Non-dissipative boundary feedback for elastic beams

Chris Guiver and Mark R. Opmeer

***Abstract*— We show that a non-dissipative feedback that has been shown in the literature to exponentially stabilize an Euler-Bernoulli beam makes a Rayleigh beam and a Timoshenko beam unstable.**

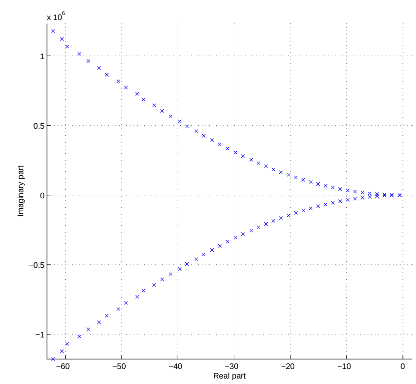
I. INTRODUCTION

Feedback control of beams is a much studied topic, in part due to its applications to the control of robot arms. The feedback control strategies used are often of the static output feedback kind and the input and output are usually chosen to make the closed loop system dissipative. An intriguing non-dissipative control strategy was however chosen in [5]. We refer to that article for the physical interpretation of their choice of feedback. As open-loop model they consider an undamped Euler-Bernoulli beam. Dissipative static output feedback strategies give rise to a closed loop system that has eigenvalues asymptotic to a line $\mathrm{Re}\lambda = -c$ for some constant $c > 0$ (see e.g. [4]). The eigenvalues of the non-dissipative closed-loop system were shown in [5] to be asymptotic to the parts of the parabolas $\mathrm{Im}\lambda = \pm c\,(\mathrm{Re}\lambda)^2$ in the left half-plane (see Fig 1(a)). This indicates that high frequencies are much better damped by the non-dissipative feedback than by dissipative feedbacks, a very attractive property.

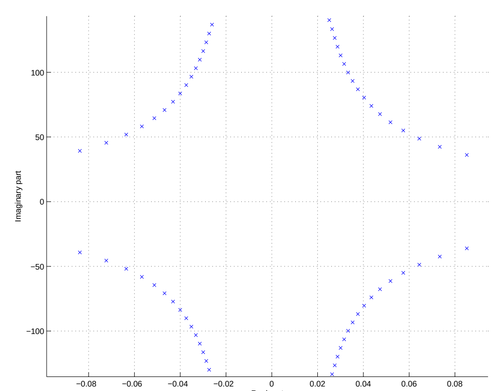
Besides the above asymptotics, [5] also showed that -as in the dissipative case- the eigenvalues of the closed loop system are all in the open left half plane. However, for partial differential equations certain pathologies may occur that prevent the stability of a system to be determined from the location of its eigenvalues. Due to this, [5] only managed to show exponential stability of the closed-loop system for smooth initial conditions in spite of the fact that all its eigenvalues are in the open left half-plane and are bounded away from the imaginary axis. Using estimates of the Green function [3] showed that the closed-loop system is a Riesz spectral system and since for Riesz spectral systems the location of the eigenvalues does determine the stability, exponential stability followed (also for non smooth initial data). Subsequently, [6] gave a more direct proof that the closed-loop system is a Riesz spectral system and [1] gave a proof of exponential stability based on microlocal analysis instead of on the Riesz basis property.

As mentioned, [5] chose an Euler-Bernoulli beam model (and the subsequent articles mentioned followed suit). This neglects the fact that the beam has a moment of inertia (and probably less importantly it neglects shear effects and non-linear effects). The Rayleigh beam model does incorporate the fact that a beam has a positive moment of inertia. The eigenvalues based on a finite element approximation of the Rayleigh beam with a non-dissipative feedback analogous to the one in [5] are given in Fig 1(b). Surprisingly, the eigenvalues are very different from those in the Euler-Bernoulli case. In particular, there are many unstable eigenvalues. In [2] we prove that indeed the Rayleigh beam with non-dissipative feedback has infinitely many unstable eigenvalues. We also prove in [2] that the addition of shear effects on top of a nonzero moment of inertia (i.e. replacing the Rayleigh model by the Timoshenko model) gives no qualitative difference: also in that case there are infinitely many eigenvalues with positive real part. We conclude that a static non-dissipative feedback as considered [5] is a *worse* choice for stability than dissipative feedback for Rayleigh and Timoshenko beam models.

Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, United Kingdom m.opmeer@maths.bath.ac.uk, cwg20@bath.ac.uk.



