On the Maximum Entropy Completion of Circulant Covariance Matrices

Francesca Carli and Tryphon T. Georgiou

***Abstract*— This paper deals with the positive-definite completion of partially specified (block-) circulant covariance matrices. In the absence of any constraint other than positivity, the maximal-determinant completion of a partially specified covariance matrix (i.e., the so-called maximum entropy completion) was shown by Dempster to have an inverse with zero-values at all locations where the original matrix was unspecified–this will be referred to as the *Dempster property*. In earlier work, Carli *etal.* [2] showed that even under the constraint of a covariance being block-circulant, as long as the unspecified elements are in a single band, the maximum entropy completion has the Dempster property. The purpose of the present paper is to prove that circulant, block-circulant, or Hermitian constraints do not interfere with the Dempster property of the maximum entropy completion. I.e., regardless of which elements are specified, the completion has the Dempster property. This fact is a direct consequence of the invariance of the determinant to the group of transformations that leave circulant, block-circulant, or Hermitian matrices invariant. A description of the set of all positive extensions is discussed and connections between this set and the factorization of certain polynomials in many variables, facilitated by the circulant structure, is highlighted.**

I. Introduction

The subject of the present paper is the maximum entropy completion of partially-specified circulant positive-definite matrices. Such matrices are covariances of wide-sense stationary periodic or reciprocal processes [15], [16], [2], [4], [14]. In turn, the completion problem is equivalent to identifying a consistent power spectrum from incomplete data. Since finite observation records of general stationary processes are often treated as records of periodic ones, the circulant-completion problem is relevant in spectral analysis of stationary time-series in general as it brings Fourier techniques to bear [2].

Maximum entropy completion of partially specified circulant matrices has been recently studied in [2] where it has been shown that, when only a single band centered along the main diagonal is specified, the constraint that enforces the circulant structure becomes redundant and thereby the maximizer shares the property of maximizers for more general problems. This property which we refer to as the *Dempster property* is that the inverse of the completion is zero at all unspecified elements of the original matrix ([6]; see also [12], [7]).

The main contribution of the present paper is to provide a simple, independent argument that explains the result of Carli *etal.* [2] and at the same time shows that, in general, the circulant, block-circulant, or Hermitian constraints do not interfere with the Dempster property. That is, the maximum entropy circulant, block-circulant and Hermitian completion of any partially-specified covariance matrix has the Dempster property. Our result is a direct consequence of the invariance of the matrix-determinant to transformations that leave circulant, block-circulant, and Hermitian matrices invariant.

The paper goes on to describe the structure of the set of all positive completions for partially specified circulant matrices. Certain connections between completions and the factorization of polynomials in many variables is also highlighted. More specifically, since every block-circulant matrix can be diagonalized by a Fourier transformation, the positivity of a partially-specified $(n\times n)$ m-block circulant matrix gives rise to n, m-order curves, that delineate the admissible competion set. These curves can be obtained by factorization of the determinant as a polynomial in the unspecified coefficients. Thus, Fourier techniques can be efficiently used to factor polynomials written as determinants of circulant matrices with variable entries.

In section II some useful facts about circulant matrices are introduced. Section III discusses the Dempster property for general covariance matrices and then presents our main result for matrices with a circulant structure. Finally in section IV we present a couple of examples that give some insight into the structure of the completion-set, and we highlight connections with the factorization of certain polynomials in many variables.

II. Technical preliminaries & notation

Let $\{x_\ell,\ \ell\in\mathbb{Z}\}$ be a wide-sense stationary periodic process on $\mathbb{Z}$ of period n or, equivalently, a wide-sense stationary process on the cyclic group $\mathbb{Z}/(n\mathbb{Z})$. This is possibly vector-valued with $x_\ell\in\mathbb{C}^m$, thought of as column vectors. Obviously the process can be identified with the vector $\left(x_0',\dots,x_{n-1}'\right)'$ of values over one period. Let R denote the covariance matrix of such a restriction. The assumption of (second-order) stationarity together with the periodicity imply that R has the following structure

$$R:=\begin{bmatrix} R_0 & R_1 & R_2 & \dots & R_2^* & R_1^*\\ R_1^* & R_0 & R_1 & & R_3^* & R_2^*\\ R_2^* & R_1^* & R_0 & & R_4^* & R_3^*\\ \vdots & & & \ddots & & \vdots\\ R_2 & R_3 & R_4 & & R_0 & R_1\\ R_1 & R_2 & R_3 & \dots & R_1^* & R_0\end{bmatrix}, \tag{1}$$

i.e. R is a block-circulant matrix [5], [11]. We now introduce some notations and briefly review some facts about circulant

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matrices which will be useful in the sequel.

