

# On the Rationality of the Feedback Connection

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**Abstract**—This paper presents a variety of necessary conditions and sufficient conditions under which the feedback interconnection of two rational input-output systems, that is, systems having bilinear state space realizations, produces a closed-loop system which is also rational.

## I. INTRODUCTION

Consider two bilinear state space systems with corresponding input-output maps  $F_c$  and  $F_d$ . Here  $c$  and  $d$  are their generating series over the noncommutative alphabet  $X = \{x_0, x_1, \dots, x_m\}$  using the Chen-Fliess formalism. It is well known that an input-output map has a bilinear realization if and only if its generating series is rational [5]. In the event that these two Fliess operators are interconnected

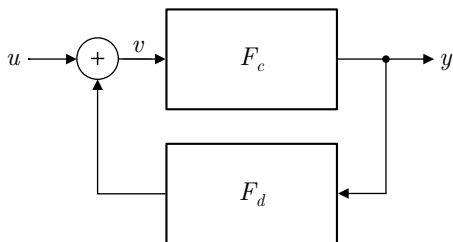


Fig. 1. Feedback connection of two Fliess operators.

to form a feedback system as shown in Figure 1, the output  $y$  must satisfy the feedback equation

$$y = F_c[u + F_d[y]]$$

for every admissible input  $u$ . It was shown in [11], [14] that there always exist a generating series  $e$  so that  $y = F_e[u]$ . In which case, the feedback equation becomes equivalent to

$$F_e[u] = F_c[u + F_{d \circ e}[u]],$$

where  $\circ$  denotes the *composition product* of two formal powers series [2], [3], [11]. The *feedback product* of  $c$  and  $d$  is thus defined as  $c@d = e$ . It was shown by counterexample in [2], [3] that in general the composition product does not preserve rationality. However, in the event that  $c$  has the additional property of being *input-limited*, a condition which is related to the nilpotency of any corresponding realization, the series  $c@d$  will be rational. Unfortunately, it was shown in [10] that this simple condition does not guarantee that  $c@d$  is rational. So the main goal of this paper is to provide a variety of necessary conditions and sufficient conditions under which the feedback product will act as a rational operation, and

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thus ensure that the closed-loop system  $F_{c@d} : u \mapsto y$  has a bilinear realization.

The paper is organized as follows. The first section briefly outlines the background for the problem. Then, in Section III, necessary conditions for the rationality of the feedback product are given and some examples are provided. Sufficient conditions are developed in the subsequent section. The main conclusions are summarized in the final section.

## II. PRELIMINARIES

A finite nonempty set of noncommuting symbols  $X = \{x_0, x_1, \dots, x_m\}$  is called an *alphabet*. Each element of  $X$  is called a *letter*, and any finite sequence of letters from  $X$ ,  $\eta = x_{i_k} \cdots x_{i_1}$ , is called a *word* over  $X$ . The *length* of  $\eta$ ,  $|\eta|$ , is the number of letters in  $\eta$ . The set of all words with length  $k$  will be denoted by  $X^k$ . The set of all words including the empty word,  $\emptyset$ , will be denoted by  $X^*$ . It forms a monoid under catenation. Any mapping  $c : X^* \rightarrow \mathbb{R}^\ell$  is called a *formal power series*. The value of  $c$  at  $\eta \in X^*$  is written as  $(c, \eta)$ . Typically,  $c$  is represented as the formal sum  $c = \sum_{\eta \in X^*} (c, \eta) \eta$ . The collection of all formal power series over  $X$  is denoted by  $\mathbb{R}^\ell \langle\langle X \rangle\rangle$ . It forms an  $\mathbb{R}$ -algebra under the Cauchy product and a commutative  $\mathbb{R}$ -algebra under the shuffle product, that is, the  $\mathbb{R}$ -bilinear mapping  $\mathbb{R}^\ell \langle\langle X \rangle\rangle \times \mathbb{R}^\ell \langle\langle X \rangle\rangle \rightarrow \mathbb{R}^\ell \langle\langle X \rangle\rangle$  uniquely specified by the shuffle product of two words  $(x_i \nu) \sqcup (x_j \xi) = x_i (\nu \sqcup (x_j \xi)) + x_j ((x_i \nu) \sqcup \xi)$  and  $\nu \sqcup \emptyset = \nu$  for all  $\nu, \xi \in X^*$ . Given  $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$ , the subset of  $X^*$  defined by  $\text{supp}(c) = \{\eta : (c, \eta) \neq 0\}$  is called the *support* of  $c$ . The subset of  $\mathbb{R}^\ell \langle\langle X \rangle\rangle$  consisting of all the series with finite support is denoted by  $\mathbb{R}^\ell \langle X \rangle$ , and its elements are called *polynomials*. A series  $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$  is called *proper* if  $\emptyset \notin \text{supp}(c)$  and *invertible* if there exists a series  $c^{-1} \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$  such that  $cc^{-1} = c^{-1}c = 1$ . In the event that  $c$  is not proper, it is always possible to write  $c = (c, \emptyset)(1 - c')$ , where  $c' \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$  is proper. It then follows that

$$c^{-1} = \frac{1}{(c, \emptyset)} (1 - c')^{-1} = \frac{1}{(c, \emptyset)} (c')^*,$$

where  $(c')^* := \sum_{i \geq 0} (c')^i$ . It can be shown that  $c$  is invertible if and only if  $c$  is not proper.

