

Quantify the Unstable

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Abstract—The Mahler measure, a notion often appearing in the number theory and dynamic system literature, provides a way to quantify the instability in a linear discrete-time system.

I. INTRODUCTION

Which of the following two first order discrete-time autonomous systems

$$x(k+1) = 2x(k) \quad (1)$$

and

$$x(k+1) = 3x(k) \quad (2)$$

is more unstable? Common sense tells us that system (2) is more unstable than system (1) since its state diverges to infinity faster. Which of the following two systems

$$x(k+1) = \begin{bmatrix} 2 & 10^{10} \\ 0 & 4 \end{bmatrix} x(k) \quad (3)$$

and

$$x(k+1) = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0.5 \end{bmatrix} x(k) \quad (4)$$

is more unstable? Now the answer is not so obvious. In this survey paper, we attempt to give an answer to this question. We will assign an instability measure to each system, and also give the historic perspective of, theoretical justification to, and various ramifications of this instability measure.

Our answer will depend only on the unstable modes of the systems involved. It has nothing to do with the stable modes, even the marginally unstable modes, and nothing to do with other eigenstructure of the system matrices, nothing to do with the norm of the system matrices.

Much of this study is powered by the recent activities in networked control systems (NCS) [2], [3], [12], though related results appeared much earlier in control system literature, often under the umbrella of fundamental performance limitations of feedback control [19]. The study is also influenced by the first Bode lecture at the 1989 IEEE Conference on Decision and Control by Gunter Stein [21].

We only examine discrete-time systems in this paper. Continuous-time systems can be studied in a rather analogous way.

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II. HISTORY

Consider a polynomial

$$\begin{aligned} a(z) &= a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n \\ &= a_0 \prod_{i=1}^n (z - r_i), \quad a_0 \neq 0, \end{aligned}$$

which has roots r_1, r_2, \dots, r_n . Kurt Mahler in 1960 [16] defined the so-called Mahler measure

$$M(a) = |a_0| \prod_{i=1}^n \max\{1, |r_i|\} = |a_0| \prod_{|r_i| > 1} |r_i|.$$

He also observed by using Jensen's formula that

$$M(a) = \exp \left[\frac{1}{2\pi} \int_0^{2\pi} \ln |a(e^{j\omega})| d\omega \right].$$

We will mostly deal with monic polynomials, for which the Mahler measure depends only on the roots, or more precisely the unstable roots (i.e., the ones outside of the unit circle), of the polynomial. Let us extend the definition to a square matrix A and call $M(\det(zI - A))$ as the Mahler measure of A , denoted by $M(A)$.

Since Mahler's definition, there were two related interesting developments. The first is the recognition of its connection to an optimization problem considered initially by Gabor Szegő [24], as well as the extension of it. Denote the unit circle of the complex plain by \mathbb{T} . Let $f : \mathbb{T} \rightarrow \mathbb{C}$ be a measurable function. Then for each $p \in [0, \infty]$ define the “ p -norm” of f by

$$\|f\|_p = \left[\frac{1}{2\pi} \int_0^{2\pi} |f(e^{j\omega})|^p d\omega \right]^{1/p}.$$

We do not wish to rule out the possibility of $p < 1$, though in such cases $\|f\|_p$ does not satisfy the triangle inequality and hence is not a norm in a strict sense. For $p = 0$ or ∞ , the above definition does not really work; we actually have

$$\|f\|_\infty = \lim_{p \nearrow \infty} \|f\|_p = \text{ess sup}_{\omega \in [0, 2\pi]} |f(e^{j\omega})|$$

and

$$\|f\|_0 = \lim_{p \searrow 0} \|f\|_p = \exp \left[\frac{1}{2\pi} \int_0^{2\pi} \ln |f(e^{j\omega})| d\omega \right].$$

It is easy to see that $\|f\|_p$ is an increasing function of p and $M(a) = \|a\|_0$ for each polynomial a .

