Quantification of model uncertainty for a state-space system

Wafa Farah, Guillaume Mercère, Thierry Poinot and Jan-Willem van Wingerden

***Abstract*— In this communication, the uncertainty domain determination problem for multi-input multi-output systems described with a linear time-invariant state-space representation is adressed. The developed method is based on a two-step approach. The first step consists in estimating the nominal model using a particular least-squares subspace algorithm. Then, the uncertainty domains are described by using a bounded error approach. Simulations are used to highlight the performance of the method.**

I. INTRODUCTION

One of the main objectives of control-oriented identification is to estimate models that are suitable for robust control design techniques [1]. For this purpose, system identification must give not only a nominal model, but also a reliable estimate of the uncertainty associated with the model. To reach this goal, two steps are generally necessary. The first one deals with the model parameters estimation. This model is required to understand, to control or to improve the system functioning. The second step consists in designing the control law from the model parameters. However, because the model is only a system approximation, it is paramount to fix some constraints so that the controller designed from the identified model achieves good performance when it is applied to the real system. In other words, the controller must be robust with respect to the estimated model parameters uncertainties. Thus, the estimated model uncertainties must be well-described.

In system identification theory, the uncertainty domain description is mainly based on prior assumptions about noise and unmodeled dynamics [2], [1]. The first works assume that the disturbances acting on the system are random variables realizations [3], [4]. Because the information on measurement noises are not often available and difficult to verify [5], a different characterization way can be considered [6]. This approach is mainly based on deterministic hypotheses, *i.e.* on the assumption that the residuals are unknown-but-bounded [7], [8], [9]. This basic idea has given rise to a number of techniques usually addressed as bounded error or set membership identification [10], [11], [12], [13]. The main drawback of this approach is its dependence on the way the bound is determined. Notice indeed that the error comes from two different sources (the unmodeled dynamics and the noise affecting the data) which makes the bound determination quite difficult. To tackle this problem, some works [4], [14], [15] propose to estimate the error modeling. In [16], the bounded error is determined through the analysis of particular iso-level curves which leads to less conservative bounds. In other words, this bounding approach uses only an assumption on the residual bound and not a stochastic noise assumption. The identification and uncertainty domain description methods developed in [16] are restricted to single-input single-output (SISO) system represented by transfer function. So, it will be interesting to extend this method to multi-input multi-output (MIMO) system.

W. Farah, G. Mercère and T. Poinot are with the University of Poitiers, Laboratoire d'Automatique et Informatique Industrielle, 40, avenue du recteur Pineau, 86022 Poitiers, France wafa.farah@etu.univ-poitiers.fr, guillaume.mercere@univ-poitiers.fr, thierry.poinot@univ-poitiers.fr

Jan-Willem van Wingerden is with the University of Delft, Delft Center of System and Control, Delft, 2628 CD The Netherlands. J.W.vanWingerden@tudelft.nl

Because it is more convenient to use the state-space representation in the MIMO case, a subspace method will be used to estimate directly a state-space realization of the system from the measured input-output (I/O). The originality of the approach proposed hereafter, is to determine the state-space parameters uncertainty using bounded error approach. To reach this goal, a particular subspace-based method, named the propagator method [17], [18], will be used to estimate directly a state-space realization of the system from the measured I/O data. Contrary to the classic subspace algorithms [19], [20], [21], this technique does not give access to a fully-parametrized form but leads to a state-space representation with a minimal number of parameters, even for MIMO systems. This method gives also a state-space model estimate in a user-defined state-space coordinates basis. In addition, thanks to the propagator method, the state vector can be estimated [22]. So, a model linear with respect to the parameters (LP) is obtained which makes the description of uncertainty areas easier. These domains are derived from the analysis of particular quadratic criteria in the optimum vicinity. The final objective is to get realistic uncertainty domains that contain all kinds of stochastic disturbances. Thus, a hard unknown-but-bounded approach is considered to reach this goal. More particularly, an easy-tuning method is proposed to fix the value of the required bound and no assumption is made on the noise.