### A. Rational Series

Let  $S$  be any subalgebra of the  $\mathbb{R}$ -algebra on  $\mathbb{R}^\ell \langle\langle X \rangle\rangle$  under the Cauchy product.  $S$  is said to be *rationally closed* when every invertible  $c \in S$  has  $c^{-1} \in S$ . The *rational closure* of any set  $E \subset \mathbb{R}^\ell \langle\langle X \rangle\rangle$  is the smallest rationally closed subalgebra of  $\mathbb{R}^\ell \langle\langle X \rangle\rangle$  containing  $E$ .

**Definition 1:** [1] A series  $c \in \mathbb{R}\langle\langle X \rangle\rangle$  is **rational** if it belongs to the rational closure of  $\mathbb{R}\langle X \rangle$ .

Thus, a given rational series can be obtained from a finite set of polynomials by performing a finite number of additions, scalar products, Cauchy products and inversions (or star operations), the fundamental *rational operations*. The following definitions and theorem provide another characterization of rational series.

**Definition 2:** A **linear representation** of a series  $c \in \mathbb{R}\langle\langle X \rangle\rangle$  is any triple  $(\mu, \gamma, \lambda)$ , where  $\mu : X^* \rightarrow \mathbb{R}^{n \times n}$  is a monoid morphism,  $\gamma, \lambda^T \in \mathbb{R}^{n \times 1}$ , and  $(c, \eta) = \lambda \mu(\eta) \gamma$  for all  $\eta \in X^*$ .

**Definition 3:** A series is called **recognizable** if it has a linear representation.

**Theorem 1:** [18] A formal power series is rational if and only if it is recognizable.

The following Hankel matrix characterization of rationality was given in [4].

**Definition 4:** For any  $c \in \mathbb{R}\langle\langle X \rangle\rangle$ , the  $\mathbb{R}$ -linear mapping  $\mathcal{H}_c : \mathbb{R}\langle X \rangle \rightarrow \mathbb{R}\langle\langle X \rangle\rangle$  uniquely specified by

$$(\mathcal{H}_c(\eta), \xi) = (c, \xi\eta), \quad \forall \xi, \eta \in X^*$$

is called the **Hankel mapping** of  $c$ .

The mapping  $\mathcal{H}_c$  has a matrix representation whose  $(\xi, \eta)$  component is given by  $[\mathcal{H}_c]_{\xi, \eta} = (\mathcal{H}_c(\eta), \xi) = (c, \xi\eta)$ . Its range space,  $\mathcal{H}_c(\mathbb{R}\langle X \rangle)$ , is an  $\mathbb{R}$ -vector subspace of  $\mathbb{R}\langle\langle X \rangle\rangle$ , which is not necessarily finite dimensional. Consider the following definition and theorem.

**Definition 5:** The **Hankel rank** of  $c \in \mathbb{R}\langle\langle X \rangle\rangle$  is  $\rho_H(c) = \dim(\mathcal{H}_c(\mathbb{R}\langle X \rangle))$ .

**Theorem 2:** A series  $c \in \mathbb{R}\langle\langle X \rangle\rangle$  is rational if and only if its Hankel rank,  $\rho_H(c)$ , is finite.

The following closely related test for rationality of a series is given in the theorem below in terms of left-shift operators, that is, the family of  $\mathbb{R}$ -linear operators  $\xi^{-1} : \mathbb{R}\langle\langle X \rangle\rangle \rightarrow \mathbb{R}\langle\langle X \rangle\rangle$  uniquely specified by  $\xi^{-1}(\eta) = \eta'$  when  $\eta = \xi\eta'$  with  $\xi, \eta' \in X^*$ , and zero otherwise [1].

**Definition 6:** A subset  $V \subset \mathbb{R}\langle\langle X \rangle\rangle$  is called **stable** when  $\xi^{-1}(c) \in V$  for all  $c \in V$  and  $\xi \in X^*$ .

**Theorem 3:** A series  $c \in \mathbb{R}\langle\langle X \rangle\rangle$  is rational if and only if there exists a stable finite dimensional  $\mathbb{R}$ -vector subspace of  $\mathbb{R}\langle\langle X \rangle\rangle$  containing  $c$ .