Theorem 1 ([7]) *Let a be a given polynomial. Then*

$$\inf_q \|aq\|_p = M(a)$$

where the infimum is taken over all monic polynomials q .

Such optimization problems are not exactly the ones we encounter in systems and control theory, but they are inspiring. In fact, as we will see later in this paper, quite many optimization problems in systems and control also have solutions given by the Mahler measure.

The second development is the connection of the Mahler measure with the entropy of linear time-invariant (LTI) autonomous system

$$x(k + 1) = Ax(k), \quad x(k) \in \mathbb{C}^n. \quad (5)$$

There were various efforts in defining an entropy for a dynamic system to capture the complexity, ergodicity, information content, or expansion rate of the solutions, for example, the Kolmogorov-Sinai measure-theoretical entropy [20] and the topological entropy [1]. It is not until Rufus Bowen [4] when a metric-theoretical entropy was defined which can be applied to an LTI autonomous system. Bowen still named his quantity the topological entropy. We choose not to repeat the wordy definition of Bowen here and refer the readers to [4] and textbooks [10], [14] for details. Let us denote the Bowen's entropy of LTI system (5) by $h(A)$. Bowen established the following connection.

Theorem 2 ([4])

$$h(A) = \ln M(A).$$

The main message of this paper is that either $M(A)$ or $h(A)$ can be used to measure the instability of a system in the form of (5). Using such a measure, we are able to answer the questions raised in Section I. System (4) which has $M(A) = 9$ is more unstable than system (3) which has $M(A) = 8$.

III. JUSTIFICATIONS I – SINGLE-INPUT SYSTEMS

Let us first consider a minimum energy control problem, illustrated in Figure 1. Here the block marked with $[A|B]$ is a state space system

$$x(k + 1) = Ax(k) + Bu(k). \quad (6)$$

In this section, we consider only the single-input case. In this case, B is a column vector. We assume that $[A|B]$ is stabilizable. The controller F is a constant feedback gain. It is said to be *stabilizing* if $A + BF$ is stable, i.e., having all eigenvalues inside the unit circle.

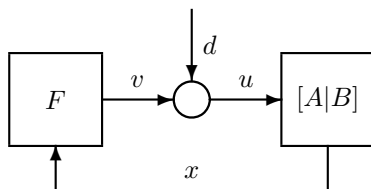


Fig. 1. State feedback stabilization

Let us first assume that the disturbance signal d is a unit impulse

$$d(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

and wish to stabilize the system with minimum energy in the controller output v . If F is stabilizing, it is well-known that the energy of v , i.e., the square of the ℓ_2 norm of v , is given by the square of the \mathcal{H}_2 norm of the transfer function $T(z)$ from d to v :

$$\|T\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |T(e^{j\omega})|^2 d\omega.$$

It is easy to see that

$$T(z) = F(zI - A - BF)^{-1}B$$

which usually is called the *complementary sensitivity function*. The smallest energy that the controller has to generate in order to stabilize the system is then given by the minimum value of $\|T\|_2^2$ over all stabilizing F .

Theorem 3

$$\inf_{F: A+BF \text{ is stable}} \|T\|_2^2 = M(A)^2 - 1.$$

The exact origin of Theorem 3 is hard to trace. A proof was given in [23].

For this optimization problem, as well as those in the sequel, the optimal F exists, i.e., the infimum is actually a minimum, if and only if A does not have an eigenvalue in \mathbb{T} . Such a case is called a regular case. Otherwise, it is called a singular case. In a regular case, the optimal F can be obtained by solving a discrete-time Lyapunov (not Riccati) equation, as done in [23].

Next let us again assume that d is a unit impulse and set to stabilize the system with minimum energy in the plant input u . If F is a stabilizing feedback, it is well-known that the energy of u is given by the square of the \mathcal{H}_2 norm of the transfer function $S(z)$ from d to u :

$$\|S\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |S(e^{j\omega})|^2 d\omega.$$

It is easy to see that

$$S(z) = 1 + F(zI - A - BF)^{-1}B = 1 + T(z)$$

which is usually called the *sensitivity function*. The smallest energy in the plant input needed in the stabilization of the plant is then given by the minimum value of $\|S\|_2^2$ over all stabilizing F .