The outline of this paper is as follows. In Section II the main problem is stated. Section III is dedicated to the system parameters estimation using a particular least-squares subspace algorithm. The uncertainty domain determination problem is studied in Section IV. In Section V, the global technique performance is emphasized thanks to numerical simulations. Section VI concludes the paper.

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II. PROBLEM FORMULATION

Consider the following linear time-invariant state-space system

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1a}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{v}(t). \tag{1b}$$

where $\mathbf{u}(t) \in \mathbb{R}^{n_u}$, $\mathbf{y}(t) \in \mathbb{R}^{n_y}$, $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ and $\mathbf{v}(t) \in \mathbb{R}^{n_y}$ are respectively the input, the output, the state and the output-noise vectors. $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ are the system matrices relatively to a certain state-space coordinate basis. The model (1) is the so-called output-error model [23]. The system order is assumed to be known *a priori*[1].

Notice that no particular assumption is made concerning the properties of $\mathbf{v}$. However, it is assumed hereafter that the model residuals, *i.e.*, the difference between the simulated and measured outputs, are bounded. This particular framework can be justified by noticing that, in many practical cases, residuals are related to complex dynamics that cannot be captured by the plant model. When the residuals contain some deterministic parts, it turns out that the classic stationary stochastic process assumption is no more suitable. On the contrary, deterministic constraint such as "bounded in magnitude residuals" may be more efficient [10].

The considered identification problem can be stated as follows: given realizations $\{\mathbf{u}(t)\}_{t=1}^{N}$ and $\{\mathbf{y}(t)\}_{t=1}^{N}$ of the input and output processes generated by a system of the form (1) on a finite but sufficiently wide time horizon N, the goal of the developed method is to estimate the matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ represented in a particular state-space form and characterize the state-space parameters precision. To reach this goal, two linked problems are solved. Firstly, the system matrices (1) are identified using a particular least-squares subspace algorithm (see Section III). Then, the estimated parameters uncertainty domains are characterized through the analysis of particular iso-level curves and an unknown-but-bounded approach (see Section IV).

III. SUBSPACE-BASED LEAST-SQUARES ALGORITHM

The proposed uncertainty domain method assumes that the model is LP. For this reason, a particular subspace-based least-squares algorithm is adopted to identify the model parameters. This algorithm has the particularity to estimate the state vector which allows to get an LP model. To reach this goal, some definitions are introduced in §III-A. Then, the subspace-based least-squares algorithm is introduced in §III-B.

[1]This assumption is not really strong because many algorithms, mainly based on SVD and information criteria [20], are now available to get a reliable estimate of this parameter. Thus, for LTI systems, the system order can be estimated beforehand.

A. Data equation

Let us define

$$\begin{aligned}
\mathbf{u}_p(t) &= \begin{bmatrix}\mathbf{u}^\top(t-p) & \cdots & \mathbf{u}^\top(t-1)\end{bmatrix}^\top \\
\mathbf{K} &= \begin{bmatrix}\mathbf{A}^{p-1}\mathbf{B} & \cdots & \mathbf{A}\mathbf{B} & \mathbf{B}\end{bmatrix} \\
\mathbf{\Gamma}_p &= \begin{bmatrix}(\mathbf{C})^\top & (\mathbf{C}\mathbf{A})^\top & \cdots & (\mathbf{C}\mathbf{A}^{p-1})^\top\end{bmatrix}^\top \\
\mathbf{H}_p &= \begin{bmatrix}\mathbf{D} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{C}\mathbf{B} & \mathbf{D} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}\mathbf{A}^{p-2}\mathbf{B} & \mathbf{C}\mathbf{A}^{p-3}\mathbf{B} & \cdots & \mathbf{D}\end{bmatrix}
\end{aligned}$$

where p is a user-defined integer such that $p \geq n_x$. With these definitions, the I/O behavior of the system (1) is now given by [21]

$$\mathbf{y}_p(t) = \mathbf{\Gamma}_p\mathbf{x}(t-p) + \mathbf{H}_p\mathbf{u}_p(t) + \mathbf{v}_p(t). \tag{2}$$

B. Identification algorithm

Because the uncertainty domain determination problem is easier when the model is LP, the state vector is firstly estimated. This vector is identified thanks to the particular property of the propagator method. Then the $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ matrices are estimated using a least-squares algorithm.

