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On asymptotic properties of MOESP-type closed-loop subspace model identification

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***Abstract*— Recently, MOESP-type closed-loop subspace model identification (CL-MOESP) has been proposed by the authors and its effectiveness has been demonstrated via both numerical simulations and real-life systems, e.g., a cart-inverted pendulum system. However, asymptotic properties of CL-MOESP has not yet been studied. The purpose of this paper is to clarify the asymptotic properties of CL-MOESP from the viewpoint of Two-stage closed-loop identification. Moreover, it is shown that CL-MOESP minimizes a truncation error due to a finite number of sampled data.**

I. INTRODUCTION

Subspace model identification has been known as one of the most powerful identification methods. In two decades, application of subspace model identification to closed-loop systems has widely been studied. If we divide closed-loop identification into three categories, namely, direct methods, indirect methods and joint input/output methods[1], SSARX[2] and PBSID[3] can be categorized into direct methods, and Verhaegen's methods[4] and Katayama's orthogonal decomposition based method[5] can be categorized into joint input/output methods. Consistency analysis on PBSID has already been studied, and asymptotic equivalence between PBSID and SSARX has been proven[3]. It is known that, for the direct method, it is important to *a priori* know information on the model structure of the stochastic part of a system to be identified. The direct method is not so good for systems modeled with Box-Jenkins type model structure.

In the indirect method, identification of a system to be identified in a closed-loop follows identification of transfer functions of the closed-loop system, such as the sensitivity, with making use of external exciting signals. Closed-loop subspace model identification methods categorized into the indirect method are, for example, TS4SID[8], Two-stage ORT method[9] and so on.

MOESP[1]-type closed-loop subspace model identification (CL-MOESP) has been developed by the authors, and the effectiveness of CL-MOESP has been demonstrated through both numerical simulations and real-life identification experiments[10], [11], [12], [13]. CL-MOESP can be categorized into the indirect method since external excitation signals are used for identification. CL-MOESP provides a state-space model via one-shot QR factorization.

In this paper, we will study asymptotic properties and optimality of CL-MOESP from the viewpoint of the Two-stage identification approach[6].

II. PROBLEM FORMULATION

Consider a closed-loop system depicted as Fig. 1. Suppose $P(q)$ to be identified be a stable discrete-time (DT) linear time-invariant (LTI) system, where q and q^{-1} denote, respectively, forward and backward shift operators[2]. Assume that a DTLTI feedback controller $K(q)$ internally stabilizes the closed-loop system. All the discrete signals in the closed loop system are assumed to be quasi-stationary[1]. Let the external signal v_t be an unknown disturbance. Suppose at least one of known external signals r_1 and r_2 be persistently exciting. For $K(q)$, v_t, r_t^1, r_t^2, we will provide the assumption as follows:

Assumption 1: Let us assume that v_t is a zero-mean colored noise uncorrelated with the external signals r_t^1 and r_t^2 in terms of

$$\lim_{M\to\infty}\frac{1}{M}\sum_{k=1}^{M} v_{i+k}(r_{j+k}^1)^T = 0, \tag{1}$$

$$\lim_{M\to\infty}\frac{1}{M}\sum_{k=1}^{M} v_{i+k}(r_{j+k}^2)^T = 0, \tag{2}$$

where i and j are arbitrary integers. Moreover, assume that either of the following statements holds.

1) Let r_t^1 be an excitation signal with a sufficient persistence of excitation property[3], and let r_t^2 be a constant reference signal. Then, it is not necessary for the controller $K(q)$ to be known when the output signal is redefined as $y_t - r_t^2$.
2) Except for the case as stated above, $K(q)$ is assumed to be known and the signal

$$r_t := r_t^1 + K(q)r_t^2 \tag{3}$$

is assumed to have a sufficient persistence of excitation property.

Note that the block diagram in Fig. 1 can equivalently be reconfigured as that in Fig. 2. Therefore, assumption 1 can be rewritten as follows:

Assumption 2: Let us assume that v_t is a zero-mean colored noiseCwhich is uncorrelated with the external signal r_t in terms of

$$\lim_{M\to\infty}\frac{1}{M}\sum_{k=1}^{M} v_{i+k}r_{j+k}^T = 0, \tag{4}$$

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[1]MOESP = Multivariable Output-Error State-sPace

[2]For a sequence w_t, $qw_t = w_{t+1}$, and $q^{-1}w_t = w_{t-1}$.

[3]The definition of PE(Persistence of Excitation) property will be described later on.

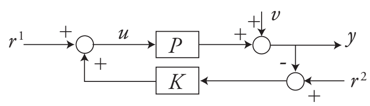

Fig. 1. A closed loop system.

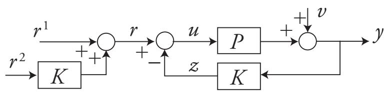

Fig. 2. A closed loop system equivalent to Fig. 1.

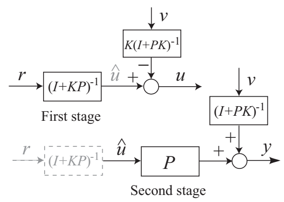

Fig. 3. Two-stage method

where i, j are arbitrary integers.

Now, the closed-loop identification problem considered in this paper is provided as follows:

Definition 1 (Closed-loop identification problem): Let M be a sufficient large integerDConsider a closed-loop system depicted as Fig. 1. Given sequences of sampled data $\{r_t\}_{t=1}^M$, $\{u_t\}_{t=1}^M$ and $\{y_t\}_{t=1}^M$, estimate the input-output relation of the system $P(q)$ to be identified.

Hereinafter, for the notational brevity, the argument of the systems (q) will be omitted if it is obvious from the context.

III. Two-stage method

A. Review of the Two-stage method

The two-stage method [6] is categorized into *indirect closed-loop identification*, since identification of the closed-loop system in the first stage is followed by identification of G in the second stage. Relations from (r,v) to (u,y) in Fig. 2 are given by

$$u_t = (I+KP)^{-1}r_t - K(I+PK)^{-1}v_t \tag{5}$$
$$=: \hat{u}_t - K(I+KP)^{-1}v_t, \tag{6}$$
$$y_t = P(I+KP)^{-1}r_t + (I+PK)^{-1}v_t$$
$$= P\hat{u}_t + (I+PK)^{-1}v_t. \tag{7}$$

Equations (5) and (7) correspond to the first stage and the second stage, respectively. Fig. 3 illustrates the procedure of the two stage method.