Theorem 4

$$\inf_{F: A+BF \text{ is stable}} \|S\|_2^2 = M(A)^2.$$

Theorem 4 follows from Theorem 3 rather easily. Since the two terms 1 and $T(z)$ in $S(z)$ are orthogonal in \mathcal{H}_2 , we have $\|S\|_2^2 = 1 + \|T\|_2^2$. Hence minimizing $\|S\|_2$ is the same

as minimizing $\|T\|_2$. The optimal feedback gain is the same for the two problems.

Minimizing the control energy for a fixed disturbance signal might not be the most reasonable thing to do. Next, we assume that the disturbance d is an unknown signal with energy bounded by 1. In this case, the worst case energy of v is then the square of the \mathcal{H}_∞ norm of the complementary sensitivity function $T(z)$

$$\sup_{\|d\|_2^2 < 1} \|v\|_2^2 = \|T\|_\infty^2 = \left[\sup_{\omega \in [0, 2\pi]} |T(e^{j\omega})| \right]^2.$$

Theorem 5 ([11])

$$\inf_{F: A+BF \text{ is stable}} \|T\|_\infty = M(A).$$

Now if we are interested in the minimum value of the worst case energy of the plant input u , then it is the \mathcal{H}_∞ norm of sensitivity function $S(z)$ that needs to be minimized.

Theorem 6 ([17])

$$\inf_{F: A+BF \text{ is stable}} \|S\|_\infty = M(A).$$

Unlike the \mathcal{H}_2 minimization case in Theorem 3-4, the \mathcal{H}_∞ minimization problems for $T(z)$ and $S(z)$ are completely different problems. The optimal or in the singular case the near optimal feedback gains are different, though rather surprisingly the optimal values of the two functions to be minimized are the same and are given by $M(A)$.

Having considered the \mathcal{H}_2 norm and \mathcal{H}_∞ norm of $T(z)$ and $S(z)$, we may consider the \mathcal{H}_0 “norm” of $T(z)$ and $S(z)$.

Theorem 7 ([22]) For each stabilizing F ,

$$\|S\|_0 = M(A).$$

This result is often called the discrete-time Bode integral formula, though it is nothing more than the Jensen’s formula from a mathematical point of view.

Theorems 4, 6, and 7 reveal that the minimum values of $\|S\|_\infty, \|S\|_2, \|S\|_0$ are all the same. Together with Theorem 1, they prompt us to take a look at the other “ p -norms”. The outcome is quite pleasant and follows directly from the increasing property of $\|S\|_p$ as a function of p .

Theorem 8 For each $p \in [0, \infty]$,

$$\inf_{F: A+BF \text{ is stable}} \|S\|_p = M(A)$$

It turns out that in the regular case, all these optimization problems for different p share a common optimal F . In the singular case, it is possible to construct a common family of stabilizing F_ϵ , parameterized by $\epsilon > 0$, such that $\|S\|_p \rightarrow M(A)$ as $\epsilon \rightarrow 0$.

The complementary sensitivity function $T(z)$ is not so lucky. We have seen that $\|T\|_2$ and $\|T\|_\infty$ have different minimum values. Curiosity drives us to consider $\|T\|_0$. It is

easy to check that $\|T\|_0$ is different for different stabilizing F . It is not clear what the minimum value of $\|T\|_0$ is when F is chosen among all stabilizing feedback gains.