The starting point of this method is the following well-known relation [24], [25], [26]

$$\mathbf{x}(t) = \mathbf{A}^p\mathbf{x}(t-p) + \mathbf{K}\mathbf{u}_p(t).$$

The key assumption in this method is that we assume that $\mathbf{A}^j \approx 0$ for all $j \geq p$. It can be shown that if the system (1) is uniformly exponential stable, the approximation error can be made arbitrarily small by making p large [24][27][25].

Under this assumption, the state $\mathbf{x}(t)$ is approximately given by [22]

$$\mathbf{x}(t) \approx \mathbf{K}\mathbf{u}_p(t) \tag{3}$$

and the output behavior can be written as

$$\mathbf{y}(t) \approx \mathbf{C}\mathbf{K}\mathbf{u}_p(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{v}(t).$$

Hence, the matrices $\mathbf{CK}$ can be estimated by solving the following linear problem

$$\min_{\mathbf{CK}} \left\| \mathbf{y}(t) - \begin{bmatrix}\mathbf{C}\mathbf{K} & \mathbf{D}\end{bmatrix} \begin{bmatrix}\mathbf{u}_p(t) \\ \mathbf{u}(t)\end{bmatrix} \right\|_F^2.$$

For finite p, the solution of this linear problem will be biased due to the approximation made in (3). In the literature, several papers have studied the effect of the window size and have proved the asymptotic properties of the algorithms (if $p \to \infty$ the bias disappears) [24][27][25].

With the approximation given in (3), we can rewrite (2) as

$$\mathbf{y}_p(t+p) \approx \mathbf{\Gamma}_p\mathbf{K}\mathbf{u}_p(t) + \mathbf{H}_p\mathbf{u}_p(t+p) + \mathbf{v}_p(t+p).$$

The product $\mathbf{\Gamma}_p\mathbf{K}$ is given by [28]

$$\mathbf{\Gamma}_p\mathbf{K} = \begin{bmatrix}\mathbf{C}\mathbf{A}^{p-1}\mathbf{B} & \mathbf{C}\mathbf{A}^{p-2}\mathbf{B} & \cdots & \mathbf{C}\mathbf{B} \\ \mathbf{C}\mathbf{A}^{p}\mathbf{B} & \mathbf{C}\mathbf{A}^{p-1}\mathbf{B} & \ddots & \mathbf{C}\mathbf{A}\mathbf{B} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{C}\mathbf{A}^{2p-2}\mathbf{B} & \cdots & \cdots & \mathbf{C}\mathbf{A}^{p-1}\mathbf{B}\end{bmatrix}.$$

Using the assumption $\mathbf{A}^j \approx 0$ for all $j \geq p$, this expression can be approximated by the following upper block diagonal matrix

$$\mathbf{\Gamma}_p\mathbf{K} \approx \begin{bmatrix} \mathbf{CA}^{p-1}\mathbf{B} & \mathbf{CA}^{p-2}\mathbf{B} & \cdots & \mathbf{CB} \\ \mathbf{0} & \mathbf{CA}^{p-1}\mathbf{B} & \ddots & \mathbf{CAB} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{CA}^{p-1}\mathbf{B} \end{bmatrix}. \quad (4)$$

By introducing zeros in this matrix, the first block row in (4) can be used to construct the other block rows. So, knowing $\mathbf{CK}$, the matrix $\mathbf{\Gamma}_p\mathbf{K}$ can be constructed.