Now, the two stage method is summarized as follows.

Algorithm 1: Two stage method[6]

1) *First stage.* According to (5), estimate the sensitivity of the closed-loop $(I+KP)^{-1}$ from the sampled data sequences $\{r_t\}_{t=1}^M$ and $\{u_t\}_{t=1}^M$.
2) Generate a sequence of the fictitious signal defined as $\hat{u}_t := (I+KP)^{-1}r_t$ with the estimated sensitivity.
3) *Second stage.* According to (7), identify P from $\{\hat{u}_t\}_{t=1}^M$ and $\{y_t\}_{t=1}^M$.

B. State-space representation of the Two-stage method

Interpretation of the Two-stage method in the context of the state-space representation will be helpful for the analysis of the asymptotic properties of CL-MOESP later in this paper.

1) State-space realization of the closed-loop system: For the closed-loop system depicted in Fig. 2, let minimum realizations of P and K be given, respectively, by

$$P: \quad x_{t+1} = Ax_t + Bu_t, \quad y_t = Cx_t + v_t, \tag{8}$$
$$K: \quad \xi_{t+1} = E\xi_t + Fy_t, \quad z_t = G\xi_t + Hy_t, \tag{9}$$

where A is assumed to be stable, and the direct feedthrough of P is assumed to be null due to the well-definedness of the closed-loop system. Then, noting the feedback input $u_t = r_t - z_t$, the closed-loop system can be described as the following state-space representation:

$$\zeta_{t+1} = A_c\zeta_t + B_{cd}r_t + B_{cs}v_t, \tag{10}$$
$$z_t = C_z\zeta_t + Hv_t, \tag{11}$$
$$u_t = C_u\zeta_t + r_t - Hv_t, \tag{12}$$
$$y_t = C_y\zeta_t + v_t, \tag{13}$$

where $\zeta_t := \begin{bmatrix} x_t^T & \xi_t^T \end{bmatrix}^T$,

$$A_c := \begin{bmatrix} A-BHC & -BG \\ FC & E \end{bmatrix}, \quad B_{cd} := \begin{bmatrix} B \\ 0 \end{bmatrix}, \tag{14}$$
$$B_{cs} := \begin{bmatrix} -BH \\ F \end{bmatrix}, \quad C_z := \begin{bmatrix} HC & G \end{bmatrix}, \tag{15}$$
$$C_u := -C_z, \quad C_y := \begin{bmatrix} C & 0 \end{bmatrix}. \tag{16}$$

2) State-space interpretation of the first stage identification and the fictitious signal $\hat{u}$: The equation (5) implies that the first stage identification to estimate the sensitivity $(I+KP)^{-1}$ of the closed-loop can be regarded as open-loop identification from r_t to u_t contaminated with the additive colored noise $-K(I+KP)^{-1}v_t$.

Note that the purpose of the first stage identification is to generate the fictitious signal $\hat{u}_t$ using the estimated sensitivity. Namely, on decomposition of the signal u_t between the so-called deterministic part that comes from r_t and the stochastic part that comes from v_t[9], $\hat{u}_t$ is equivalent to the deterministic part. Therefore, we will derive state-space representations of the deterministic subsystem and the stochastic subsystem, respectively.

Henceforth, the deterministic parts and the stochastic parts of ζ_t, z_t, u_t and y_t in the equations (10)-(13) will be denoted by ζ_t^d, z_t^d, u_t^d, y_t^d, and ζ_t^s, z_t^s, u_t^s, y_t^s, respectively[4]. For example, $z_t = z_t^d + z_t^s$.

Lemma 1: Let us consider the state-space representation (10) and (12) of the input-output relation (5) on the first stage identification. Then, using $\zeta_t = \zeta_t^d + \zeta_t^s$ and $u_t = u_t^d + u_t^s$, the state-space representations of the deterministic part

[4]The superscripts d and s, respectively, stand for "deterministic" and "stochastic."

$(I+KP)^{-1}r_t$ and the stochastic part $-K(I+PK)^{-1}v_t$ in (5) can be given as

$$\zeta_{t+1}^d = A_c\zeta_t^d + B_{cd}r_t, \quad u_t^d = C_u\zeta_t^d + r_t, \tag{17}$$
$$\zeta_{t+1}^s = A_c\zeta_t^s + B_{cs}v_t, \quad u_t^s = C_u\zeta_t^s - Hv_t. \tag{18}$$

Moreover, the fictitious input $\hat{u}_t$ is the deterministic part of u_t, i.e., $\hat{u}_t = u_t^d$ and its state-space representation is given as (17).

Proof: From Fig. 2, we can find the sensitivity $(I+KP)^{-1}$ as the coefficient in the first term on the right hand side of (5). We know that the sensitivity is the transfer function from r_t to u_t, and its state-space representation is given as (17), which can be derived from the equations (10) and (12) with $v_t \equiv 0$.

Similarly from Fig. 2, the transfer function $K(I+PK)^{-1}$, which is the coefficient in the second term on the right hand side of (5), is the transfer function from v_t to z_t. Its state-space representation can be derived from the equations (10) and (11) with $r_t \equiv 0$, and it results in

$$\zeta_{t+1}^s = A_c\zeta_t^s + B_{cs}v_t, \quad z_t^s = C_z\zeta_t^s + Hv_t. \tag{19}$$

Note that $u_t = r_t - z_t$. If we focus on the stochastic part which comes from v_t, the equation (18) can be derived from (19) with $u_t^s = -z_t^s$.

Now, the deterministic part of z_t can be written as $z_t^d = z_t - K(I+PK)^{-1}v_t$, and its state-space representation can be derived from (10) and (11) with subtraction of the stochastic part (19), and which turns out to be

$$\zeta_{t+1}^d = A_c\zeta_t^d + B_{cd}r_t, \tag{20a}$$
$$z_t^d = C_z\zeta_t^d, \tag{20b}$$

Then, substitution of $u_t = r_t - z_t$ for (6) gives

$$\hat{u}_t = r_t - \left(z_t - K(I+PK)^{-1}v_t\right). \tag{21}$$

Therefore, from (20) and (21), the state-space representation of the system which generates the fictitious signal $\hat{u}_t$ is given as (17). ■

3) State-space interpretation of the second stage identification: According to the relation (7), the second stage identification can be regarded as an open-loop identification with $\hat{u}_t$ as the input, y_t as the output, which is contaminated with the additive colored noise $(I+PK)^{-1}v_t$.