## B. Rational Systems

For each  $c \in \mathbb{R}\langle\langle X \rangle\rangle$ , one can formally associate a causal  $m$ -input,  $\ell$ -output operator,  $F_c$ , in the following manner. Let  $\mathfrak{p} \geq 1$  and  $t_0 < t_1 < \infty$  be given. For a measurable function  $u : [t_0, t_1] \rightarrow \mathbb{R}^m$ , define  $\|u\|_{\mathfrak{p}} = \max\{\|u_i\|_{\mathfrak{p}} : 1 \leq i \leq m\}$ , where  $\|u_i\|_{\mathfrak{p}}$  is the usual  $L_{\mathfrak{p}}$ -norm for a measurable real-valued function,  $u_i$ , defined on  $[t_0, t_1]$ . Let  $L_{\mathfrak{p}}^m[t_0, t_1]$  denote the set of all measurable functions defined on  $[t_0, t_1]$  having a finite  $\|\cdot\|_{\mathfrak{p}}$  norm and  $B_{\mathfrak{p}}^m(R)[t_0, t_1] := \{u \in L_{\mathfrak{p}}^m[t_0, t_1] : \|u\|_{\mathfrak{p}} \leq R\}$ . Define recursively for each  $\eta \in X^*$  the mapping  $E_{\eta} : L_{\mathfrak{p}}^m[t_0, t_1] \rightarrow \mathcal{C}[t_0, t_1]$  by setting  $E_{\emptyset}[u] = 1$ , and

$$E_{x_i \bar{\eta}}[u](t, t_0) = \int_{t_0}^t u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0) d\tau,$$

where  $x_i \in X$ ,  $\bar{\eta} \in X^*$ , and  $u_0 = 1$ . The causal input-output operator corresponding to  $c$  is then

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta) E_{\eta}[u](t, t_0),$$

which is referred to as a *Fliess operator* [5]–[7], [11], [13]–[15], [17]. When there exist real numbers  $K, M > 0$  such that  $|(c, \eta)| \leq KM^{|\eta|} |\eta|!$  for all  $\eta \in X^*$ , where  $|z| := \max\{|z_1|, |z_2|, \dots, |z_{\ell}|\}$  for  $z \in \mathbb{R}^{\ell}$ , then  $F_c$  constitutes a well defined operator from  $B_{\mathfrak{p}}^m(R)[t_0, t_0 + T]$  into  $B_{\mathfrak{q}}^{\ell}(S)[t_0, t_0 + T]$  for sufficiently small  $R, S, T > 0$ , where the numbers  $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$  are conjugate exponents, i.e.,  $1/\mathfrak{p} + 1/\mathfrak{q} = 1$  [13]. Such a power series  $c$  is said to be *locally convergent*.

A Fliess operator  $F_c$  defined on  $B_{\mathfrak{p}}^m(R)[t_0, t_0 + T]$  is said to be *realized* by a state space realization when there exists a system of  $n$  analytic differential equations and  $\ell$  output equations

$$\dot{z} = f(z) + \sum_{i=1}^m g_i(z) u_i, \quad z(t_0) = z_0 \quad (1)$$

$$y = h(z) \quad (2)$$

such that (1) has a well defined solution  $z(t)$ ,  $t \in [t_0, t_0 + T]$  in a neighborhood  $\mathcal{W} \subseteq \mathbb{R}^n$  of  $z_0$  for any given input  $u \in B_{\mathfrak{p}}^m(R)[t_0, t_0 + T]$ , and

$$F_c[u](t) = h(z(t)), \quad t \in [t_0, t_0 + T].$$

It is well known in this situation that the generating series  $c$  is related to the realization  $(f, g, h, z_0)$  by

$$(c, \eta) = L_{g_{\eta}} h(z_0), \quad \forall \eta \in X^*,$$

where the iterated Lie derivatives are defined by

$$L_{g_{\eta}} h = L_{g_{i_1}} \cdots L_{g_{i_k}} h, \quad \eta = x_{i_k} \cdots x_{i_1} \in X^*$$

with  $L_{g_i} : h \mapsto \partial h / \partial z \cdot g_i$  and  $L_{\emptyset} h = h$  [5]–[7], [15]. The analyticity of the maps  $f, g$  and  $h$  ensure that  $c$  is locally convergent [20].  $F_c$  will be referred to as a *rational operator* whenever  $c$  is rational. It can be easily shown via Theorem 1 that every rational series satisfies a stricter *global* growth condition of the form  $|(c, \eta)| \leq KM^{|\eta|}$ ,  $\forall \eta \in X^*$ . Given any linear representation  $(\mu, \gamma, \lambda)$  of  $c$ , it follows that

$$c = \sum_{k=0}^{\infty} \sum_{i_1, \dots, i_k=0}^m (\lambda N_{i_k} \cdots N_{i_1} \gamma) x_{i_k} \cdots x_{i_1},$$