(a) Euler-Bernoulli beam



(b) Rayleigh beam

Fig. 1. Numerical approximations for eigenvalues of the Euler-Bernoulli and Rayleigh beam models

ISBN 978-963-311-370-7

II. Main results

A. Rayleigh beam case.

We consider first the following Rayleigh beam problem:

$$\begin{aligned} &EIw_{\xi\xi\xi\xi} + \rho w_{tt} - I_\rho w_{\xi\xi tt} = 0, \\ &w = w(\xi,t),\ t \in \mathbb{R}_+,\ \xi \in [a,b] \subset \mathbb{R}, \end{aligned} \tag{1a}$$

where $w(\xi,t)$ is the transverse displacement of the beam at position ξ and time t. We use the notation $w_t = \frac{\partial w}{\partial t}$ and $w_\xi = \frac{\partial w}{\partial \xi}$. The constants EI, ρ and I_ρ are physical parameters associated with the beam, for details see [7], or most elementary vibration textbooks. The choice of boundary feedbacks are analogous to the choice in [5], [3], [6] and [1] and are for $t \geq 0$:

$$\begin{aligned} w(a,t) &= 0 \\ w_\xi(a,t) &= 0 \\ -k_1 w_t(b,t) &= w_{\xi\xi}(b,t) \\ -k_2 w_{\xi t}(b,t) &= (I_\rho w_{\xi tt} - EIw_{\xi\xi\xi})(b,t) \end{aligned} \tag{1b}$$

where $k_1, k_2 \geq 0$ are the feedback constants.

The beam is clamped at the left endpoint which is described by the first two equations in (1b). To help understand the motivation for the third and fourth equations in (1b), recall that the energy of the Rayleigh beam is given by:

$$E(t) = \frac{1}{2}\int_a^b EI|w_{\xi\xi}|^2 + \rho|w_t|^2 + I_\rho|w_{t\xi}|^2 \,\mathrm{d}\xi.$$

Differentiating with respect to t, substituting using (1a), integrating by parts and then applying the boundary conditions at $\xi = a$ gives:

$$\begin{aligned} E_t(t) &= \left\langle \begin{pmatrix} w_t(b,t) \\ w_{\xi t}(b,t) \end{pmatrix}, \begin{pmatrix} I_\rho w_{\xi tt}(b,t) - EIw_{\xi\xi\xi}(b,t) \\ EIw_{\xi\xi}(b,t) \end{pmatrix} \right\rangle \\ &=: \langle y(t), u(t) \rangle, \end{aligned} \tag{2}$$

where $\langle \cdot,\cdot \rangle$ denotes the inner product on $\mathbb{R}^2$ and $u(t)$ is the input. From Lyapunov theory, it is sensible to choose u such that $E_t(t) < 0$ along solutions w. Therefore, an obvious choice of u is

$$u(t) = Ky(t), \tag{3}$$

with K negative definite, which is the so-called dissipative boundary feedback. Inserting (3) into (2) gives:

$$E_t(t) = \langle y(t), Ky(t) \rangle < 0.$$

The canonical negative definite matrix is

$$K = \begin{pmatrix} -k_1 & 0 \\ 0 & -k_2 \end{pmatrix},\ k_1, k_2 > 0.$$

The choice of boundary conditions in [3] for the Euler-Bernoulli case (i.e. (1a) and (1b) with $I_\rho = 0$) is to instead take

$$K = \begin{pmatrix} 0 & -k_2 \\ -k_1 & 0 \end{pmatrix}, \tag{4}$$

which is an indefinite matrix (and leads to non-dissipative boundary feedback). Exponential stability is proven when $k_1 = 0$ and $k_2 > 0$. The same result also holds in the alternate case with $k_1 > 0$, $k_2 = 0$ which follows by a duality argument.

The choice of feedback matrix (4) in the Rayleigh case gives the third and fourth equations in (1b).

Denote by (1) the partial differential equation (1a) and the boundary conditions (1b). In [2] it is proven that not only is the Rayleigh system (1) *not* exponentially stable, but further that the system is in fact *unstable*.