Assumption 3: For the state-space representation (17), an initial state of ζ_t^d, denoted by ζ_0^d, is assumed to be independent of the noise v_t.

Then, we know that the fictitious input $\hat{u}_t$ is uncorrelated with the noise v_t due to assumptions 2 and 3.

Now, we would like to derive a state-space representation to describe the input/output relation from $\hat{u}_t$ to y_t^d. To begin with, we will derive the inverse representation of the input/output relation from r_t to $\hat{u}_t$.

Lemma 2: The state-space representation (17) describes the input/output relation from r_t to $\hat{u}_t$. Then, its inverse representation, which describes the relation from $\hat{u}_t$ to r_t, can be given as

$$\zeta_{t+1}^d = A_I\zeta_t^d + B_{cd}\hat{u}_t, \tag{22a}$$
$$r_t = C_z\zeta_t^d + \hat{u}_t, \tag{22b}$$

where

$$A_I := A_c + B_{cd}C_z = \begin{bmatrix} A & 0 \\ FC & E \end{bmatrix}. \tag{23}$$

Proof: From (17), we have $r_t = \hat{u}_t - C_u\zeta_t^d = C_z\zeta_t^d + \hat{u}_t$. Then, to substitute this equation for r_t in (17), we have

$$\begin{aligned}\zeta_{t+1}^d &= A_c\zeta_t^d + B_{cd}\left(C_z\zeta_t^d + \hat{u}_t\right)\\ &= (A_c + B_{cd}C_z)\,\zeta_t^d + B_{cd}\hat{u}_t = A_I\zeta_t^d + B_{cd}\hat{u}_t.\end{aligned}$$

Hence, this concludes the proof. ■

Lemma 3: A state-space representation which describes the input/output relation from $\hat{u}_t$ to y_t^d can be derived as

$$\zeta_{t+1}^d = A_I\zeta_t^d + B_{cd}\hat{u}_t, \quad y_t^d = C_y\zeta_t^d. \tag{24}$$

Moreover, if we remove the unobservable mode from (24), we have the following state-space representation:

$$x_{t+1}^d = Ax_t^d + B\hat{u}_t, \quad y_t^d = Cx_t^d. \tag{25}$$

Proof: The representation (24) can obviously be obtained from the deterministic part of (13) and (22a). For the equivalence between (24) and (25) in terms of the transfer functionCwe can easily obtain the relation $C_y(sI - A_I)^{-1}B_{cd} = C(sI-A)^{-1}B$. ■

Lemmas 1 and 3 suggest that the key point of the Two-stage identification is to remove the stochastic parts, namely, the effects of the noise v_t, from both u_t and y_t, by making the best use of uncorrelation property between r_t and v_t. For example, in the literature[8], we have proposed a closed-loop identification method developed by integrating PI-MOESP[7] into Two-stage method[6]. PI-MOESP is known as effective in identification of systems with output-error model structure contaminated with colored noise.

IV. MOESP-TYPE CLOSED-LOOP SUBSPACE MODEL IDENTIFICATION

In this section, we will briefly review the MOESP-type closed-loop subspace model identification (CL-MOESP) method[11], which has been proposed by the author.

A. Notations and assumptions

Given a sampled data sequence $\{u_i\}$, the block Hankel matrix of s block rows, $\mathscr{U}_{i,j}$, is defined as

$$\mathscr{U}_{i,j} \;:=\; \overbrace{\begin{bmatrix} u_i & u_{i+1} & \cdots & u_{i+j-1} \\ u_{i+1} & u_{i+2} & \cdots & u_{i+j} \\ \vdots & \vdots & & \vdots \\ u_{i+s-1} & u_{i+s} & \cdots & u_{i+j+s-2} \end{bmatrix}}^{j \text{ columns}}, \qquad (26)$$

where the subscripts on $\mathscr{U}$, i.e., i and j, respectively, denote the subscript of the first element of the first column and the number of columns. Given sampled data sequences $\{y_i\}$, $\{r_i\}$ and $\{v_i\}$, the block Hankel matrices, $\mathscr{Y}_{i,j}$, $\mathscr{R}_{i,j}$ and $\mathscr{V}_{i,j}$, respectively, are defined in a manner similar to (26). The superscripts, ^{d} and ^{s}, are used also for block Hankel matrices and they correspond to the deterministic part and the stochastic part, respectively. Note that k is a user-defined index and should be chosen to be greater than the orders of P and $(I+KP)^{-1}$.

Hereinafter, for M in Definition 1, let $M = N+2k-1$. For the persistence of excitation property, we will additionally adopt the following two assumptions:

Assumption 4: For a sufficiently large N, the following matrix is invertible:

$$\frac{1}{N}\begin{bmatrix} \mathscr{R}_{1,N} \\ \mathscr{R}_{k+1,N} \end{bmatrix}\begin{bmatrix} \mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T \end{bmatrix}.$$

Assumption 5: The following matrix is of full row rank:

$$\begin{bmatrix} \mathscr{U}_{1,N} \\ \mathscr{U}_{k+1,N} \end{bmatrix} \Big/ \begin{bmatrix} \mathscr{R}_{k+1,N} \\ \mathscr{R}_{1,N} \end{bmatrix}, \qquad (27)$$

where, according to the literature[14], the notation / denotes the orthogonal projection. Namely, given two matrices $W \in \mathbb{R}^{p\times q}$ and $X \in \mathbb{R}^{r\times q}$, the orthogonal projection of the row space of W on the row space of X is denoted by

$$W/X := W\Pi_X := WX^T(XX^T)^{-1}X, \qquad (28)$$

where X^T denotes the transpose of X.