where  $N_i = \mu(x_i)$ . Thus, the corresponding rational operator is realized by the bilinear realization

$$\dot{z} = N_0 z + \sum_{i=1}^m N_i z u_i, \quad z(t_0) = \gamma \quad (3)$$

$$y = \lambda z. \quad (4)$$

The stronger growth condition on  $c$  guarantees that  $z(t)$  is well defined for all  $t \geq t_0$  and any  $u \in L_{\mathfrak{p}, \epsilon}^m(t_0) := \{u : [t_0, \infty) \rightarrow \mathbb{R}^m : u_{[t_0, t_0+T]} \in L_{\mathfrak{p}}^m[t_0, t_0+T], 0 < T < \infty\}$  [13, Corollary 4.1].

### C. System Interconnections

For any  $d \in \mathbb{R}^m \langle\langle X \rangle\rangle$ , define the family of mappings

$$D_{x_i} : \mathbb{R} \langle\langle X \rangle\rangle \rightarrow \mathbb{R} \langle\langle X \rangle\rangle : e \mapsto x_0(d_i \sqcup e),$$

where  $i = 0, 1, \dots, m$  and  $d_0 := 1$ . Assume  $D_0$  is the identity map on  $\mathbb{R} \langle\langle X \rangle\rangle$ . Such maps can be composed in an obvious way so that  $D_{x_i x_j} := D_{x_i} D_{x_j}$  provides an  $\mathbb{R}$ -algebra which is isomorphic to the usual  $\mathbb{R}$ -algebra on  $\mathbb{R} \langle\langle X \rangle\rangle$  under the catenation product. The following definition is useful for describing the cascade connection of two Fliess operators.

**Definition 7:** [2], [9], [11] The **composition product** of a word  $\eta \in X^*$  and a series  $d \in \mathbb{R}^m \langle\langle X \rangle\rangle$  is defined as

$$\underbrace{(x_{i_k} x_{i_{k-1}} \cdots x_{i_1})}_\eta \circ d = D_{x_{i_k}} D_{x_{i_{k-1}}} \cdots D_{x_{i_1}}(1) = D_\eta(1).$$

For any  $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$  define

$$c \circ d = \sum_{\eta \in X^*} (c, \eta) D_\eta(1).$$

The composition product is linear in its left argument, distributes to the left over the shuffle product and has the following key properties.

**Theorem 4:** [2], [11] For any  $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$  and  $d \in \mathbb{R}^m \langle\langle X \rangle\rangle$ , the identity  $F_c \circ F_d = F_{c \circ d}$  is satisfied. In addition, if  $c$  and  $d$  are locally convergent then  $c \circ d$  is also locally convergent.

It can be shown by a simple counterexample that global convergence, in general, is *not* preserved under composition. In addition, the mapping  $d \mapsto c \circ d$  is a contraction on  $\mathbb{R}^m \langle\langle X \rangle\rangle$  in the ultrametric sense [2], [11]. Finally, the following definition, given by Ferfera in [2], [3], describes a sufficient condition under which the composition product preserves rationality when it applies to its left argument.

**Definition 8:** A series  $c \in \mathbb{R} \langle\langle X \rangle\rangle$  is **limited relative to**  $x_i$  if there exists an integer  $\mathcal{N}_i \geq 0$  such that

$$\max_{\eta \in \text{supp}(c)} |\eta|_{x_i} = \mathcal{N}_i < \infty.$$

Here  $|\eta|_{x_i}$  denotes the number of times the letter  $x_i$  appears in  $\eta$ . If  $c$  is limited relative to  $x_i$  for every  $i = 1, \dots, m$  then  $c$  is said to be **input-limited**. In such cases, let  $\mathcal{N}_c := \max_i \mathcal{N}_i$ . A series  $c \in \mathbb{R} \langle\langle X \rangle\rangle$  is input-limited if each component series,  $c_j$ , is input-limited for  $j = 1, \dots, \ell$ . In this case,  $\mathcal{N}_c := \max_j \mathcal{N}_{c_j}$ .

Concerning the feedback product,  $c@d$ , it was shown in [11], [14], [16] to be the unique fixed point of the contractive iterated map

$$\tilde{S} : e_i \mapsto e_{i+1} = (c\tilde{\circ}d) \circ e_i,$$

where  $\tilde{\circ}$  denotes the *modified* composition product. That is, the product

$$c\tilde{\circ}d = \sum_{\eta \in X^*} (c, \eta) \tilde{D}_\eta(1),$$

where

$$\tilde{D}_{x_i} : \mathbb{R} \langle\langle X \rangle\rangle \rightarrow \mathbb{R} \langle\langle X \rangle\rangle : e \mapsto x_i e + x_0(d_i \sqcup e)$$