In order to prove the system is unstable we make the ansatz that a non-trivial solution to (1) has the form:

$$w(\xi,t) = \mathrm{e}^{st}\mathrm{e}^{\lambda(\xi-a)}, \quad s, \lambda \in \mathbb{C}. \tag{5}$$

Throughout this paper we will assume that $s \neq 0$. In order for such an ansatz (5) to be a solution λ, s must satisfy an algebraic condition given by the PDE (1a) and a characteristic equation given by the boundary conditions (1b). The algebraic condition is:

$$\lambda^4 - \frac{s^2 I_\rho}{EI}\lambda^2 + \frac{s^2\rho}{EI} = 0, \tag{6}$$

giving

$$\begin{aligned} \lambda_1 &= \sqrt{\frac{\frac{s^2 I_\rho}{EI} + \sqrt{\frac{s^4 I_\rho^2}{EI^2} - 4\frac{s^2\rho}{EI}}}{2}}, \\ \lambda_2 &= \sqrt{\frac{\frac{s^2 I_\rho}{EI} - \sqrt{\frac{s^4 I_\rho^2}{EI^2} - 4\frac{s^2\rho}{EI}}}{2}} \\ \lambda_3 &= -\lambda_1, \\ \lambda_4 &= -\lambda_2. \end{aligned} \tag{7}$$

It follows that a non-trivial solution to (1a) is given by

$$w(\xi,t) = \mathrm{e}^{st}\sum_{i=1}^4 c_i \mathrm{e}^{\lambda_i(s)(\xi-a)},\ s \in \mathbb{C},\ c_i \in \mathbb{R}, \tag{8}$$

with c_i not all zero. The boundary conditions (1b) applied to (8) yields the second condition for λ, s in the form of a linear system for the c_i. Namely if the matrix P is defined as

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & -\lambda_1 & -\lambda_2 \\ \varepsilon_1\mathrm{e}^{\lambda_1\Delta} & \varepsilon_2\mathrm{e}^{\lambda_2\Delta} & \varepsilon_1\mathrm{e}^{-\lambda_1\Delta} & \varepsilon_2\mathrm{e}^{-\lambda_2\Delta} \\ \lambda_1\eta_1\mathrm{e}^{\lambda_1\Delta} & \lambda_2\eta_2\mathrm{e}^{\lambda_2\Delta} & -\lambda_1\eta_1\mathrm{e}^{-\lambda_1\Delta} & -\lambda_2\eta_2\mathrm{e}^{-\lambda_2\Delta} \end{bmatrix} \tag{9}$$

and $c := \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$ then

$$Pc = 0, \tag{10}$$

where $\Delta := b - a$, $\varepsilon_i = \lambda_i^2 + k_1 s$ and $\eta_i = (-k_2 s - s^2 I_\rho + EI\lambda_i^2)$. Equation (10) has a non-trivial solution c if and only if $\det P = 0$. Computing $\det P = 0$ and dividing through by s^5 results in the following characteristic equation:

$$
\begin{aligned}
0 = \lambda_1\lambda_2 \Bigg[& \frac{I_\rho^2}{EIs} + \frac{k_2 I_\rho}{EIs^2} \\
& + \frac{k_1 I_\rho}{s^2} + 2\frac{k_1k_2-\rho}{s^3}\Bigg] \cosh(\lambda_1\Delta)\cosh(\lambda_2\Delta) \\
- \Bigg[& \frac{\rho I_\rho}{EIs} + \frac{k_1k_2 I_\rho}{EIs} + \frac{2k_2\rho}{EIs^2} \\
& + \frac{2k_1\rho}{s^2}\Bigg] \sinh(\lambda_1\Delta)\sinh(\lambda_2\Delta) \\
- \lambda_1\lambda_2 & \left[\frac{k_2I_\rho}{EIs^2} + \frac{k_1 I_\rho}{s^2} + 2\frac{\rho + k_1k_2}{s^2}\right]. \qquad (11)
\end{aligned}
$$

Instability of the system (1) is proven in [2] by investigating the sign of $\operatorname{Re} s$, for s a zero of (11) and ultimately proving (11) has zeros with positive real part. In this case there are solutions of (1) in the form (8) with $\operatorname{Re} s > 0$, and instability follows. We mention again that in [3] only one of the feedback parameters is required to be non-zero in order to achieve exponential stability. To give full generality we consider all three possible cases. These are where exactly one of k_1 and k_2 is zero, and also where both k_1, k_2 are positive. Our main results are now stated beneath, and are all proved in [2]:

Theorem 1: For all $k_1, k_2 \geq 0$ with $k_1 + k_2 > 0$ the equation (11) has zeros $s_n \in \mathbb{C}, n \in \mathbb{N}$ which satisfy

$$\left| s_n - \frac{(\pi n + \frac{\pi}{2})\mathrm{i}}{b-a}\sqrt{\frac{EI}{I_\rho}}\right| \to 0 \quad \text{as } n \to \infty.$$

Further, Re $s_n > 0$ for infinitely many $n \in \mathbb{N}$.
We then deduce the following corollary.