Evidently, a (block-) circulant matrix is completely specified by its first row (or, column) entries. In particular a generic circulant matrix is Hermitian if its first row entries, say a_i, satisfy $a_0 = a_0^*$ as well as

$$a_k = a_{n-k}^* \text{ for } k = 1, \dots \left\lfloor \frac{n}{2} \right\rfloor.$$

where a^* denotes the complex-conjugate-transpose. Thus, when n is even, both a_0 and $a_{\lfloor \frac{n}{2} \rfloor}$ need to be Hermitian for the matrix to be Hermitian. Let S denote the circulant (up) $(n \times n)$-shift

$$S := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix},$$

and I_n the $(n \times n)$-identity matrix. Clearly $S^n = I_n$ and, as is well-known [5] and easy to check, S^k has the (eigenvalue-eigenvector) decomposition

$$S^k U = U W^k,$$

where U is the Fourier-matrix with elements $U_{p,q} = w^{pq}$ for $j := \sqrt{-1}$, $w := e^{-j\frac{2\pi}{n}}$ and

$$W \quad := \quad \text{diag}\{1, w, w^2, \dots, w^{n-1}\}.$$

where $\text{diag}\{\cdot\}$ denotes a diagonal (or, possibly, block-diagonal) matrix with its entries listed within the brackets. We are now ready to intoduce the following two spaces. Let $\mathcal{C}_n$ denote the space of $(n \times n)$-circulant matrices,

$$\mathcal{C}_n := \left\{ a(S) := \sum_{k=0}^{n-1} S^k a_k \mid a_k \in \mathbb{C} \right\},$$

and $\mathcal{C}_{n;m}$ the space of $(n \times n)$-circulant m-block matrices,

$$\mathcal{C}_{n;m} := \left\{ a(S) := \sum_{k=0}^{n-1} S^k \otimes a_k \mid a_k \in \mathbb{C}^{m \times m} \right\}.$$

where $\otimes$ denotes the Kronecker product. Moreover let $\mathcal{C}_n^+$ and $\mathcal{C}_{n;m}^+$ be the cones of Hermitian non-negative elements in the two spaces $\mathcal{C}_n$ and $\mathcal{C}_{n;m}$, respectively, i.e.

$$\mathcal{C}_n^+ := \{a(S) \geq 0 \mid a(S) \in \mathcal{C}_n\}$$
$$\mathcal{C}_{n;m}^+ := \{a(S) \geq 0 \mid a(S) \in \mathcal{C}_{n;m}\}.$$

The following facts about circulant/block-circulant matrices hold. We refer the reader to [3] for the proofs.

Proposition 1: The matrix $M \in \mathbb{C}^{nm \times nm}$ is in $\mathcal{C}_{n;m}$ if and only if

$$(S \otimes I_m) M (S^* \otimes I_m) = M. \tag{2}$$

Proposition 2: The matrix $a(S) \in \mathcal{C}_{n;m}$ is invertible if and only if the determinant of the polynomial-matrix

$$a(x) = \sum_{k=0}^{n-1} x^k a_k$$

does not vanish at the n-th roots of unity $\{w^k \mid k = 0, 1, \dots, n-1\}$. Moreover, if $a(S) \in \mathcal{C}_{n;m}$ is invertible, then its inverse is also in $\mathcal{C}_{n;m}$.

An immediate consequence of Proposition 2 is the following.

Corollary 3: A Hermitian matrix $a(S) \in \mathcal{C}_{n;m}$ is positive semidefinite if and only if the $m \times m$ matrices

$$a(e^{-j2\pi\ell/n}) = \sum_{k=0}^{n-1} e^{-j2\pi\ell k/n} a_k$$

are positive semidefinite for $\ell = 0, \dots, n-1$.

Finally, we introduce a useful characterization of circulant Hermitian matrices. Let $\mathcal{M}_n$ denote the set of all $n \times n$ matrices over $\mathbb{C}$ and $\mathcal{H}_n \subset \mathcal{M}_n$ the subset of all Hermitian matrices. Consider the pair $\mathcal{G} := (\{\text{conj}, \text{shift}\}, \circ)$ where conj and shift are the maps

$$\begin{aligned} \text{conj} : \mathcal{M}_n &\longrightarrow \mathcal{M}_n \\ M &\longmapsto M^* \\ \text{shift} : \mathcal{M}_n &\longrightarrow \mathcal{M}_n \\ M &\longmapsto (S \otimes I_m) M (S^* \otimes I_m) \end{aligned}$$

and $\circ$ denote the usual composition of maps. It is easy to check that the pair $\mathcal{G}$ is a commutative group and the following characterization of circulant Hermitian matrices holds.