TABLE I

SELECTED SEQUENCES FROM THE OEIS

sequence	OEIS number	$n = 0, 1, 2, \dots$
$e_n$ (Ex. 2)	A000110	1, 1, 2, 5, 15, 52, 203, 877, 4140, ...
$e_n$ (Ex. 3)	A000111	1, 1, 2, 5, 16, 61, 272, 1385, 7936, ...
$e_n$ (Ex. 4)	A000629	1, 2, 6, 26, 150, 1082, 9366, 94586, ...

with  $d_0 := 0$ . Therefore,  $e = c@d$  satisfies the fixed point equation  $e = (c\tilde{\circ}d) \circ e$ . In the case of a unity feedback system, denoted by  $c@d$ , this equation reduces to  $e = c\tilde{\circ}e$ . The output of a self-excited feedback loop (i.e., when  $u = 0$ ) is described by the fixed point  $(c@d)_0 \in \mathbb{R}^m[[X_0]]$ ,  $X_0 := \{x_0\}$ , of

$$S : e_i \mapsto e_{i+1} = (c \circ d) \circ e_i.$$

Therefore,  $e = (c@d)_0$  is the solution of  $e = (c \circ d) \circ e$ , which becomes  $e = c \circ e$  for the unity feedback case. It was shown in [11], [12] that the feedback connection preserves local convergence in the self-excited case. It will be evident in the next section that global convergence is *not* always preserved in this situation. Similar results hold for the general case.

### III. NECESSARY CONDITIONS

In this section, two closely related necessary conditions for the rationality of the feedback product are given. While elementary in nature, they are very useful for quickly assessing whether a given feedback connection has any hope of producing a rational closed-loop system. The single-input, single-output case will be assumed for brevity. Thus, hereafter,  $X = \{x_0, x_1\}$ .

**Theorem 5:** Let  $c, d \in \mathbb{R} \langle\langle X \rangle\rangle$  be rational series. Then  $c@d$  is rational only if  $(c@d)_0$  is rational. Specifically, there must exist a linear representation  $(\lambda_0, N_0, \gamma_0)$  such that  $(c@d)_0 = \sum_{k \geq 0} \lambda_0 N_0^k \gamma_0$ , or equivalently,  $F_{c@d}[0](t) = \lambda_0 \exp(N_0 t) \gamma_0$ ,  $t \geq t_0$ .

*Proof:* Observe that the projection

$$P_0 : \mathbb{R} \langle\langle X \rangle\rangle \rightarrow \mathbb{R}[[X_0]] : c \mapsto c_0 = c \circ x_0^*$$

preserves rationality since the Hurwitz product, denoted by  $\odot$ , is a rational operation [5]. The claim then follows directly from the fact that  $(c@d)_0 = P_0(c@d)$ . ■

**Corollary 1:** Let  $c, d \in \mathbb{R} \langle\langle X \rangle\rangle$  be rational series. Then  $c@d$  is rational only if  $y = F_{c@d}[0]$  is well defined on  $[t_0, \infty)$ .

**Example 1:** Suppose  $c = 1 + x_0^* x_1$ . The self-excited unity feedback equation is

$$e = (1 + x_0^* x_1) \circ e = 1 + x_0^* x_0 e,$$

which can be solved directly to give  $e = (2x_0)^*(1 - x_0)$ . Clearly,  $e$  is rational, and, in particular,

$$F_{c@d}[0](t) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \exp \left( \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} t \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \geq 0.$$

This result is consistent with the fact that  $c$  is the generating series for a linear, time-invariant system, and unity feedback preserves these characteristics.

□

*Example 2:* Suppose  $c = 1 + x_1 x_0^*$ . The self-excited unity feedback equation in this case is

$$\begin{aligned} e &= (1 + x_1 x_0^*) \circ e \\ &= 1 + \sum_{k=0}^{\infty} x_1 x_0^k \circ e = 1 + \sum_{k=0}^{\infty} x_0 (e \sqcup x_0^k) \\ &= 1 + x_0 (e \sqcup x_0^*). \end{aligned}$$

Since  $x_0^{-1}(e) = e \sqcup x_0^*$  and  $(e, \emptyset) = 1$ , it follows that the exponential generating function for  $e$  satisfies

$$f'(x_0) = f(x_0) e^{x_0}, \quad f(0) = 1,$$

which has the solution

$$f(x_0) = e^{e^{x_0}-1}. \quad (5)$$

The coefficients of  $e$  correspond to the Bell numbers, which is sequence A000110 in the OEIS [19] (see Table I). It is known that

$$e_n := (e, x_0^n) \sim \frac{1}{r^n} \frac{e^{e^r-1}}{\sqrt{2\pi r^2 e^r}} n!,$$

where  $r$  is the positive root of  $re^r = n$ . While the solution in (5) is clearly entire, as required by Corollary 1, it does not have the exponential form required by Theorem 5. Thus,  $c @ \delta$  is not rational. □