Let us now consider several implications of the above results. The first is the stabilization problem with signal to noise ratio (SNR) constraint. Such a problem was considered in [5]. In Figure 1, assume now that d is a zero-mean white noise with variance σ_d^2 . With F stabilizing, v is then a stationary process with variance $\sigma_v^2 = \|T\|_2^2 \sigma_d^2$. In order to implement the stabilizing control corresponding to the F , the signal to noise ratio of the input channel, defined as

$$\text{SNR} = \frac{\sigma_v^2}{\sigma_d^2},$$

has to be $\|T\|_2^2$. In certain applications, the SNR of the input channel is constrained due to hardware limitations. Then the controller F can be implemented only if the SNR of the input channel hardware is greater than $\|T\|_2^2$. What is the smallest SNR requirement on the hardware so that the stabilization is possible? It follows from Theorem 3 immediately that the stabilization is possible if and only if the SNR is greater than $M(A)^2 - 1$. Here finding the smallest SNR requirement is nothing more than minimizing the \mathcal{H}_2 norm of $T(z)$. However this new interpretation lays the foundation of the extension to multiple-input systems.

In the above problem, the input signal is subject to an additive noise, whereas in the problem to be considered in the sequel, the input signal is subject to a multiplicative noise, as shown in Figure 2. Such a problem was considered in [8]. Here $\kappa(k)$ is a white noise with unit mean and variance σ^2 . The variance gauges the unreliability of the channel. The closed-loop system now is not an usual LTI system. It is governed by the following stochastic difference equation

$$x(k+1) = (A + B\kappa(k)F)x(k).$$

This system is said to be mean square stable if for each $x(0) = x_0$, possibly random with finite second moment $\mathcal{E}[x_0 x_0^*]$, the second moment $\mathcal{E}[x(k)x(k)^*] \rightarrow 0$ as $k \rightarrow \infty$. Here we used the notation $\mathcal{E}[\cdot]$ to mean the expectation. It can be proved [15] that this system is mean square stable if and only if $\sigma^2 \|T\|_2^2 < 1$. Therefore, the system can tolerate more unreliability of the input channel if $\|T\|_2^2$ is small. The maximum amount of unreliability tolerable by designing F is then given by $[M(A)^2 - 1]^{-1}$.

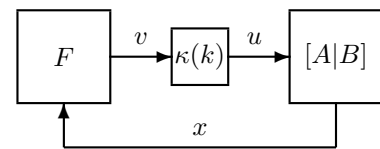


Fig. 2. State feedback system with stochastic time-varying gain

Let us now consider two robust stabilization problems. The first problem concerns the uncertain system shown in Figure 3. The uncertainty Δ in the input channel is a possibly

nonlinear, time-varying, dynamic system with an ℓ_2 induced norm bound:

$$\|\Delta\| = \sup_{v \in \ell_2, v \neq 0} \frac{\|e\|_2}{\|v\|_2} \leq \delta. \quad (7)$$

The uncertain input channel now has an input/output map $(I + \Delta)$ and it introduces a multiplicative uncertainty to the plant. We say that F is *robustly stabilizing* if it is stabilizing and the closed-loop system in Figure 3 is internally stable for all possible uncertainty Δ satisfying the norm bound (7). By using the small gain theorem [27], one can see that F is robust stabilizing if and only if $\delta < \|T\|_\infty^{-1}$. Hence $\|T\|_\infty^{-1}$ gives a *stability margin* of the closed-loop system. What is the largest stability margin obtainable by designing F ? Immediately from Theorem 5, we see that the largest stability margin is $M(A)^{-1}$.

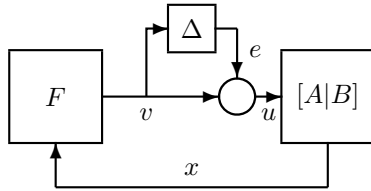


Fig. 3. State feedback system with plant multiplicative uncertainty

The second robust stabilization problem concerns the uncertain system shown in Figure 4. The uncertainty again satisfies the norm bound

$$\|\Delta\| = \sup_{u \in \ell_2, u \neq 0} \frac{\|e\|_2}{\|u\|_2} \leq \delta, \quad (8)$$

but now it is connected to the input channel in a feedback manner. The uncertain input channel now has an input/output map $(I - \Delta)^{-1}$ and it introduces a relative uncertainty to the plant. In this setup, we say that F is *robustly stabilizing* if it is stabilizing and the closed-loop system in Figure 4 is internally stable for all possible uncertainty Δ satisfying the norm bound (8). By using the small gain theorem again, one can see that F is robustly stabilizing if and only if $\delta < \|S\|_\infty^{-1}$. Hence $\|S\|_\infty^{-1}$ gives a *stability margin* of the closed-loop system. What is the largest stability margin obtainable by designing F ? Immediately from Theorem 6, we see that the largest stability margin is again $M(A)^{-1}$.