Because our objective is to get an estimate of $\mathbf{x}(t)$, the propagator method will be used [18][17]. Firstly, thanks to the estimate of $\mathbf{\Gamma}_p\mathbf{K}$, the signal

$$\mathbf{q} = \mathbf{\Gamma}_p\mathbf{K}\mathbf{u}_p(t) \approx \mathbf{K}\mathbf{x}(t)$$

can be simulated. Then, assuming that the system (1) is observable, $\mathbf{\Gamma}_p$ has got, at least, n_x linearly independent rows. Introducing a permutation matrix $\mathbf{S} \in \mathbb{R}^{n_y p \times n_y p}$ such that the extended observability matrix can be reordered as follows

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} = \mathbf{S}\mathbf{\Gamma}_p\mathbf{K}\mathbf{u}_p = \begin{bmatrix} \mathbf{\Gamma}_1 \\ \mathbf{\Gamma}_2 \end{bmatrix} \mathbf{K}\mathbf{u}_p = \begin{bmatrix} \mathbf{I}_{n_x} \\ \mathbf{P} \end{bmatrix} \mathbf{\Gamma}_1\mathbf{K}\mathbf{u}_p$$

where $\mathbf{\Gamma}_1$ is a matrix block containing a set of n_x independent rows, $\mathbf{\Gamma}_2$ is a matrix block containing the other rows and $\mathbf{P}$ is a unique operator named the propagator [17].
It holds that

$$\hat{\mathbf{x}}(t) \approx \mathbf{\Gamma}_1\mathbf{K}\mathbf{u}_p(t). \quad (5)$$

So, The state vector can be estimated in a user-defined coordinate basis.

Knowing $\mathbf{y}$, $\mathbf{u}$ and $\hat{\mathbf{x}}$, the system matrices ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$) can be estimated by solving the following linear problem

$$\min_{\mathbf{ABCD}} \left\| \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} - \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right\|_F^2.$$

Using $\mathbf{\Gamma}_1$ as a similarity transformation, the state-space matrices satisfy

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} \mathbf{I}_{n_x} \\ \mathbf{P} \end{bmatrix} (2 : n_x + 1, :) \quad (6a)$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n_x-1} \end{bmatrix} \quad (6b)$$

$$\boldsymbol{c}_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$\boldsymbol{c}_j = \begin{bmatrix} \mathbf{I}_{n_x} \\ \mathbf{P} \end{bmatrix} ((j-1)i + 1, :) \text{ for } j \in [2, n_y] \quad (6c)$$

with a_j, $j \in [0, n_x - 1]$, the coefficients of the characteristic polynomial of $\mathbf{A}$ [29]. Knowing $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{c}$, the matrices $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{D}}$ can be estimated by using a least squares regression [21]

$$\widehat{\boldsymbol{\mathcal{B}}}, \widehat{\boldsymbol{\mathcal{D}}} = \arg\min_{\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{D}}} \sum_t [\mathbf{y}(t) - \begin{bmatrix} \mathbf{u}^\top(t) \otimes \mathbf{I}_{n_y} & \sum_{k=0}^{t-1} \mathbf{u}^\top(k) \otimes \widehat{\boldsymbol{c}}\widehat{\boldsymbol{\mathcal{A}}}^{t-k-1} \end{bmatrix} \begin{bmatrix} \text{vec}(\boldsymbol{\mathcal{D}}) \\ \text{vec}(\boldsymbol{\mathcal{B}}) \end{bmatrix}]$$

where vec(.) stands for the vectorization operator and $\otimes$ is the Kronecker product [30].

Thanks to the propagator method, $(\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{D}})$ are expressed in LP form. So, the proposed method to describe the uncertainty domain can be used to determin the $(\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{D}})$ parameters uncertainty.

IV. UNCERTAINTY DOMAINS DESCRIPTION

As soon as the user has access to the estimated state-space matrices, the uncertainty domains determination problem can be considered. More precisely, the uncertainty domain description of the coefficients a_i, b_i^j, c_i^k and d_j^l, $i \in [0, n_x - 1]$, $j \in [1, n_u]$, $k \in [2, n_y]$ and $l \in [1, n_y]$ (the state-space matrices $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{D}}$ parameters) is performed. The developed approach basic idea is to study particular ellipsoidal iso-level curves by adapting a bounded error method in which only bounded residual hypothesis is required [16]. This original method has been developed assuming that the system model is described via a linear regression.