*Example 3:* Suppose  $c = 1 + x_1 + x_1^2$ . The self-excited unity feedback equation is

$$e = (1 + x_1 + x_1^2) \circ e = 1 + x_0 e + x_0 (e \sqcup x_0 e),$$

where  $(e, \emptyset) = (e, x_0) = 1$ . Applying a left-shift to both sides of this equation gives  $x_0^{-1}(e) = e + e \sqcup x_0 e$ . Thus, the exponential generating series for  $e$  satisfies

$$f'(x_0) = f(x_0) + f(x_0) \int_0^{x_0} f(\xi) d\xi$$

with  $f(0) = f'(0) = 1$ , or equivalently,

$$f''(x_0) f(x_0) = (f'(x_0))^2 + f^2(x_0), \quad f(0) = f'(0) = 1.$$

The solution of this equation is

$$f(x_0) = \tan(x_0) + \sec(x_0).$$

In this case, the coefficients for  $e$  corresponding to the integer sequence OEIS A000111 as shown in Table I. This system clearly has a finite-escape time  $t_{esc} = \pi/2 \approx 1.5708$ . Thus, by Corollary 1,  $c @ \delta$  is not rational. Note that this example and the previous one illustrate that the input-limited property of  $c$  is not sufficient for producing a rational closed-loop system. Furthermore, these examples illustrate that global convergence in general is not preserved under feedback. □

*Example 4:* Consider the series  $c = x_0^* \sqcup x_1^* = \sum_{\eta \in X^*} \eta$ . The self-excited unity feedback equation is

$$e = c \circ e$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \sum_{\substack{r_0, r_1 \geq 0 \\ r_0 + r_1 = k}} (x_0^{r_0} \circ e) \sqcup (x_1^{r_1} \circ e) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\substack{r_0, r_1 \geq 0 \\ r_0 + r_1 = k}} \frac{k!}{r_0! r_1!} (x_0)^{\sqcup r_0} \sqcup (x_0 e)^{\sqcup r_1} \\ &= \sum_{k=0}^{\infty} \frac{(x_0 + x_0 e)^{\sqcup k}}{k!}. \end{aligned}$$

Therefore,

$$x_0^{-1}(e) = (1 + e) \sqcup e.$$

The exponential generating series for  $e$  satisfies

$$f'(x_0) = f(x_0) + f^2(x_0), \quad f(0) = 1,$$

which has the solution

$$f(x_0) = \frac{e^{x_0}}{2 - e^{x_0}}.$$

The coefficients of  $e$  correspond to OEIS sequence A000629 as shown in Table I. The corresponding finite escape time is  $t_{esc} = 1/M_e = \ln(2) \approx 0.6931$ . Thus, this closed-loop system is also not rational. But it will be shown in the next section that  $x_0^* \sqcup (-x_1)^*$  *does* produce a rational self-excited closed-loop system. This means that simple conditions involving the support of  $c$ , like the input-limited condition, will not in general be effective for determining whether rationality is preserved under feedback. □

#### IV. SUFFICIENT CONDITIONS

In this section, sufficient conditions to ensure the rationality of a closed-loop system are considered. The first test is based on the following theorems due to Fliess and Reutenauer [8].

*Theorem 6:* The output  $y$  of system (3)-(4), whose generating series  $c$  has Hankel rank  $n$ , satisfies the  $n$  order differential equation

$$p_n(u) y^{(n)} + p_{n-1}(u) y^{(n-1)} + \dots + p_0(u) y = 0, \quad (6)$$

where each  $p_i(u)$  is a polynomial in  $u, \dot{u}, \dots, u^{(n-1)}$ . Furthermore, no equation of this form with a lower order is satisfied by  $y$ .

*Theorem 7:* Let  $y = F_c[u]$  for some  $c \in \mathbb{R}\langle\langle X \rangle\rangle$  such that (6) is satisfied. Then  $c$  is rational, and hence, realizable by a system of the form (3)-(4).

The first sufficient condition is given below.

*Theorem 8:* Let  $c$  be a rational series. If  $y = F_{c @ \delta}[u]$  still satisfies an equation of the form (6) after each  $p_i(u)$  is replaced with  $p_i(u + y)$  then  $c @ \delta$  is rational.

*Proof:* Since  $c$  is rational, it follows that the open-loop system satisfies identity (6). It was shown in [14] that there always exists a series  $e \in \mathbb{R}\langle\langle X \rangle\rangle$  such that  $y = F_e[u] = F_{c @ \delta}[u]$ . Hence from Theorem 7, if the stated assumption is met, then the closed-loop system is rational. ■

It should be noted that the condition above is only sufficient because it is possible, as will be shown in an example

below, that the closed-loop system can still be rational but satisfy an equation of the form (6) with a different order than that satisfied by the open-loop system.