B. Procedure of CL-MOESP

Algorithm 2 (CL-MOESP@[11]): Suppose that sampled data sequences $\{r_t\}_{t=1}^{N+2k-1}$, $\{u_t\}_{t=1}^{N+2k-1}$ and $\{y_t\}_{t=1}^{N+2k-1}$ be obtained from the closed-loop system depicted in Fig. 1. Then, a state-space model which represents the input/output relation of P can be obtained according to the procedure as follows:

1. Execute the QR factorization of the following matrix:

$$\begin{bmatrix} \mathscr{R}_{1,N} \\ \mathscr{R}_{k+1,N} \\ \mathscr{U}_{1,N} \\ \mathscr{U}_{k+1,N} \\ \mathscr{Y}_{k+1,N} \end{bmatrix} = \begin{bmatrix} L_{11} & & & & \\ L_{21} & L_{22} & & & \\ L_{31} & L_{32} & L_{33} & & \\ L_{41} & L_{42} & L_{43} & L_{44} & \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix}\begin{bmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \\ Q_4^T \\ Q_5^T \end{bmatrix}. \qquad (29)$$

2. Compute $\Upsilon^{\frac{1}{2}}$ as follows:

$$\Upsilon^{\frac{1}{2}} := \left(L_{51}P_1^T + L_{52}P_2^T\right)\left(P_1P_1^T + P_2P_2^T\right)^{-\frac{1}{2}}, \qquad (30)$$

where P_j $(j=1,2)$ are defined as

$$P_j := L_{3j} - \left(L_{31}L_{41}^T + L_{32}L_{42}^T\right)\left(L_{41}L_{41}^T + L_{42}L_{42}^T\right)^{-1}L_{4j}.$$

3. To estimate the extended observability matrix of P, execute SVD of $\Upsilon^{\frac{1}{2}}$ and we have

$$\Upsilon^{\frac{1}{2}} = \begin{bmatrix} U & U^\perp \end{bmatrix}\begin{bmatrix} \Sigma & \\ & \tilde{\Sigma} \end{bmatrix}\begin{bmatrix} V \\ V^\perp \end{bmatrix}^T,$$

where the diagonal matrix $\Sigma \in \mathbb{R}^{n\times n}$ has n dominant singular values as its diagonal entries, and n corresponds to the order of P.

4. An estimate of the quadruple of (A,B,C,D) of a state-space representation of P can be obtained according to the procedure similarly to the literature[15] as follows. (A,C) are estimated by $\hat{C} = U(1:l,:)$,

$$\hat{A} = U^\dagger(1:(k-1)l,:)\;U(l+1:kl,:),$$

where $X^\dagger$ denotes the Moore-Penrose generalized inverse matrix of a matrix X. To estimate (B,D), we define α and β as follows:

$$\begin{aligned}\alpha &:= \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_k \end{bmatrix} := \left(U^\perp\right)^T, \\ \beta &:= \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_k \end{bmatrix} \\ &:= \left(U^\perp\right)^T\left(L_{51}L_{41}^T + L_{52}L_{42}^T\right)\left(L_{41}L_{41}^T + L_{42}L_{42}^T\right)^{-1}.\end{aligned}$$

Then, using $U_1 := U(1:(k-1)l,:)$, an estimate of the pair (B,D) gives

$$\begin{bmatrix} \hat{D} \\ \hat{B} \end{bmatrix} = \begin{bmatrix} I_l & 0 \\ 0 & U_1 \end{bmatrix}^\dagger \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \alpha_2 & & \alpha_k & \\ \vdots & \alpha_k & & \\ \alpha_k & & & O \end{bmatrix}^\dagger \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}.$$

V. On the asymptotic properties of CL-MOESP

In subsection III-B, for the closed-loop identification considered here, we learn that it is important to extract the deterministic parts of the observed signals u_t and y_t, and to use them. In this section, we will study asymptotic properties of CL-MOESP to unveil the mechanism for extracting the deterministic parts of the signals. We will start with u_t, and then deal with y_t. Henceforth, unless stated otherwise, all in the lemmas and theorems, the assumptions and the conditions are automatically fulfilled.

A. Analysis of u_t

Lemma 4: For any i and j, the following asymptotic relation holds:

$$\lim_{N\to\infty}\frac{1}{N}\mathscr{V}_{i,N}\mathscr{R}_{j,N}^T=0. \tag{31}$$

Proof: It is obvious from the uncorrelation property between v_t and r_t in assumption 2. ■

Lemma 5: The following asymptotic relation holds:

$$\lim_{\substack{N\to\infty\\k\to\infty}}\frac{1}{N}\mathscr{U}_{i+k,N}^s\mathscr{R}_{j+k,N}^T=0 \tag{32}$$

Proof: Note the block Hankel matrix structure and use (18) repeatedly, and we have

$$\mathscr{U}_{i+k,N}^s=\mathscr{O}_c\mathscr{L}_{cs}\mathscr{V}_{i,N}+\mathscr{K}_c\mathscr{V}_{i+k,N}+\mathscr{O}_cA_c^k\mathscr{Z}_{i,N}^s, \tag{33}$$

where $\mathscr{Z}_{i,N}^s:=\begin{bmatrix}\zeta_i^s & \cdots & \zeta_{i+N-1}^s\end{bmatrix}$,

$$\mathscr{O}_c:=\begin{bmatrix}C_u^T & (C_uA_c)^T & \cdots & (C_uA_c^{k-1})^T\end{bmatrix}^T,$$

$$\mathscr{L}_{cs}:=\begin{bmatrix}A_c^{k-1}B_{cs} & \cdots & A_cB_{cs} & B_{cs}\end{bmatrix},$$

$$\mathscr{K}_c:=\begin{bmatrix}-H & & & 0\\ C_uB_{cs} & -H & & \\ \vdots & \vdots & \ddots & \\ C_uA_c^{k-2}B_{cs} & C_uA_c^{k-3}B_{cs} & \cdots & -H\end{bmatrix}.$$

Now, the both sides of (33) are multiplied from the right by $\frac{1}{N}\mathscr{R}_{j+k,N}^T$. Then, from lemma 4, we have

$$\lim_{N\to\infty}\frac{1}{N}\mathscr{V}_{i,N}\mathscr{R}_{j+k,N}^T=0,\ \lim_{N\to\infty}\frac{1}{N}\mathscr{V}_{i+k,N}\mathscr{R}_{j+k,N}^T=0.$$

Since A_c is stable due to the internal stability of the closed-loop system,

$$\lim_{k\to\infty}\frac{1}{N}\mathscr{O}_cA_c^k\mathscr{Z}_{i,N}^s\mathscr{R}_{j+k,N}^T=0$$

holds. Hence, it concludes the proof. ■

Immediately, lemma 5 yields the following theorem:

Theorem 1: For the fictitious signal $\hat{u}_t$,

$$\lim_{\substack{N\to\infty\\k\to\infty}}\frac{1}{N}\begin{bmatrix}\widehat{\mathscr{U}}_{1,N}\\ \widehat{\mathscr{U}}_{k+1,N}\end{bmatrix}\begin{bmatrix}\mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T\end{bmatrix}$$
$$=\lim_{\substack{N\to\infty\\k\to\infty}}\frac{1}{N}\begin{bmatrix}\mathscr{U}_{1,N}\\ \mathscr{U}_{k+1,N}\end{bmatrix}\begin{bmatrix}\mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T\end{bmatrix}, \tag{34}$$

where $\widehat{\mathscr{U}}_{i,j}$ is a block Hankel matrix with every entry of (26) replaced by $\hat{u}_t$ $(t=i,\cdots,i+j+k-2)$.