For this reason, knowing $\mathbf{y}$, $\mathbf{u}$ and $\hat{\mathbf{x}}$, the model (1) can be transformed as follow

$$\underbrace{\text{vec}\begin{bmatrix} \mathbf{x}(t+1)^\top & \mathbf{y}(t)^\top \end{bmatrix}}_{\mathbf{y}_M} = \underbrace{\begin{bmatrix} \mathbf{I}_{n_x+n_y} \otimes \begin{bmatrix} \mathbf{x}(t)^\top & \mathbf{u}(t)^\top \end{bmatrix}^\top \end{bmatrix}}_{\mathbf{\Phi}^\top(t)} \text{vec} \underbrace{\begin{bmatrix} \boldsymbol{\mathcal{A}}^\top & \boldsymbol{\mathcal{C}}^\top \\ \boldsymbol{\mathcal{B}}^\top & \boldsymbol{\mathcal{D}}^\top \end{bmatrix}}_{\boldsymbol{\theta}} + \text{vec}(\mathbf{v}(t)).$$

Remark 1 *The first $n_x - 1$ rows of $\boldsymbol{\mathcal{A}}$ and the first row of $\boldsymbol{\mathcal{C}}$ are made up of 0 and 1. For this reason, $\boldsymbol{\theta}$ can be restricted as*

$$\tilde{\boldsymbol{\theta}} = vec \begin{bmatrix} \boldsymbol{\mathcal{A}}(n_x, :)^\top & \boldsymbol{\mathcal{C}}(j, :)^\top \\ \boldsymbol{\mathcal{B}}^\top & \boldsymbol{\mathcal{D}}^\top \end{bmatrix} \quad for \quad j \in [2, n_y].$$

$\mathbf{y}_M$ and $\mathbf{\Phi}$ can be deduced in a compatible way with $\tilde{\boldsymbol{\theta}}$.

It is obvious that the least-squares estimates of these matrices can be obtained by minimizing the cost function [1, Appendix II][2]

$$J(\widehat{\boldsymbol{\theta}}) = \frac{1}{2}\left(\mathbf{y}_M - \mathbf{\Psi}_M\widehat{\boldsymbol{\theta}}\right)^\top \left(\mathbf{y}_M - \mathbf{\Psi}_M\widehat{\boldsymbol{\theta}}\right) = \frac{1}{2}\boldsymbol{\varepsilon}^\top\boldsymbol{\varepsilon}$$

[2] In this method, We don't take into account the uncertainty of the state sequence estimate.

where $\widehat{\boldsymbol{\theta}}$ is an arbitrary estimate of $\boldsymbol{\theta}$,

$$\mathbf{y}_M = \begin{bmatrix} \text{vec}[\mathbf{x}(2)^\top & \mathbf{y}(1)^\top] \\ \vdots & \\ \text{vec}[\mathbf{x}(M+1)^\top & \mathbf{y}(M)] \end{bmatrix}$$

$$\text{and} \quad \boldsymbol{\Psi}_M = \begin{bmatrix} \boldsymbol{\Phi}^\top(1) \\ \vdots \\ \boldsymbol{\Phi}^\top(M) \end{bmatrix}.$$

