of Σ_0 is given by the vector field $f = Ax\frac{\partial}{\partial x} = \sum_{i=1}^n A_i x \frac{\partial}{\partial x_i}$ on X, where A_i denotes the i-th row of the matrix A. The output function of Σ_0 is the map $h(x) = Cx = \begin{pmatrix} C_1 x \\ \vdots \\ C_r x \end{pmatrix}$, where C_i denotes the i-th row of the matrix C. By applying the vector field f to the components of the output map h we derive that

$$\begin{aligned} f(C_j x) &= \sum_{i=1}^n A_i x \frac{\partial}{\partial x_i}(C_j x) = \sum_{i=1}^n A_i x C_{j,i} = C_j A x \\ &\vdots \\ \underbrace{f \cdots f}_{k-times}(C_j x) &= f^k(C_j x) = C_j A^k x \end{aligned}$$

for $j = 1, \ldots, r$ and $k \in \mathbb{N}$. Therefore, the observation algebra $A_{obs}(\Sigma_0)$ equals the algebra $\mathbb{R}[\{C_j x, f^k(C_j x) \mid j = 1, \ldots, r; k \in \mathbb{N}\}] = \mathbb{R}[\{C_j x, C_j A^k x \mid j = 1, \ldots, r; k \in \mathbb{N}\}]$. So, Σ_0 is algebraically observable if $A_{obs}(\Sigma_0) = \mathbb{R}[\{C_j x, C_j A^k x \mid j = 1, \ldots, r; k \in \mathbb{N}\}] = \mathbb{R}[x_1, \ldots, x_n] = A$, and it is rationally observable if $Q_{obs}(\Sigma_0) = \mathcal{Q}(A_{obs}(\Sigma_0)) = \mathbb{R}(x_1, \ldots, x_n) = Q$.

$(i) \Leftrightarrow (ii)$ We show that the observability rank condition for the system Σ is satisfied if and only if $A_{obs}(\Sigma_0) = \mathbb{R}[\{C_j x, C_j A^k x \mid j = 1, \ldots, r; k \in \mathbb{N}\}] = \mathbb{R}[x_1, \ldots, x_n] = A$. The Cayley-Hamilton theorem implies that for proving the equality $A_{obs}(\Sigma_0) = A$ it is sufficient to prove that $\mathbb{R}[\{C_j x, C_j A x, \ldots, C_j A^{n-1} x \mid j = 1, \ldots, r\}] = \mathbb{R}[x_1, \ldots, x_n]$.

Note that for all $i = 1, \ldots, n$ it holds that $x_i \in \mathbb{R}[\{C_jx, C_jAx, \ldots, C_jA^{n-1}x \mid j = 1, \ldots, r\}]$ if and only if $x_i \in < \{C_jx, C_jAx, \ldots, C_jA^{n-1}x \mid j = 1, \ldots, r\} >$. Here $< \{a_1, \ldots, a_s\} >$ denotes the linear vector space over $\mathbb{R}$ generated by the elements $a_1, \ldots, a_s$. Indeed, if $x_i \in \mathbb{R}[\{C_jx, C_jAx, \ldots, C_jA^{n-1}x \mid j = 1, \ldots, r\}]$ then there exists a polynomial $p_i \in \mathbb{R}[X_1, \ldots, X_{rn}]$ such that $x_i = p_i(C_1x, \ldots, C_rA^{n-1}x)$. Therefore, $x_i = \sum_{j=1,\ldots,r,\ k=0,\ldots,n-1,\ l\in\mathbb{N}\cup\{0\}} a_{j,k,l}(C_jA^kx)^l$ with finitely many non-zero coefficients $a_{j,k,l} \in \mathbb{R}$. Because the degree of every monomial of $(C_jA^kx)^l$ equals l and because x_i is a monomial of degree 1, it follows that $a_{j,k,l} = 0$ for every $l \in (\mathbb{N} \cup \{0\}) \setminus \{1\}$. Thus, $x_i = \sum_{j=1,\ldots,r,\ k=0,\ldots,n-1} a_{j,k,1}C_jA^kx$ which implies that x_i belongs to $< \{C_jx, C_jAx, \ldots, C_jA^{n-1}x \mid j = 1, \ldots, r\} >$. The converse implication is obvious.

The rank condition rank $\begin{pmatrix} C \\ \vdots \\ CA^{n-1} \end{pmatrix} = n$ is satisfied if and only if $e_i \in <\{C_j, C_jA, \ldots, C_jA^{n-1} \mid j = 1, \ldots, r\}>$ for $i = 1, \ldots, n$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the unit vector of $\mathbb{R}^n$ with 1 on the i-th place. This is equivalent to $x_i \in < \{C_jx, C_jAx, \ldots, C_jA^{n-1}x \mid j = 1, \ldots, r\} >$ for $i = 1, \ldots, n$, and further, by the paragraph above, to $x_i \in \mathbb{R}[\{C_jx, C_jAx, \ldots, C_jA^{n-1}x \mid j = 1, \ldots, r\}]$ for $i = 1, \ldots, n$, i.e. to $A_{obs}(\Sigma_0) = \mathbb{R}[x_1, \ldots, x_n] = A$. Therefore, rank $\begin{pmatrix} C \\ \vdots \\ CA^{n-1} \end{pmatrix} = n$ if and only if $A_{obs}(\Sigma_0) = A$, and thus, the observability rank condition for Σ is satisfied if and only if Σ_0 is algebraically observable.

