Admissibilty for Volterra systems with scalar kernels

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***Abstract*— Volterra observations systems with scalar kernels are studied. New sufficient conditions for admissibility of observation operators are developed.**

I. INTRODUCTION

The purpose of this article is to present conditions for the admissibility of observation operators to parabolic Volterra equations, that is, we consider the 'observed' system

$$\begin{aligned} x(t) &= x_0 + \int_0^t a(t-s)\,Ax(s)\,ds, \qquad (1) \\ y(t) &= Cx(t), \end{aligned}$$

where $t \geq 0$. Here, the operator A is supposed to be a closed operator with dense domain on a Banach space X, $x_0 \in X$, the kernel function $a \in L^1_{\text{loc}}$ is supposed to be of sub-exponential growth and 1-regular, and it is assumed that (1) is parabolic. In Prüss [7] it is shown that under these assumptions, equation (1) admit a unique *solution family*, i.e. a family of bounded linear operators $(S(t))_{t\geq 0}$ on X.

For some results we need in addition that $-A$ a sectorial operator of type $\omega \in (0, \pi)$ or that the kernel a is *sectorial of angle* $\theta \in (0, \pi)$. The kernel a is called *sectorial of angle* $\theta \in (0, \pi)$ if

$$\widehat{a}(\lambda) \in \Sigma_\theta \qquad \text{for all } \lambda \text{ with positive real part.}$$

In particular, when $-A$ and a are both sectorial in the respective sense with angles that sum up to a constant strictly inferior to π, the Volterra equation is parabolic.

The operator C is supposed to be an operator from X into another Banach space Y that acts as a bounded operator from $X_1 \to Y$ where $X_1 = \mathcal{D}(A)$ is endowed by the graph norm of A. In order to guarantee that the output function lies locally in L_2 we are interested in the following property.

Definition 1. *A bounded linear operator* $C : X_1 \to Y$ *is called* finite-time admissible *for the Volterra equation (1) if there are constants* $\eta, K > 0$ *such that*

$$\left(\int_0^t \|CS(r)x\|^2\,dr\right)^{1/2} \leq Ke^{\eta t}\|x\|$$

for all $t \geq 0$ *and all* $x \in \mathcal{D}(A)$.

The notion of admissible observation operators is well studied in the literature for Cauchy systems, that is, $a \equiv 1$, see for example [3], [8], and [9]. Admissible observation operators for Volterra systems are studied in [2], [4], [5] and [6].

The Laplace transform of S, denoted by H, is given by

$$H(\lambda)x = \frac{1}{\lambda}(I - \hat{a}(\lambda)A)^{-1}x, \qquad \operatorname{Re}\lambda > 0.$$

Our first main result, Theorem 2 provides a subordination argument to obtain admissibility for the observed Volterra equation from the admissibility of the observation operator for the underlying Cauchy problem.

Theorem 2. *Let* A *generate an exponentially stable strongly continuous semigroup* $(T(t))_{t\geq 0}$ *and let* $C : X_1 \to Y$ *be bounded. Further we assume that the kernel* $a \in L^1_{loc}(\mathbb{R}_+)$ *is of sub-exponential growth,* 1*-regular and sectorial of angle* $\theta < \pi/2$*. Then finite-time admissibility of* C *for the semigroup* $(T(t))_{t\geq 0}$ *implies that of* C *for the solution family* $(S(t))_{t\geq 0}$.

This allows a large number of corollaries, based on positive results for the Weiss conjecture. Here we only mention the following.

Corollary 3. *Assume in addition to the hypotheses of the theorem that* A *admits a Riesz basis of eigenfunctions* (e_n) *on a Hilbert space* X *with corresponding eigenvalues* λ_n*. If* $Y = \mathbb{C}$ *and if*

$$\mu = \sum_n |Ce_n|^2 \delta_{-\lambda_n}$$

is a Carleson measure on $\mathbb{C}_+$*, then* C *is finite-time admissible for the solution family* $(S(t))_{t\geq 0}$.

This corollary improves a direct Carleson measure criterion from Haak, Jacob, Partington and Pott [2]. Our second main result provides a sufficient condition for admissibility.

Theorem 4. *Assume that* A *is a closed operator with dense domain on* X*, the kernel function* $a \in L^1_{loc}$ *is of sub-exponential growth, 1-regular, and (1) is parabolic. Let* $C : X_1 \to Y$ *be bounded and assume that for some* $\alpha > 1/2$,

$$\sup_{r>0} \left\| \, (1 + \log^+ r)^\alpha r^{1/2} CH(r) \, \right\| < \infty. \qquad (2)$$

Then C *is finite-time admissible for* $(S(t))_{t\geq 0}$.

The obtained results are applied to time-fractional diffusion equations of distributed order and are compared with other results on Volterra systems known so far.

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References

[1] B. Haak, B. Jacob, *Observation of Volterra systems with scalar kernels*, Journal of Integral Equations and Applications, to appear.

[2] B. Haak, B. Jacob, J.R. Partington, and S. Pott, Admissibility and Controllability of diagonal Volterra equations with scalar inputs, *J. Differential Equations*, 246 (2009), 4423-4440.

[3] B. Jacob and J.R. Partington, Admissibility of control and observation operators for semigroups: a survey, in J.A. Ball, J.W. Helton, M. Klaus and L. Rodman: Current Trends in Operator Theory and its Applications, Proceedings of IWOTA 2002, Operator Theory: Advances and Applications, Vol. 149, Birkhäuser (2004), 199-221.

[4] B. Jacob and J.R. Partington, Admissible control and observation operators for Volterra integral equations, *Journal of Evolution Equations*, 4 (2004), 333-343.

[5] B. Jacob and J.R. Partington, A resolvent test for admissibility of Volterra observation operators, *J. Math. Anal. and Appl.*, 332 (2007), 346-355.

[6] M. Jung, Admissibility of control operators for solution families to Volterra integral equa- tions, *SIAM J. Control Optim.*, 38 (2000), 1323-1333.

[7] J. Prüss, *Evolutionary integral equations and applications*, vol. 87 of Monographs in Mathematics, Birkhäuser Verlag, Basel, 1993.

[8] O. Staffans, *Well-Posed Linear Systems*, no. 103 in Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2005.

[9] G. Weiss, Admissible observation operators for linear semigroups, *Israel J. Math.*, 65 (1989), pp. 17-43.