Then, assuming that $\mathbf{R}_{\boldsymbol{\Psi}} = \boldsymbol{\Psi}_M^\top \boldsymbol{\Psi}_M$ is invertible, it is easy to see that

$$\begin{aligned} J(\widehat{\boldsymbol{\theta}}) &= \frac{1}{2}\mathbf{y}_M^\top \left(\mathbf{I} - \boldsymbol{\Psi}_M \left(\boldsymbol{\Psi}_M^\top \boldsymbol{\Psi}_M\right)^{-1} \boldsymbol{\Psi}_M^\top\right) \mathbf{y}_M \quad (7) \\ &+ \frac{1}{2}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_{ls}\right)^\top \boldsymbol{\Psi}_M^\top \boldsymbol{\Psi}_M \left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_{ls}\right) \\ &= J_{min} + \frac{1}{2} d\widehat{\boldsymbol{\theta}}^\top \mathbf{R}_{\boldsymbol{\Psi}} d\widehat{\boldsymbol{\theta}} \end{aligned}$$

with $\boldsymbol{\theta}_{ls} = \left(\boldsymbol{\Psi}_M^\top \boldsymbol{\Psi}_M\right)^{-1} \boldsymbol{\Psi}_M^\top \mathbf{y}_M$ and $d\widehat{\boldsymbol{\theta}}$ the variation of the estimate $\widehat{\boldsymbol{\theta}}$ around $\boldsymbol{\theta}_{ls}$. This relation shows that $J(\widehat{\boldsymbol{\theta}})$ has a unique minimum at the least-squares solution $\boldsymbol{\theta}_{ls}$ and the first term of the lhs of Eq. (7) is the minimum value. Furthermore, if we introduce $V(\widehat{\boldsymbol{\theta}}) = J(\widehat{\boldsymbol{\theta}}) - J_{min}$, it is obvious that

$$V(\widehat{\boldsymbol{\theta}}) = \frac{1}{2} d\widehat{\boldsymbol{\theta}}^\top \mathbf{R}_{\boldsymbol{\Psi}} d\widehat{\boldsymbol{\theta}} \quad (8)$$

is an ellipsoid centered in $\boldsymbol{\theta}_{ls}$ whose main directions are given by $\mathbf{R}_{\boldsymbol{\Psi}}$.

The goal of the developed method is to determine an uncertainty domain $\mathcal{D}$ such that $\boldsymbol{\theta} \in \mathcal{D}$. Here, $\mathcal{D}$ will be an ellipsoidal surface depending on a user-defined criterion level $J_{\mathcal{D}}$. This threshold must be chosen such that the system parameters $\boldsymbol{\theta}$ is surely included in $\mathcal{D}$ without leading to a too conservative solution. This level is obtained hereafter by using an analogy with the stochastic framework. To reach this goal, this link will be presented firstly in gaussian case and then in general case.

Under the Gaussian assumptions (no modeling error and zero-mean white Gaussian output noise), it is well-known that [1]

$$\frac{1}{2}\left(\boldsymbol{\theta}_{ls} - \boldsymbol{\theta}\right)^\top \boldsymbol{\Psi}_M^\top \boldsymbol{\Psi}_M \left(\boldsymbol{\theta}_{ls} - \boldsymbol{\theta}\right) = n^2\sigma^2$$

where σ^2 is the noise variance and $n \in \mathbb{R}^+$. This is again an ellipsoid with the same shape as the one given by (8). This analogy leads to choose the level $J_{\mathcal{D}}$ as follows

$$J_{\mathcal{D}} - J_{min} = n^2\sigma^2 = \ell^2.$$

In the general case, when the disturbance $\mathbf{v}$ is not Gaussian but only zero-mean and modeling error is considered, we can say that

$$-n\sigma < \mathbf{v} < n\sigma \text{ with } n \geq 3 \text{ and } \ell^2 = n^2\sigma^2. \quad (9)$$

So, an ellipsoid domain $\mathcal{D}$ including all the possible values of $\boldsymbol{\theta}$ is defined by

$$J_{\mathcal{D}} = J_{min} + \ell^2 \quad \text{with} \quad \ell = n\sigma. \quad (10)$$