$(ii) \Leftrightarrow (iii)$ If $A_{obs}(\Sigma_0) = A$ then $Q_{obs}(\Sigma_0) = Q$. To complete the proof we prove the converse implication. If $Q_{obs}(\Sigma_0) = Q$ then $\mathbb{R}(\{C_jx, C_jA^kx \mid j = 1, \ldots, r; k \in \mathbb{N}\}) = \mathbb{R}(x_1, \ldots, x_n)$. Since all monomials of C_jx and C_jA^kx, for $j = 1, \ldots, r, k \in \mathbb{N}$, are of degree 1, it follows that by taking the quotients of the elements of $A_{obs}(\Sigma_0)$ we do not introduce any polynomial which would not be an element of $A_{obs}(\Sigma_0)$. Hence, since $\mathcal{Q}(A_{obs}(\Sigma_0)) = Q_{obs}(\Sigma_0) = Q = \mathcal{Q}(A)$, we conclude that $A_{obs}(\Sigma_0) = A$.

$(iv) \Rightarrow (i)$ Let Σ_0 be semi-algebraically observable. From [12, Proposition IV.11], $A_{obs}(\Sigma_0) = \mathbb{R}[\{C_jx, C_jAx, \ldots, C_jA^{n-1}x \mid j = 1, \ldots, r\}]$ distinguishes the points of $\mathbb{R}^n$ and it follows that

$$\text{Ker} \begin{pmatrix} C_1 \\ \vdots \\ C_rA^{n-1} \end{pmatrix} = (0). \tag{3}$$

If Ker $\begin{pmatrix} C_1 \\ \vdots \\ C_rA^{n-1} \end{pmatrix} \neq (0)$ then there was $0 \neq z \in \mathbb{R}^n$ such that

$$C_1z = \cdots = C_rz = \cdots = C_rA^{n-1}z = 0 \tag{4}$$

and consequently $\forall g \in A_{obs}(\Sigma_0) : g(z) = 0$. Note that for all $g \in A_{obs}(\Sigma_0)$ there exists $p_g \in \mathbb{R}[X_1, \ldots, X_{rn}]$ such that $g(x) = p_g(C_1x, \ldots, C_rA^{n-1}x)$ for $x \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ be arbitrary. From (4) it follows that $g(x + z) = p_g(C_1(x + z), \ldots, C_rA^{n-1}(x + z)) = p_g(C_1x, \ldots, C_rA^{n-1}x) = g(x)$ for all $g \in A_{obs}(\Sigma_0)$, which contradicts the fact that $A_{obs}(\Sigma_0)$ distinguishes the points of $\mathbb{R}^n$.

From (3), the dimension of the kernel (nullity) of the observability matrix of Σ is zero, i.e.

$$\text{null} \begin{pmatrix} C_1 \\ \vdots \\ C_rA^{n-1} \end{pmatrix} = 0.$$

By the rank-nullity theorem,

$$\text{null} \begin{pmatrix} C_1 \\ \vdots \\ C_rA^{n-1} \end{pmatrix} + \text{rank} \begin{pmatrix} C_1 \\ \vdots \\ C_rA^{n-1} \end{pmatrix} = n$$

Therefore, rank $\begin{pmatrix} C_1 \\ \vdots \\ C_rA^{n-1} \end{pmatrix} = n$ which proves the observability of Σ.

$(ii) \Rightarrow (iv)$ As Σ_0 is algebraically observable, it holds that $A_{obs}(\Sigma_0) = A$. Then $A^{Nash}_{obs}(\Sigma_0) = A^{Nash} = \mathcal{N}(\mathbb{R}^n)$ and one concludes that Σ_0 is semi-algebraically observable. ■

VI. CONCLUSION

The characterization of the dependencies between the existence of polynomial, rational and Nash realizations derived in this paper allows one to choose the easiest way to check the existence of respective realizations for a given response map. In particular, it is shown that the necessary and sufficient conditions for the existence of polynomial realizations are sufficient for the existence of rational realizations, and that the necessary and sufficient conditions for the existence of rational realizations are sufficient for the existence of Nash realizations. The example of a response map which is realizable by a rational system but which is not realizable by a polynomial system proves that realization theory of rational systems is not a trivial extension of realization theory of polynomial systems. Further, the finitness of input-space is shown to be a sufficient condition under which the existence of Nash realizations implies the existence of rational realizations. However, the question whether the existence of a Nash realization implies the existence of a rational realization is still open.

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