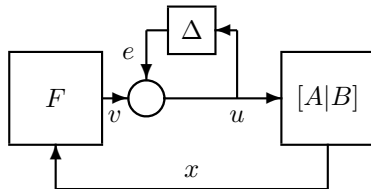


Fig. 4. State feedback system with plant relative uncertainty

All results in this section demonstrate that the Mahler measure $M(A)$ gives a degree of difficulty in controlling an unstable single-input system $[A|B]$. However, extending these results to multiple-input systems cannot be done in a naive way. The next section is dedicated to this purpose.

IV. JUSTIFICATIONS II – MULTIPLE-INPUT SYSTEMS

The climax of the current activities in this line of research is in the extension of the results in the last section to the multiple-input case. The extension is motivated by issues in networked control systems. The presentation here is sketchy.

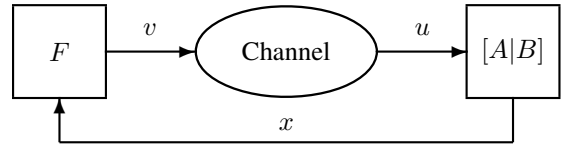


Fig. 5. A state feedback networked control system

Now assume that the system $[A|B]$ has m inputs, i.e., B is an $n \times m$ matrix. Each input channel is now considered to be a communication channel with certain capacity constraint. How to model a communication channel in a feedback loop is a big unsettled issue. We will consider several possible models. Since there are m input channels, we consider the i -th channel, transmitting the i -th element v_i of v to the i -th element u_i of u .

The first model is the commonly used additive white Gaussian noise (AWGN) channel model, as shown in Figure 6. This model is strongly influenced by the existing information theory [6]. Here d_i is a white Gaussian process with zero

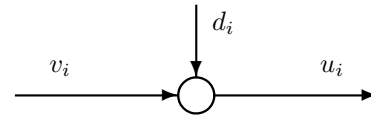


Fig. 6. An SNR channel model

mean and variance $\sigma_{d_i}^2$, and v_i is a stationary process with zero mean and variance $\sigma_{v_i}^2$. The signal-to-noise ratio of the channel is defined as

$$\text{SNR}_i = \frac{\sigma_{v_i}^2}{\sigma_{d_i}^2},$$

and the capacity of this channel as

$$C_i = \frac{1}{2} \ln(1 + \text{SNR}_i).$$

The second model is the so-called fading channel model shown in Figure 7. Here instead of an additive white Gaussian noise, the transmitted signal is subject to a multiplicative

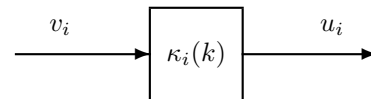


Fig. 7. A fading channel model

noise $\kappa_i(k)$ which is a white process with mean μ_i and variance σ_i^2 . The channel capacity is defined as

$$C_i = \frac{1}{2} \ln \left(1 + \frac{\mu_i^2}{\sigma_i^2} \right).$$

This model is strongly motivated by the packet drop phenomenon in a communication network and $\kappa_i(k)$ is often taken as a Bernoulli process [8].

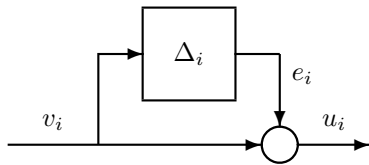


Fig. 8. An SER channel model

The third model is the so-called signal-to-error ratio (SER) model shown in Figure 8. Here Δ_i is a possibly nonlinear, time-varying, dynamic uncertain system satisfying $\|\Delta_i\| \leq \delta_i$. We call δ_i^{-1} the SER which measures the transmission accuracy and define the capacity as $C_i = \ln \delta_i^{-1}$. This model has been used to describe a logarithmic quantization in a networked channel [9], [11], [18].