Proof: Note $u_t=\hat{u}_t+u_t^s$ and $\mathscr{U}_{i,N}=\widehat{\mathscr{U}}_{i,N}+\mathscr{U}_{i,N}^s$. The proof is obvious from lemma 5. ■

Note that theorem 1 corresponds to the first stage identification of the Two-stage method. Note also that this theorem guarantees that only the first and second columns L_{*1} and L_{*2} of L on the right hand side of (29) are used for the calculations of (30) and subsequent equations in the procedure of CL-MOESP.

B. Analysis of y_t

Note the block Hankel structure and use (10) and (13) repeatedly, and we have

$$\mathscr{Y}_{k+1,N}=\mathscr{O}_yA_c^k\mathscr{Z}_{1,N}+\mathscr{O}_y\mathscr{L}_{cd}\mathscr{R}_{1,N}+\mathscr{H}_y\mathscr{R}_{k+1,N}$$
$$+\mathscr{O}_y\mathscr{L}_{cs}\mathscr{V}_{1,N}+\mathscr{K}_y\mathscr{V}_{k+1,N}, \tag{35}$$

where $\mathscr{L}_{cd}:=\begin{bmatrix}A_c^{k-1}B_{cd} & \cdots & A_cB_{cd} & B_{cd}\end{bmatrix}$,

$$\mathscr{O}_y:=\begin{bmatrix}(C_y)^T & (C_yA_c)^T & \cdots & (C_yA_c^{k-1})^T\end{bmatrix}^T,$$

$$\mathscr{H}_y:=\begin{bmatrix}0 & & & 0\\ C_yB_{cd} & 0 & & \\ \vdots & \vdots & \ddots & \\ C_yA_c^{k-2}B_{cd} & C_yA_c^{k-3}B_{cd} & \cdots & 0\end{bmatrix},$$

$$\mathscr{K}_y:=\begin{bmatrix}I & & & 0\\ C_yB_{cs} & I & & \\ \vdots & \vdots & \ddots & \\ C_yA_c^{k-2}B_{cs} & C_yA_c^{k-3}B_{cs} & \cdots & I\end{bmatrix}.$$

Now, note that $y_t=y_t^d+y_t^d$, and that $\zeta_t=\zeta_t^d+\zeta_t^s$, the stochastic part of (35) can be described as follows:

$$\mathscr{Y}_{k+1,N}^s=\mathscr{O}_yA_c^k\mathscr{Z}_{1,N}^s+\mathscr{O}_y\mathscr{L}_{cs}\mathscr{V}_{1,N}+\mathscr{K}_y\mathscr{V}_{k+1,N}. \tag{36}$$

Then, for the matrix $\mathscr{Y}_{k+1,N}^s$, we have the following lemma:

Lemma 6: The following asymptotic relation holds:

$$\lim_{\substack{N\to\infty\\k\to\infty}}\frac{1}{N}\mathscr{Y}_{k+1,N}^s\begin{bmatrix}\mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T\end{bmatrix}=0. \tag{37}$$

Proof: The both sides of (36) are multiplied from the right by $\frac{1}{N}\begin{bmatrix}\mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T\end{bmatrix}$. Then, the proof can be completed by the context similar to that in lemma 5 together with lemma 4. ■

From lemma 6, multiply the both side of (35) from the right by $\frac{1}{N}[\mathscr{R}_{1,N}^T\ \ \mathscr{R}_{k+1,N}^T]$ and then take the limit, and we have

$$\begin{aligned}
&\lim_{\substack{N\to\infty\\k\to\infty}} \frac{1}{N}\mathscr{Y}_{k+1,N}\begin{bmatrix} \mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T \end{bmatrix}\\
&\quad= \lim_{\substack{N\to\infty\\k\to\infty}} \frac{1}{N}\mathscr{Y}^d_{k+1,N}\begin{bmatrix} \mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T \end{bmatrix}\\
&\quad= \lim_{\substack{N\to\infty\\k\to\infty}} \frac{1}{N}\mathscr{O}_y A_c^k \mathscr{X}^d_{1,N}\begin{bmatrix} \mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T \end{bmatrix}\\
&\qquad+ \lim_{\substack{N\to\infty\\k\to\infty}} \frac{1}{N}\left(\mathscr{O}_y\mathscr{L}_{cd}\mathscr{R}_{1,N} + \mathscr{H}_y\mathscr{R}_{k+1,N}\right)\\
&\qquad\qquad\cdot\begin{bmatrix} \mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T \end{bmatrix}.
\end{aligned} \tag{38}$$

Now, substitute the following equations, which are derived from (22),

$$\begin{aligned}
\mathscr{R}_{1,N} &= \mathscr{O}_I\mathscr{X}^d_{1,N} + \mathscr{H}_I\widehat{\mathscr{U}_{1,N}}\\
\mathscr{R}_{k+1,N} &= \mathscr{O}_I A_I^k\mathscr{X}^d_{1,N} + \mathscr{O}_I\mathscr{L}_I\widehat{\mathscr{U}_{1,N}} + \mathscr{H}_I\widehat{\mathscr{U}_{k+1,N}},
\end{aligned}$$

for $\mathscr{O}_y\mathscr{L}_{cd}\mathscr{R}_{1,N} + \mathscr{H}_y\mathscr{R}_{k+1,N}$ in (38), and we have