Corollary 2: For all $k_1, k_2 \geq 0$ with $k_1 + k_2 > 0$ the system (1) is unstable.

B. *Timoshenko beam case.*

We consider next the Timoshenko beam equation:

$$
\begin{aligned}
& EIw_{\xi\xi\xi\xi} + \rho w_{tt} \\
& \qquad - (I_\rho + \frac{EI\rho}{K})w_{\xi\xi tt} + \frac{I_\rho\rho}{K}w_{tttt} = 0, \qquad (12) \\
& \qquad w = w(\xi,t),\ t \in \mathbb{R}_+,\ \xi \in [a,b] \subset \mathbb{R},
\end{aligned}
$$

where K is an additional physical parameter, the shear modulus. It is also convenient to write (12) as the coupled wave equations

$$
\begin{aligned}
\rho w_{tt} &= K w_{\xi\xi} - K\phi_\xi, \\
I_\rho \phi_{tt} &= EI\phi_{\xi\xi} - K\phi + Kw_\xi,
\end{aligned} \qquad (13a)
$$

where ϕ is the angular displacement. Note that as the parameter K tends to infinity the equation (12) collapses to (1a), the PDE for the Rayleigh beam, which represents the beam becoming rigid to shear. The non-dissipative boundary feedbacks for the Timoshenko beam are:

$$
\begin{aligned}
w_t(a,t) = \phi_t(a,t) &= 0, \\
w_\xi(b,t) - \phi(b,t) &= -k_1 I_\rho \phi_t(b,t), \\
\phi_\xi(b,t) &= -k_2\rho w_t(b,t),
\end{aligned} \qquad (13b)
$$

where $k_1, k_2 \geq 0$ are the feedback constants.

There is an elegant formulation of the Timoshenko beam problem using state variables x_1, x_2, x_3, x_4 where

$$
\begin{aligned}
x_1 &= w_\xi - \phi, \\
x_2 &= \rho w_t, \\
x_3 &= \phi_\xi, \\
x_4 &= I_\rho \phi_t.
\end{aligned}
$$

In these variables the energy of the Timoshenko beam is

$$E(t) = \frac{1}{2}\int_a^b K|x_1|^2 + \frac{1}{\rho}|x_2|^2 + EI|x_3|^2 + \frac{1}{I_\rho}|x_4|^2 \, \mathrm{d}\xi.$$

Arguing as in the Rayleigh case it is not difficult to see that (13b) are indeed the analogous choice of non-dissipative boundary conditions for this problem. For more information on the state variable approach to the Timoshenko beam we refer the reader to [8].

Let (13) denote the PDE (13a) and boundary conditions (13b). We proceed as in the Rayleigh case and make the ansatz for a solution of (13)

$$
\begin{aligned}
w(\xi,t) &= \mathrm{e}^{st}\sum_{i=1}^4 c_i \mathrm{e}^{\lambda_i(s)(\xi-a)}, \\
\phi(\xi,t) &= \mathrm{e}^{st}\sum_{i=1}^4 c_i \mathrm{e}^{\lambda_i(s)(\xi-a)}\left(\lambda_i - \frac{\rho s^2}{K\lambda_i}\right),
\end{aligned} \qquad (14)
$$

for $c_i \in \mathbb{R}$ not all zero. The λ, s satisfy algebraic conditions from the PDE (13a) and the boundary conditions (13b). For each $s \in \mathbb{C}$, the λ_i are the four roots of

$$EI\lambda^4 - \left(I_\rho + \frac{EI\rho}{K}\right)s^2\lambda^2 + \left(\rho s^2 + s^4\frac{\rho I_\rho}{K}\right) = 0. \quad (15)$$