Proposition 4: Let $M \in \mathcal{M}_{mn}$. Then $M \in \mathcal{C}_{n;m} \cap \mathcal{H}_{mn}$ if and only if the orbit of M under the action of $\mathcal{G}$ is M itself.

That is, $M \in \mathcal{C}_{n;m} \cap \mathcal{H}_{mn}$ if and only if M stay invariant under the action of the group $\mathcal{G}$ and the orbit contains no additional elements.

III. The covariance completion problem

Let $M \in \mathcal{H}_n$ be a partially specified matrix and consider the problem

$$\max \left\{ \det(M) \mid M \in \mathcal{H}_n^+ \right\} \tag{3a}$$

subject to

$$e_k M e_\ell^* = m_{k,\ell} \text{ for } (k,\ell) \in \mathcal{S} \text{ and } m_{k,\ell} \in \mathcal{M}, \tag{3b}$$

where $\mathcal{H}_n^+ \subset \mathcal{H}_n$ denotes the cone of positive-definite $n \times n$ matrices, e_k is the row-vector with a 1 at the kth entry

$$e_k := [\overbrace{0, \dots 0, 1}^{k}, 0, \dots 0],$$

$\mathcal{S}$ is a symmetric selection of pairs of indices (i.e., if $(k,\ell) \in \mathcal{S}$, then $(\ell, k) \in \mathcal{S}$), and the data

$$\mathcal{M} := \{m_{k,\ell} \mid (k,\ell) \in \mathcal{S}\}$$

is consistent with the hypothesis that M is Hermitian, i.e., $m_{k,\ell} = m_{\ell,k}^*$ for all entries in $\mathcal{M}$. Clearly as long as a positive-definite completion for M is at all possible, and as long as the set of such positive completions is bounded, a completion with maximal determinant is uniquely determined since the determinant is a strictly log-concave function of its argument. Let us denote by M_{me} this unique maximizer, i.e.

$$M_{\rm me} := \operatorname{argmax}\{\det(M) \mid M \in \mathcal{H}_n^+ \text{ satisfies } (3b)\}.$$

For ease of notation, we also introduce the subspace

$$\mathcal{L}_n := \{M \in \mathcal{H}_n \mid (3b) \text{ holds}\}$$

which contains matrices with the specified elements.

Theorem 5 (Dempster): Consider an index set $\mathcal{S}$ and a corresponding data set $\mathcal{M}$ consistent with the hypothesis $M \in \mathcal{H}_n$. Assume that $\mathcal{H}_n^+ \cap \mathcal{L}_n \neq \emptyset$ and bounded. Then

$$e_k M_{\rm me}^{-1} e_\ell^* = 0 \text{ for } (k,\ell) \notin \mathcal{S}.$$

Proof: By hypothesis M_{me} exists and it is uniquely defined. Moreover, if M_{me} maximizes $\det M$, it also maximizes $\log\det M$. Computing the Lagrangian

$$\mathbb{L}(M, \lambda_{k,\ell}) := \log\det(M) + \sum_{(k,\ell)\in\mathcal{S}} \lambda_{k,\ell}(m_{k,\ell} - e_k M e_\ell^*) \tag{4}$$

and setting the derivative of $\mathbb{L}$ with respect to the entries of M equal to zero, we readily obtain

$$M^{-1} = \sum_{(k,\ell)\in\mathcal{S}} \lambda_{k,\ell} e_k^* e_\ell. \tag{5}$$

Therefore, the inverse of the maximizer has the claimed zero-pattern, i.e., the Dempster property. ■

The above result goes back to Dempster [6]. Conditions on partially-specified matrices to have a positive-definite completion (i.e., conditions for $\mathcal{H}_n^+ \cap \mathcal{L}_n \neq \emptyset$) are addressed in [12] and analogous questions for multivariable moment problems are being discussed in [8].

Remark 1: When additional restrictions are placed on M then, in general, this property of $M_{\rm me}$ no longer holds. Suppose, for instance, that M, besides satisfying (3b), is required to have a Toeplitz structure. In this case, an additional set of Lagrange multipliers $(k,\ell) \notin \mathcal{S}$ is needed to enforce the Toeplitz structure via terms of the form $\lambda_{k,\ell}(e_k^* M e_\ell - e_{k+1}^* M e_{\ell+1})$. As a consequence, the statement of Theorem 5 fails in such cases.

A noticeable exception is when the additional constraint on M requires this to be circulant. In such a case, in fact, M_{me}^{-1} still has the zero-pattern of Proposition 4, as shown by a direct algebraic verification in Carli etal. [2]. Theorem 6 below gives an independent proof and, at the same time, explains how this generalizes to appropriate sets of interpolation conditions on the circulant structure.