Unfortunately, this relation depends on the prior knowledge of σ^2. In order to circumvent this difficulty, a bounded error approach is used[3]. To estimate this bound, symbolized hereafter by ℓ, the first idea could be to measure the upper bound of the noise acting on the system when the system is not excited. Practically, this methodology cannot involve the modeling error brought out when the system is excited. Then, it is proposed to use the information contained in the residuals. Indeed, the following result can be proved

$$J_{min} = \frac{1}{2}\mathbf{v}^\top \left(\mathbf{I} - \boldsymbol{\Psi}_M \left(\boldsymbol{\Psi}_M^\top \boldsymbol{\Psi}_M\right)^{-1} \boldsymbol{\Psi}_M^\top\right) \mathbf{v} = \frac{1}{2}\boldsymbol{\varepsilon}^\top \boldsymbol{\varepsilon}$$

where $\mathbf{v}$ is the system disturbance and $\boldsymbol{\varepsilon} = \mathbf{y}_M - \hat{\mathbf{y}}_M(\boldsymbol{\theta})$ are the residuals. Thus, a good noise effect approximation can be deduced from $\boldsymbol{\varepsilon}$ and we fix ℓ as follows

$$\ell = \max\left\{|\boldsymbol{\varepsilon}(t)|\right\}.$$

ℓ depend only on the residual and not on the noise acting on the output.

V. Simulation example

In order to show the performance of the method described beforehand, the following state-space matrices are used

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0 & -1 \\ 1 & 0.3 & 5 \\ -2 & -0.4 & -0.6 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 5 & 0 & 1 \\ -3 & 1 & 1 \end{bmatrix}.$$

These matrices can be rewritten as

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.1 & 0.2 & -0.1 \end{bmatrix} \quad \boldsymbol{\mathcal{B}} = \begin{bmatrix} 7.3 & 0.2 \\ -0.3 & -6.9 \\ -14.6 & -2.0 \end{bmatrix}$$

$$\boldsymbol{\mathcal{C}} = \begin{bmatrix} 1.0 & 0 & 0 \\ 1.0 & -0.3 & 0.6 \end{bmatrix}$$

by using

$$\boldsymbol{\Gamma}_1 = \begin{bmatrix} 5 & 0 & 1 \\ -1 & -0.4 & -5.6 \\ 10.6 & 2.12 & 2.36 \end{bmatrix}$$

as similarity transformation.

The input signal is a pseudo random binary sequences (PRBS) of length 1000. A Monte Carlo simulation of size 1000 is carried out. The output noise signal $\mathbf{v}$ is a zero-mean white Gaussian noise such that the signal to noise ratio (SNR) equals $20dB$. To satisfy the assumption $\mathbf{A}^j \approx 0$ for all $j \geq p$, the past index is choosen as $p = 20$. The least-squares method is applied to estimate the system matrices. The results of this identification method are presented in Table I. This table shows that the subspace-based least-squares algorithm gives a good $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ matrices estimation.

[3]In this method, the only assumption is to have bounded residuals.

	True parameters	Estimated parameters
$-a_0$	0.1640	$0.1640 \pm 2.60\ 10^{-7}$
$-a_1$	0.2400	$0.2400 \pm 3\ 10^{-7}$
$-a_2$	-0.1000	$-0.1000 \pm 3.30\ 10^{-7}$
c_0^2	1.0406	$1.0406 \pm 3.2\ 10^{-7}$
c_1^2	-0.3068	$-0.3068 \pm 4.3\ 10^{-7}$
c_2^2	0.6707	$0.6706 \pm 6.4\ 10^{-7}$
b_0^1	7.3777	$7.3775 \pm 8.57\ 10^{-5}$
b_0^2	0.2006	$0.2007 \pm 8.93\ 10^{-5}$
b_1^1	-0.3933	$-0.3936 \pm 9.77\ 10^{-5}$
b_1^2	-6.9678	$-6.9682 \pm 8.53\ 10^{-5}$
b_2^1	-14.6091	$-14.6093 \pm 9.02\ 10^{-5}$
b_2^2	-2.0070	$-2.0070 \pm 8.09\ 10^{-5}$

TABLE I

ESTIMATED PARAMETERS USING THE LEAST-SQUARES METHOD.