The fourth model is the received-signal-to-error ratio (R-SER) model shown in Figure 9. Here Δ_i is a possibly nonlinear, time-varying, dynamic uncertain system satisfying $\|\Delta_i\| \leq \delta_i$. We call δ_i^{-1} the R-SER which measures the transmission accuracy and define the capacity as $C_i = \ln \delta_i^{-1}$. This model can be used to describe an alternative version of logarithmic quantization [17].

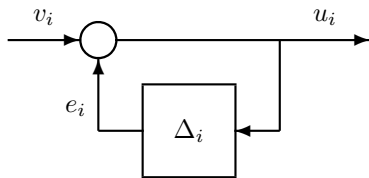


Fig. 9. An R-SER channel model

Now let the m input channels be all modelled by one of the above ways. Assume that there is no mixed modelling. The total capacity of the m channels is then given by

$$C = C_1 + C_2 + \dots + C_m.$$

Since the capacities are finite, one may not be able to design a feedback controller to stabilize the feedback system in the respective sense corresponding to the modelling method. One possible question to ask is that for given C_1, C_2, \dots, C_m , whether it is possible to design an F so that the closed-loop system is stable. This is a bad problem leading to a dead end. We have to take over the power of allocating the individual channel capacities C_1, C_2, \dots, C_m when the total capacity C is given. The question now becomes that for a given total capacity C , whether it is possible to allocate it among different channels and also design a feedback matrix such that the closed-loop system is stable. Here the feedback gain F is not the only design parameter. The allocation of the total capacity C among the m input channels is also a design freedom. Since the whole design involves both the channel resource allocation and the controller design, we call it the channel/controller co-design. There is a very nice answer to

the latter question which gives the smallest total capacity needed to make the co-design possible.

Theorem 9 (Universal minimum capacity) *Assume that the multiple input channels are modelled by any of the four channel models. The system can be stabilized by the channel/controller co-design if and only if $C > h(A)$.*

This theorem needs to be made more precise and proved for each model individually. For the first, third and fourth models, the proofs are given in [17]. For the third model, the proof is also given in [18]. For the second model, the proof is given in [26]. It is worth pointing out that all these proofs connect back to the very basic idea of Murray Wonham in doing multiple-input state feedback pole placement [25]. We expect that this theorem is also true for other possible channel models not listed here. It might also hold true for some more sophisticated models capturing some combined channel features. This is why the word universal is used. More research is underway.

V. CONCLUSIONS

It is the hope of the author that this paper provides a convincing story on $M(A)$ or $h(A)$ as an instability measure for discrete-time LTI systems of the form (5) or (6).

There are other theoretical justifications for $M(A)$ to be taken as a measure of instability of a discrete-time LTI system, that we do not survey in this paper. In particular, there is a body of literature on the minimum data rate requirement in the input channel in stabilizing a system. In this body of studies, an input channel in Figure 5 is modelled as a data communication device which can only transmit certain number of bits of data per unit time. In several different setups, it has been shown that the smallest total data rate of all input channels required for the stabilization of a system $[A|B]$ is given by $h(A) = \ln M(A)$ in a rather simple way. There were an extensive list of references on such studies. We refer to the two special issues [2], [3], the papers therein and the references listed in the papers.

This paper only addresses the control issues for system $[A|B]$, implying that the state feedback is used for control. Roughly speaking, $M(A)$ quantifies how difficult it is to stabilize a system using state feedback control. More generally we wish to treat a complete state space system $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, i.e., one described by difference equation

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k). \end{aligned}$$

In this case, the controller is implicitly assumed to be an output feedback controller. What is the difficulty in stabilization? What is the appropriate measure of instability? These are intriguing problems in front of us.

VI. ACKNOWLEDGMENTS

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