The second condition is the corresponding linear system for the c_i and is $Qc = 0$. Here $Q = Q(s)$ is given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \varepsilon_1 & \varepsilon_2 & -\varepsilon_1 & -\varepsilon_2 \\ \eta_1\mathrm{e}^{\lambda_1\Delta} & \eta_2\mathrm{e}^{\lambda_2\Delta} & -\eta_1\mathrm{e}^{-\lambda_1\Delta} & -\eta_2\mathrm{e}^{-\lambda_2\Delta} \\ \chi_1\mathrm{e}^{\lambda_1\Delta} & \chi_2\mathrm{e}^{\lambda_2\Delta} & \chi_1\mathrm{e}^{-\lambda_1\Delta} & \chi_2\mathrm{e}^{-\lambda_2\Delta} \end{bmatrix}, \qquad (16)$$

with $\Delta := b - a$ and for $i \in \{1, 2\}$

$$
\begin{aligned}
\varepsilon_i &= \lambda_i - \frac{\rho s^2}{K\lambda_i}, \\
\eta_i &= k_1 I_\rho\lambda_i + \frac{\rho s}{K\lambda_i} - \frac{k_1 I_\rho \rho s^2}{K\lambda_i}, \\
\chi_i &= \lambda_i^2 - \frac{\rho s^2}{K} + k_2\rho s.
\end{aligned} \qquad (17)
$$

Again, we seek s such that $\det Q = 0$. The resulting characteristic equation is:

$$
\begin{aligned}
0 = {} & R(s,\lambda_1,\lambda_2)\cosh(\lambda_1\Delta)\cosh(\lambda_2\Delta) \\
& + P(s,\lambda_1,\lambda_2)\sinh(\lambda_1\Delta)\sinh(\lambda_2\Delta) + T(s,\lambda_1,\lambda_2)
\end{aligned} \qquad (18)
$$

where P, R and T are polynomials in several variables and are given in more detail in [2].

As before, zeros of the characteristic equation (18) will give a solution to the Timoshenko beam system (13) in the form of our ansatz (14). It is proven in [2] that (13) is *not* exponentially stable by proving (18) has zeros with positive real part.

Theorem 3: For all positive ρ, EI, I_ρ and K with $\frac{I_\rho}{EI} \neq \frac{\rho}{K}$ and all non-negative k_1, k_2 with $k_1 + k_2 > 0$ and $k_1 k_2 \neq \frac{1}{KI_\rho}$, the equation (18) has infinitely many zeros, $s_n \in \mathbb{C}$, with Re $s_n > 0$.

If $\frac{I_\rho}{EI} = \frac{\rho}{K}$ and $k_1 k_2 > 0, k_1 k_2 \neq \frac{1}{KI_\rho}$ then the above result holds. If $\frac{I_\rho}{EI} = \frac{\rho}{K}$ and $k_1 k_2 = 0$ then the above result holds provided that additionally $\cos\left(\frac{(b-a)}{2}\sqrt{\frac{\rho}{I_\rho}}\right) \neq 0$.

We deduce the following corollary.

Corollary 4: Assuming the hypotheses of Theorem 3, the system (13) is unstable.

References

[1] Boris Belinskiy and Irena Lasiecka. Gevrey's and trace regularity of a semigroup associated with beam equation and non-monotone boundary conditions. *J. Math. Anal. Appl.*, 332(1):137–154, 2007.

[2] Chris Guiver and Mark R. Opmeer. Non-dissipative boundary feedback for Rayleigh and Timoshenko beams. *Systems Control Lett.* Submitted August 2009.

[3] Bao-Zhu Guo, Jun-min Wang, and Siu-Pang Yung. On the C_0-semigroup generation and exponential stability resulting from a shear force feedback on a rotating beam. *Systems Control Lett.*, 54(6):557–574, 2005.

[4] Bao-Zhu Guo and Runyi Yu. The Riesz basis property of discrete operators and application to a Euler-Bernoulli beam equation with boundary linear feedback control. *IMA J. Math. Control Inform.*, 18(2):241–251, 2001.

[5] Zheng-Hua Luo and Bao-Zhu Guo. Shear force feedback control of a single-link flexible robot with a revolute joint. *IEEE Trans. Automat. Control*, 42(1):53–65, 1997.

[6] Marianna A. Shubov. Generation of Gevrey class semigroup by non-selfadjoint Euler-Bernoulli beam model. *Math. Methods Appl. Sci.*, 29(18):2181–2199, 2006.

[7] William T. Thomson. *Theory of Vibrations with applications, fourth edition*. Nelson Thornes Ltd, Delta Place, 27 Bath Road, Cheltenham, GL53 7TH, UK, 2003.

[8] Javier A. Villegas. *A port-Hamiltonian approach to distributed parameter systems*. PhD thesis, Department of Applied Mathematics, Universiteit Twente, Enschede, http://doc.utwente.nl/57842/, May 2007.