Identification results are plotted also in Figure 1. Firstly, these three graphs show that the estimates obtained using the least-squares method are accurate. Indeed, in each plot, the black cross (+), symbolizing the mean value of the estimated parameters, almost matches the red one (×) which corresponds to the real parameters.

Figure 1 shows the system parameters (×), the mean value of the estimated parameters (+) and the estimated parameters (∗) calculated during the 1000 realizations of the Monte Carlo simulation for $(-a_0, -a_1)$, $(-a_2, b_0^1)$ and $(-a_2, b_0^1)$.

To assess the quality of these uncertainty domains and to get rid of the drawing of all the uncertainty domains, a failure rate measure is used. This failure rate is defined as the percentage of realizations for which the system parameters are outside of the ellipsoid $\mathcal{D}$ centered in the estimated parameters vector, (see black disc (•) in Fig. 1). The failure ratio equals 0.8 % for the couple $(-a_0, -a_1)$, 2 % for $(-a_2, b_0^1)$ and 6.7 % for $(-a_2, b_0^1)$. These values show that the way the threshold is fixed leads to reliable uncertainty domains.

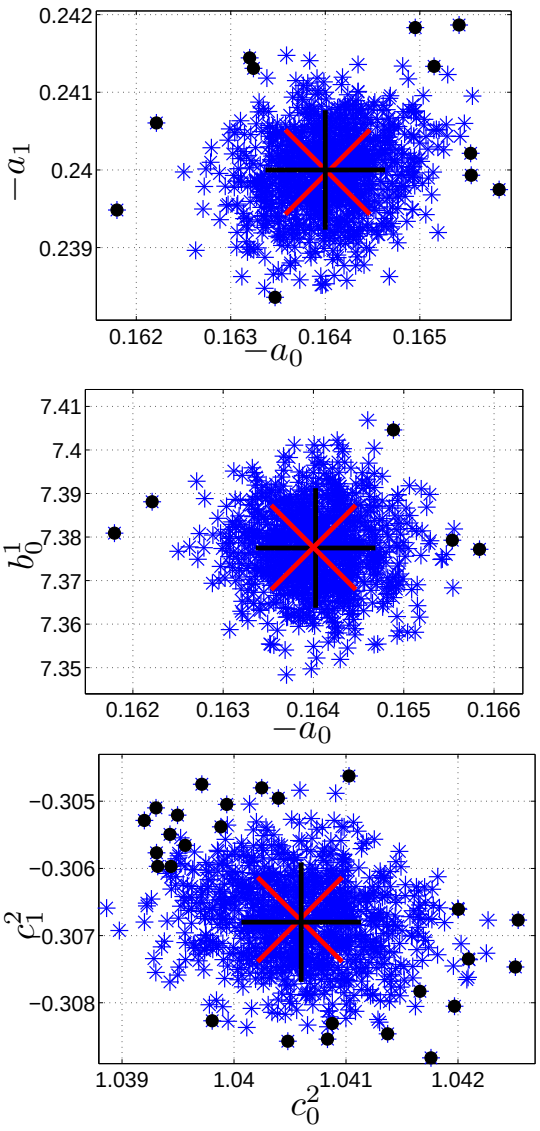

Fig. 1. Level surfaces of the cost function $J(\bar{\theta})$. The real parameters are symbolized by a red cross (×), the estimated parameters by a blue cross (∗) and the mean value of the estimated parameters by a black cross (+). Black discs (•) are finally the failure draws.

VI. CONCLUSIONS

In this paper, the problem of the state-space coefficients uncertainty domain determination for MIMO LTI state-space systems is solved. Because the ordinary subspace method provide an estimation of the system matrices up to a similarity transformation, a particular least-squares method is used. This identification method contains two steps. The first leads to estimate the state vector thanks to the propagator method in a user-define coordinate basis. Then, $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ are estimated using a least-squares algorithm. The uncertainty domains are described using a bounded error approach. This approach supposes that the residual is bounded and no stochastic assumption on the noise is made. The experimental results have emphasized the reliability of the developed method.

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