*Example 5:* Consider the rational series  $c = (x_0 - x_1)^* = x_0^* \sqcup (-x_1)^*$ . In this case,  $\rho_H(c) = 1$  since  $x_0^{-1}(c) = c$  and  $x_1^{-1}(c) = -c$ . Furthermore,  $y = F_c[u]$  satisfies

$$\dot{y} + (u - 1)y = 0.$$

The unity feedback system is thus described by substituting  $u + y$  for  $u$  in the equation above, namely,

$$\dot{y} + (u - 1)y + y^2 = 0.$$

The form of this equation clearly indicates that the closed-loop system does not have a rational solution with Hankel rank 2. The self-excited system has two rational solutions of lower rank:  $y(t) = 0$  and  $y(t) = 1$ .  $\square$

*Example 6:* Consider the rational series  $c = (x_1 x_0)^*(1 + x_1 - x_1^2)$ . It can be readily verified that  $\rho_H(c) = 3$  with

$$\begin{aligned} \text{Range}(\mathcal{H}_c) &= \text{span}_{\mathbb{R}}\{c, x_1^{-1}(c), x_1^{-2}(c)\} \\ &= \text{span}_{\mathbb{R}}\{c, (1 - x_1) + x_0 c, 1\}. \end{aligned}$$

Repeated differentiation of  $y = F_c[u]$  with back substitution gives the corresponding input-output equation

$$u^2 \ddot{y} - (3u\dot{u})\dot{y} + (3\dot{u}^2 - 2u^2\dot{u} - u^3 - u\ddot{u})\dot{y} + (3u^2\dot{u})y = 0,$$

which is clearly consistent with Theorem 6. The self-excited unity feedback system is thus described by

$$y^2 \ddot{y} - 4y\dot{y}\dot{y} + 3\dot{y}^3 - 2y^2\dot{y}^2 + 2y^3\dot{y} = 0. \quad (7)$$

It is clear that no solution having Hankel rank 3 exists, but unlike the previous example, it is not obvious whether there exists other finite Hankel rank solutions. A method suitable for the analysis of this example is presented next.  $\square$

Define the mappings

$$\begin{aligned} \Lambda_{x_0} &: \mathbb{R}\langle\langle X \rangle\rangle \rightarrow \mathbb{R}\langle\langle X \rangle\rangle : c \mapsto x_0^{-1}(c) + c \sqcup x_1^{-1}(c) \\ \Lambda_{x_1} &: \mathbb{R}\langle\langle X \rangle\rangle \rightarrow \mathbb{R}\langle\langle X \rangle\rangle : c \mapsto x_1^{-1}(c), \end{aligned}$$

and let  $\Lambda_{x_i x_j} = \Lambda_{x_j} \Lambda_{x_i}$  for any  $x_i, x_j \in X^*$  (note the order reversal). Take  $\Lambda_\emptyset$  as the identity map on  $\mathbb{R}\langle\langle X \rangle\rangle$ .

*Theorem 9:* Let  $c \in \mathbb{R}\langle\langle X \rangle\rangle$  be a rational series. Then the solution  $e \in \mathbb{R}\langle\langle X \rangle\rangle$  of  $e = c\tilde{e}$  is rational if and only if the  $\mathbb{R}$ -vector space

$$\tilde{V}_e = \text{span}_{\mathbb{R}}\{\Lambda_\eta(c)\tilde{e}, \eta \in X^*\}$$

is finite dimensional.

*Proof:* The result follows directly from Theorem 3 and the identities:

$$\begin{aligned} x_0^{-1}(c\tilde{e}d) &= x_0^{-1}(c)\tilde{e}d + c \sqcup [x_1^{-1}(c)\tilde{e}d] \\ x_1^{-1}(c\tilde{e}d) &= x_1^{-1}(c)\tilde{e}d. \end{aligned}$$

In particular, note that

$$x_0^{-1}(e) = x_0^{-1}(c\tilde{e}e)$$

$$\begin{aligned} &= x_0^{-1}(c)\tilde{e}e + c \sqcup [x_1^{-1}(c)\tilde{e}e] \\ &= x_0^{-1}(c)\tilde{e}e + [c\tilde{e}e] \sqcup [x_1^{-1}(c)\tilde{e}e] \\ &= \Lambda_{x_0}(c)\tilde{e}e. \end{aligned}$$

Similarly,

$$x_1^{-1}(e) = \Lambda_{x_1}(c)\tilde{e}e.$$

Proceeding inductively gives

$$\eta^{-1}(e) = \Lambda_\eta(c)\tilde{e}e,$$

and the theorem is proved.  $\blacksquare$

*Corollary 2:* Let  $c \in \mathbb{R}\langle\langle X \rangle\rangle$  be a rational series. Then the solution  $e \in \mathbb{R}[[X_0]]$  of  $e = c \circ e$  is rational if and only if the  $\mathbb{R}$ -vector space

$$V_e = \text{span}_{\mathbb{R}}\{e, \Lambda_{x_0}(c) \circ e, \Lambda_{x_0^2}(c) \circ e, \dots\}$$

is finite dimensional.