$$\begin{aligned}
&\mathscr{O}_y\mathscr{L}_{cd}\mathscr{R}_{1,N} + \mathscr{H}_y\mathscr{R}_{k+1,N}\\
&\quad= \left(\mathscr{O}_y\mathscr{L}_{cd}\mathscr{O}_I + \mathscr{H}_y\mathscr{O}_I A_I^k\right)\mathscr{X}^d_{1,N}\\
&\qquad+ \left(\mathscr{O}_y\mathscr{L}_{cd}\mathscr{H}_I + \mathscr{H}_y\mathscr{O}_I\mathscr{L}_I\right)\widehat{\mathscr{U}_{1,N}}\\
&\qquad+ \mathscr{H}_y\mathscr{H}_I\widehat{\mathscr{U}_{k+1,N}},
\end{aligned} \tag{39}$$

where the matrices $\mathscr{H}_I$, $\mathscr{L}_I$ and $\mathscr{O}_I$ are defined as follows:

$$\mathscr{H}_I := \begin{bmatrix} I & & & 0\\ C_zB_{cd} & I & & \\ \vdots & \vdots & \ddots & \\ C_zA_I^{k-2}B_{cd} & C_zA_I^{k-3}B_{cd} & \cdots & I \end{bmatrix}, \tag{40}$$

$$\mathscr{L}_I := \begin{bmatrix} A_I^{k-1}B_{cd} & \cdots & A_IB_{cd} & B_{cd} \end{bmatrix}, \tag{41}$$

$$\mathscr{O}_I := \begin{bmatrix} (C_z)^T & (C_zA_I)^T & \cdots & (C_zA_I^{k-1})^T \end{bmatrix}^T. \tag{42}$$

Then, with respect to the second term on the right hand side of (39) we have the following lemma:

Lemma 7: The following relation holds:

$$\mathscr{L}_{cd}\mathscr{H}_I = \mathscr{L}_I. \tag{43}$$

Moreover,

$$\mathscr{O}_y\mathscr{L}_{cd}\mathscr{H}_I + \mathscr{H}_y\mathscr{O}_I\mathscr{L}_I = \left(\mathscr{O}_y + \mathscr{H}_y\mathscr{O}_I\right)\mathscr{L}_I = \mathscr{O}\mathscr{L} \tag{44}$$

holds, where

$$\mathscr{O} := \begin{bmatrix} C\\ CA\\ \vdots\\ CA^{k-1} \end{bmatrix}, \mathscr{L} := \begin{bmatrix} A^{k-1}B & \cdots & AB & B \end{bmatrix}.$$

Proof: First, we will show the relation (43) by induction. Note that $A_I = A_c + B_{cd}C_z$. Calculate the most right column of $\mathscr{L}_{cd}\mathscr{H}_I$ on the left hand side, and obviously we have B_{cd} and it coincides with the right hand side. Let $i \geq 2$. Now, suppose that with respect to the i-th column from the right of $\mathscr{L}_{cd}\mathscr{H}_I$ on the left hand side, we can calculate

$$A_c^{i-1}B_{cd} + \cdots + B_{cd}C_zA_I^{i-2}B_{cd} = A_I^{i-1}B_{cd}.$$

Actually, it holds for the case of $i = 2$. Then, calculate the $(i+1)$-th column from the right, and we have

$$\begin{aligned}
&A_c^iB_{cd} + A_c^{i-1}B_{cd}C_zB_{cd} + \cdots + B_{cd}C_zA_I^{i-1}B_{cd}\\
&\quad= A_c(A_c^{i-1}B_{cd} + \cdots + B_{cd}C_zA_I^{i-2}B_{cd})\\
&\qquad+ B_{cd}C_zA_I^{i-1}B_{cd}\\
&\quad= A_cA_I^{i-1}B_{cd} + B_{cd}C_zA_I^{i-1}B_{cd}\\
&\quad= A_I^iB_{cd}.
\end{aligned}$$

Hence, we have shown the relation (43).

Second, with respect to (44), substitute (14)-(16) and (23) for $\mathscr{O}_y$, $\mathscr{H}_y$ and $\mathscr{O}_I$ with some calculuses, and we have

$$\mathscr{O}_y + \mathscr{H}_y\mathscr{O}_I = \begin{bmatrix} C & 0\\ CA & 0\\ \vdots & \vdots\\ CA^{k-1} & 0 \end{bmatrix}. \tag{45}$$

Then, note that A_I is a lower-triangular block matrix, and we derive

$$\begin{aligned}
&\left(\mathscr{O}_y + \mathscr{H}_y\mathscr{O}_I\right)\mathscr{L}_I\\
&= \begin{bmatrix} C & 0\\ CA & 0\\ \vdots & \vdots\\ CA^{k-1} & 0 \end{bmatrix}\begin{bmatrix} A^{k-1}B & A^{k-2}B & \cdots & B\\ * & * & \cdots & * \end{bmatrix}.
\end{aligned}$$

It concludes the proof. ■

The following lemma concerns the coefficient of the third term on the right hand side of (39).

Lemma 8: The following relation holds:

$$\mathscr{H}_y\mathscr{H}_I = \mathscr{H}, \tag{46}$$

where

$$\mathscr{H} := \begin{bmatrix} 0 & & & 0\\ CB & 0 & & \\ \vdots & \ddots & \ddots & \\ CA^{k-2}B & \cdots & CB & 0 \end{bmatrix}$$

Proof: Note that $A_I = A_c + B_{cd}C_z$. Then we have

$$\mathscr{H}_y\mathscr{H}_I = \begin{bmatrix} 0 & & & 0 \\ C_yB_{cd} & 0 & & \\ \vdots & \ddots & \ddots & \\ C_yA_I^{k-2}B_{cd} & \cdots & C_yB_{cd} & 0 \end{bmatrix} = \begin{bmatrix} 0 & & & 0 \\ CB & 0 & & \\ \vdots & \ddots & \ddots & \\ CA^{k-2}B & \cdots & CB & 0 \end{bmatrix}.$$