*Example 7:* Reconsider the series  $c = (x_0 - x_1)^*$  in Example 5. In this case,

$$\begin{aligned} \Lambda_{x_0}(c) \circ e &= e - e \sqcup^2 \\ \Lambda_{x_0^2}(c) \circ e &= e - 3e \sqcup^2 + 2e \sqcup^3 \\ &\vdots \end{aligned}$$

It is known that  $e = 0$  and  $e = 1$  correspond to rational solutions of the self-excited unity feedback equation. Thus,  $\dim(V_e) = 0$  and  $\dim(V_e) = 1$  for these cases, respectively.  $\square$

*Example 8:* Reconsider the series  $c = (x_1 x_0)^*(1 + x_1 - x_1^2)$  in Example 6. A direct calculation gives

$$\Lambda_{x_0^k}(c) = c \sqcup (x_1^{-1}(c)) \sqcup^k.$$

In which case,

$$\Lambda_{x_0^k}(c) \circ e = e \sqcup (x_1^{-1}(c) \circ e) \sqcup^k.$$

A key observation is that

$$\begin{aligned} x_1^{-1}(c) \circ e &= [(1 - x_1) + x_0 c] \circ e \\ &= 1 - x_0 e + x_0 e = 1. \end{aligned}$$

Thus,  $\Lambda_{x_0^k}(c) \circ e = e$  for all  $k \geq 1$  and  $\dim(V_e) = 1$ . Furthermore, observe that  $e \in \mathbb{R}[[X_0]]$  with  $(e, \emptyset) = 1$  and  $x_0^{-k}(e) = e$  for all  $k \geq 1$ . In which case,  $e = c \circ e$  has the rational solution  $e = x_0^*$ . Thus, the self-excited unity feedback response satisfies  $y^{(k)}(t) = e^t$ ,  $k \geq 0$ . It can be easily verified that this satisfies (7).  $\square$

*Example 9:* Consider the series  $c = (x_1 x_0)^*(b + ax_1 - x_1^2)$ , where  $a, b \in \mathbb{R}[X_0]$ . Note that the special case where  $a(x_0) = b(x_0) = 1$  corresponds to the  $c$  in Examples 6 and 8. It is immediate from the structure of  $c$  that

$$c = b + ax_1 - x_1^2 + (x_1 x_0)c.$$

Thus, from the identity  $e = c \circ e$  it follows that

$$e = b + ax_0 e - \underbrace{x_1^2 \circ e + ((x_1 x_0)c) \circ e}_0$$

or the self-excited unity feedback equation has the rational solution

$$e = (ax_0)^*b.$$

□

In light of the previous examples, the following theorem clearly provides a sufficient but not necessary condition for a rational unity feedback solution.

*Theorem 10:* Let  $c \in \mathbb{R}\langle\langle X \rangle\rangle$  be a rational series. Then the solution  $e \in \mathbb{R}\langle\langle X \rangle\rangle$  of  $e = c\tilde{o}e$  is rational if the  $\mathbb{R}$ -vector space

$$\tilde{V}_c = \text{span}_{\mathbb{R}}\{\Lambda_\eta(c), \eta \in X^*\}$$

has finite dimension.

*Corollary 3:* Let  $c \in \mathbb{R}\langle\langle X \rangle\rangle$  be a rational series. Then the solution  $e \in \mathbb{R}[[X_0]]$  of  $e = c\circ e$  is rational if the  $\mathbb{R}$ -vector space

$$V_c = \text{span}_{\mathbb{R}}\{c, \Lambda_{x_0}(c), \Lambda_{x_0^2}(c), \dots\}$$

has finite dimension.

*Example 10:* Let  $c = 1 + x_1$ . Observe

$$\begin{aligned} \Lambda_{x_0}(c) &= c \\ \Lambda_{x_1}(c) &= 1 \end{aligned}$$

such that  $\dim(\widehat{V}_c) = 2$ . Trivially then

$$e = (1 + x_1)\tilde{o}e = 1 + x_1 + x_0e,$$

so that  $e = x_0^*(1 + x_1)$ . □

## V. CONCLUSIONS AND FUTURE RESEARCH

In this paper, a variety of necessary conditions and sufficient conditions were given under which the feedback product will act as a rational operation, and thus, the closed-loop system will have a bilinear realization. It does not appear that any simple test for rationality involving only the support of the generating series for the subsystems exists, as in the case for cascade connections. A direction not explored in this paper was the state space approach. In some cases, where rationality is preserved, it is almost certain that the feedback is rendering some portion of the open-loop system unobservable due to internal cancelation. Whether this is always necessary for preserving rationality would be an interesting topic to pursue.

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