It concludes the proof. ■

Therefore, taking account of (39) and lemmas 7 and 8, the right hand side of (38) can be rewritten as

$$\begin{aligned}
&\lim_{\substack{N\to\infty\\k\to\infty}} \frac{1}{N}\mathscr{O}_yA_c^k\mathscr{Z}_{1,N}^d \begin{bmatrix} \mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T \end{bmatrix} \\
&\quad + \lim_{\substack{N\to\infty\\k\to\infty}} \frac{1}{N}\left(\mathscr{O}_y\mathscr{L}_{cd}\mathscr{R}_{1,N} + \mathscr{H}_y\mathscr{R}_{k+1,N}\right) \\
&\qquad \cdot \begin{bmatrix} \mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T \end{bmatrix} \\
&= \lim_{\substack{N\to\infty\\k\to\infty}} \frac{1}{N}\left(\mathscr{O}_yA_c^k + \mathscr{O}_y\mathscr{L}_{cd}\mathscr{O}_I + \mathscr{H}_y\mathscr{O}_IA_I^k\right)\mathscr{Z}_{1,N}^d \\
&\qquad \cdot \begin{bmatrix} \mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T \end{bmatrix} \\
&\quad + \lim_{\substack{N\to\infty\\k\to\infty}} \frac{1}{N}\left(\mathscr{O}\mathscr{L}\widehat{\mathscr{U}_{1,N}} + \mathscr{H}\widehat{\mathscr{U}_{k+1,N}}\right)\left[\mathscr{R}_{1,N}^T \;\; \mathscr{R}_{k+1,N}^T\right] \\
&= \lim_{\substack{N\to\infty\\k\to\infty}} \frac{1}{N}\left(\mathscr{O}_y + \mathscr{H}_y\mathscr{O}_I\right)A_I^k\mathscr{Z}_{1,N}^d \begin{bmatrix} \mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T \end{bmatrix} \\
&\quad + \lim_{\substack{N\to\infty\\k\to\infty}} \frac{1}{N}\left(\mathscr{O}\mathscr{L}\widehat{\mathscr{U}_{1,N}} + \mathscr{H}\widehat{\mathscr{U}_{k+1,N}}\right)\left[\mathscr{R}_{1,N}^T \;\; \mathscr{R}_{k+1,N}^T\right] \\
&= \lim_{\substack{N\to\infty\\k\to\infty}} \frac{1}{N}\mathscr{O}A^k\mathscr{X}_{1,N}^d \begin{bmatrix} \mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T \end{bmatrix} \\
&\quad + \lim_{\substack{N\to\infty\\k\to\infty}} \frac{1}{N}\left(\mathscr{O}\mathscr{L}\widehat{\mathscr{U}_{1,N}} + \mathscr{H}\widehat{\mathscr{U}_{k+1,N}}\right)\left[\mathscr{R}_{1,N}^T \;\; \mathscr{R}_{k+1,N}^T\right] \\
&= \lim_{\substack{N\to\infty\\k\to\infty}} \frac{1}{N}\begin{bmatrix} \mathscr{O}\mathscr{L} & \mathscr{H} \end{bmatrix}\begin{bmatrix} \mathscr{U}_{1,N} \\ \mathscr{U}_{k+1,N} \end{bmatrix}\left[\mathscr{R}_{1,N}^T \;\; \mathscr{R}_{k+1,N}^T\right].
\end{aligned}$$

Note that A is stable. The last equality is due to theorem 1. On the second equality, the lemma as follows is used for the calculus of $A_c^k + \mathscr{L}_{cd}\mathscr{O}_I$:

Lemma 9: the following relation holds:

$$A_c^k + \mathscr{L}_{cd}\mathscr{O}_I = A_I^k$$

Proof: Using $A_I = A_c + B_{cd}C_z$,

$$\begin{aligned}
A_c^k + \mathscr{L}_{cd}\mathscr{O}_I &= A_c^k + A_c^{k-1}B_{cd}C_z + A_c^{k-2}B_{cd}C_zA_I \\
&\quad + A_c^{k-3}B_{cd}C_zA_I^2 + \cdots + B_{cd}C_zA_I^{k-1} \\
&= A_c^{k-1}A_I + A_c^{k-2}B_{cd}C_zA_I \\
&\quad + A_c^{k-3}B_{cd}C_zA_I^2 + \cdots + B_{cd}C_zA_I^{k-1} \\
&\;\;\vdots \\
&= A_cA_I^{k-1} + B_{cd}C_zA_I^{k-1} = A_I^k.
\end{aligned}$$

Hence, it concludes the proof. ■

To sum up, we can derive the following theorem with respect to the second stage identification of the Two-stage method:

Theorem 2: The following relation holds:

$$\begin{aligned}
&\lim_{\substack{N\to\infty\\k\to\infty}} \frac{1}{N}\mathscr{Y}_{k+1,N} \begin{bmatrix} \mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T \end{bmatrix} \qquad (47)\\
&= \lim_{\substack{N\to\infty\\k\to\infty}} \frac{1}{N}\begin{bmatrix} \mathscr{O}\mathscr{L} & \mathscr{H} \end{bmatrix}\begin{bmatrix} \mathscr{U}_{1,N} \\ \mathscr{U}_{k+1,N} \end{bmatrix}\begin{bmatrix} \mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T \end{bmatrix}
\end{aligned}$$

Proof: See the above discussion. ■

An instinctive interpretation of theorem 2 can be given as follows. After the orthogonal projection of the closed-loop signals, u_t and y_t, onto the subspace spanned by the row vectors in $\mathscr{R}$ is used as preprocessing, we can apply an open-loop identification technique to the processed data. This does not contradict to the discussion in the literature [9].

VI. On the optimality of CL-MOESP

On real-life identification experiments, both the numbers of data N and the number of block rows in block Hankel matrices are finite numbers. Therefore, in place of (47), the following relation holds:

$$\begin{aligned}
&\frac{1}{N}\mathscr{Y}_{k+1,N} \begin{bmatrix} \mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T \end{bmatrix} \\
&= \frac{1}{N}\begin{bmatrix} \mathscr{O}\mathscr{L} & \mathscr{H} \end{bmatrix}\begin{bmatrix} \mathscr{U}_{1,N} \\ \mathscr{U}_{k+1,N} \end{bmatrix}\begin{bmatrix} \mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T \end{bmatrix} \\
&\quad + \varepsilon(N,k), \qquad (48)
\end{aligned}$$

where $\varepsilon(N,k)$ denotes a bias due to the finite number of data. In order to reduce the bias effect, CL-MOESP will provide estimates of $\mathscr{O}\mathscr{L}$ and $\mathscr{H}$ which minimize the following square-error criterion:

Theorem 3: CL-MOESP provides the optimal estimates in the sense that they minimize the following square-error criterion:

$$J(\mathscr{O}\mathscr{L},\mathscr{H}) := \left\| \mathscr{Y}_{k+1,N} - [\mathscr{O}\mathscr{L} \;\; \mathscr{H}]\begin{bmatrix} \mathscr{U}_{1,N} \\ \mathscr{U}_{k+1,N} \end{bmatrix} \right\|_{F,\Pi}^2, \qquad (49)$$

where $\|\cdot\|_{F,\cdot}$ denotes a weighted Frobenius norm defined as follows:

$$\|X\|_{F,Y}^2 := \mathrm{trace}(XYX^T)$$

The weight Π is defined as

$$\Pi := \begin{bmatrix}\mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T\end{bmatrix}\left(\begin{bmatrix}\mathscr{R}_{1,N}\\ \mathscr{R}_{k+1,N}\end{bmatrix}\begin{bmatrix}\mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T\end{bmatrix}\right)^{-1} \cdot \begin{bmatrix}\mathscr{R}_{1,N}\\ \mathscr{R}_{k+1,N}\end{bmatrix}. \quad (50)$$

Proof: $\widehat{\mathscr{OL}}$ and $\widehat{\mathscr{H}}$ denote the minimizers of $\mathscr{OL}$ and $\mathscr{H}$ which minimizes (49). When we solve the weighted least squares problem, immediately we have,

$$\begin{bmatrix}\widehat{\mathscr{OL}} & \widehat{\mathscr{H}}\end{bmatrix} = \mathscr{Y}\Pi\mathscr{U}^T\left(\mathscr{U}\Pi\mathscr{U}^T\right)^{-1}, \quad (51)$$

where, for the notational brevity, the notations $\mathscr{Y} := \mathscr{Y}_{k+1,N}$ and $\mathscr{U} := [\mathscr{U}_{1,N}^T\ \mathscr{U}_{k+1,N}^T]^T$ are adopted. Here, due to assumptions 4 and 5, the matrix $\mathscr{U}\Pi\mathscr{U}^T$ is invertible[11].

Taking lemmas 10 and 11 in Appendix into account, we will extract the counterpart of $\widehat{\mathscr{OL}}$ from the right hand side of (51). Note that $(1,1)$ and $(2,1)$ block entries of the inverse of the right-hand side of (53) in Appendix can be given by

$$\begin{aligned}(1,1) &= (P_1P_1^T + P_2P_2^T)^{-1}\\ (2,1) &= -(L_{41}L_{41}^T + L_{42}L_{42}^T)^{-1}(L_{31}L_{41}^T + L_{32}L_{42}^T)^T\\ &\quad \cdot(P_1P_1^T + P_2P_2^T)^{-1},\end{aligned}$$

respectively, where

$$\begin{aligned}P_1P_1^T + P_2P_2^T &= L_{31}L_{31}^T + L_{32}L_{32}^T\\ &\quad -(L_{31}L_{41}^T + L_{32}L_{42}^T)(L_{41}L_{41}^T + L_{42}L_{42}^T)^{-1}\\ &\quad \cdot(L_{31}L_{41}^T + L_{32}L_{42}^T)^T.\end{aligned}$$

Therefore, using (54), we have

$$\begin{aligned}\widehat{\mathscr{OL}} &= (L_{51}L_{31}^T + L_{52}L_{32}^T)(P_1P_1^T + P_2P_2^T)^{-1}\\ &\quad -(L_{51}L_{41}^T + L_{52}L_{42}^T)(L_{41}L_{41}^T + L_{42}L_{42}^T)^{-1}\\ &\quad \cdot(L_{31}L_{41}^T + L_{32}L_{42}^T)^T(P_1P_1^T + P_2P_2^T)^{-1}\\ &= (L_{51}P_1^T + L_{52}P_2^T)(P_1P_1^T + P_2P_2^T)^{-1}\\ &= \Upsilon^{\frac{1}{2}}(P_1P_1^T + P_2P_2^T)^{\frac{1}{2}}. \quad (52)\end{aligned}$$

It concludes the proof. ■

Note 1: From the relation (52), we obtain

$$\Upsilon^{\frac{1}{2}} = \widehat{\mathscr{OL}}(P_1P_1^T + P_2P_2^T)^{-\frac{1}{2}}.$$

VII. Conclusion

We have considered the asymptotic properties and the optimality of the MOESP-type closed-loop subspace model identification (CL-MOESP). The results in this paper gives theoretical guarantee of CL-MOESP, which has several successful real-life identification experiments.

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Appendix

Lemma 10: The following relation holds:

$$\begin{aligned}&\begin{bmatrix}\mathscr{U}_{1,N}\\ \mathscr{U}_{k+1,N}\end{bmatrix}\begin{bmatrix}\mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T\end{bmatrix}\\ &\cdot\left(\begin{bmatrix}\mathscr{R}_{1,N}\\ \mathscr{R}_{k+1,N}\end{bmatrix}\begin{bmatrix}\mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T\end{bmatrix}\right)^{-1}\\ &\cdot\begin{bmatrix}\mathscr{R}_{1,N}\\ \mathscr{R}_{k+1,N}\end{bmatrix}\begin{bmatrix}\mathscr{U}_{1,N}^T & \mathscr{U}_{k+1,N}^T\end{bmatrix}\\ &= \begin{bmatrix}L_{31}L_{31}^T + L_{32}L_{32}^T & L_{31}L_{41}^T + L_{32}L_{42}^T\\ L_{41}L_{31}^T + L_{42}L_{32}^T & L_{41}L_{41}^T + L_{42}L_{42}^T\end{bmatrix}\end{aligned} \quad (53)$$

Lemma 11: The following relation holds:

$$\begin{aligned}&\mathscr{Y}_{k+1,N}\begin{bmatrix}\mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T\end{bmatrix}\\ &\cdot\left(\begin{bmatrix}\mathscr{R}_{1,N}\\ \mathscr{R}_{k+1,N}\end{bmatrix}\begin{bmatrix}\mathscr{R}_{1,N}^T & \mathscr{R}_{k+1,N}^T\end{bmatrix}\right)^{-1}\\ &\cdot\begin{bmatrix}\mathscr{R}_{1,N}\\ \mathscr{R}_{k+1,N}\end{bmatrix}\begin{bmatrix}\mathscr{U}_{1,N}^T & \mathscr{U}_{k+1,N}^T\end{bmatrix}\\ &= \begin{bmatrix}L_{51}L_{31}^T + L_{52}L_{32}^T & L_{51}L_{41}^T + L_{52}L_{42}^T\end{bmatrix}\end{aligned} \quad (54)$$