Let $M \in \mathcal{C}_{n;m}$ be a partially specified block-matrix and consider again Problem (3a)-(3b) where now the index set $\mathcal{S}$ and the data-set $\mathcal{M}$ are chosen to be *consistent with the $\mathcal{C}_{n;m}$-circulant Hermitian structure*, i.e. $\mathcal{S}$ is such that

$$(k,\ell) \in \mathcal{S} \Rightarrow (\ell,k) \in \mathcal{S}, \tag{6a}$$

$$(k,\ell) \in \mathcal{S} \Rightarrow ((\ell+m)_{\operatorname{mod} nm}, (k+m)_{\operatorname{mod} nm}) \in \mathcal{S}. \tag{6b}$$

and the values $m_{k,\ell}$ in $\mathcal{M}$ satisfy

$$m_{k,\ell} = m_{\ell,k}^* \tag{7a}$$

$$m_{k,\ell} = m^*_{(\ell+m)_{\operatorname{mod} nm}, (k+m)_{\operatorname{mod} nm}} \tag{7b}$$

for all pairs of indices.

Theorem 6: Let $\mathcal{S}, \mathcal{M}$ be sets of indices and corresponding values consistent with the $\mathcal{C}_{n;m}$-circulant structure and assume that there exists a positive completion, i.e., that $\mathcal{H}_{nm}^+ \cap \mathcal{L}_{nm} \neq \emptyset$, and that this set is bounded. Then

i) there is positive completion in $\mathcal{C}_{n;m}$, i.e., $\mathcal{C}_{n;m} \cap \mathcal{H}_{nm}^+ \cap \mathcal{L}_{nm} \neq \emptyset$,

ii) the (*maximum entropy*) completion $M_{\rm me}$ over $\mathcal{H}_n^+$ is an element in $\mathcal{C}_{n;m}$,

iii) $e_k M_{\rm me}^{-1} e_\ell^* = 0$ for $(k,\ell) \notin \mathcal{S}$.

Clearly ii) implies i) as well as, by Theorem 5, iii). Thus the only thing to be proven is that M_{me} is indeed circulant.

Proof: Both the objective function and the constraints (3b) are invariant under the group $\mathcal{G}$ generated by $\{\text{conj}, \text{shift}\}$. Thus also the maximizer has to be invariant under the group $\mathcal{G}$, for otherwise, there would be multiple maxima. By Corollary 4, this implies that $M_{\rm me}$ is in $\mathcal{C}_{n;m}$, which concludes the proof. ■

Remark 2: The above argument applies to maximizers that may be restricted further to lie in a convex set in a way that is consistent with the circulant structure. Thus, the essence of this result is a rather general invariance principle that the maximizer of a concave functional when restricted to points that individually remain invariant under the action of a certain group, it is identical to the unconstrained one – assuming that the domain of the functional is convex and invariant under the group.

IV. Structure of solutions and factorization of polynomials in several variables

In this last section we provide insight into the shape of the set of all positive-definite block-circulant completions of a partially specified covariance matrix. An immediate consequence of Corollary 3 is the following.

Corollary 7: Let M be a partially specified $n \times n$ m-block circulant matrix. The set of all positive-definite block-circulant completions of M is delineated by the intersection of the m-order surfaces defined by the non-negativity of the matrices $a(e^{-j2\pi\ell/n})$, for $\ell = 0, 1, \ldots, n-1$.

Let us consider the following clarifying examples.

Example 1: Let $a(S) = \sum_{k=0}^{3} S^k \otimes a_k \in \mathcal{C}_{4;2}$, with

$$a_0 = \begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{bmatrix}, \; a_1 = a_3^\top = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

while the block

$$a_2 = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$$

is left completely unspecified. The maximum entropy solution, computed using a general semi-definite programming solver (e.g., `SDPT`, `SeDuMi`, etc.) by means of the interface in cvx [9], [10], results to have $x = z = 0.4853$,

$y = 0.4789$ and its inverse, say $\lambda(S)$, is given by

$$\lambda_0 = \begin{bmatrix} 1.1707 & -0.0163 \\ -0.0163 & 1.1707 \end{bmatrix}, \tag{8a}$$

$$\lambda_1 = \begin{bmatrix} -0.4469 & -0.4394 \\ 0.3335 & -0.4469 \end{bmatrix} = \lambda_3^\top, \tag{8b}$$

$$\lambda_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{8c}$$

i.e. λ_2 is the 2×2 zero matrix, as claimed.

In order to characterize the set of all positive-definite solutions, we consider the 2×2 polynomial matrices

$$a(w^0) = \begin{bmatrix} 4+x & \frac{3}{2}+y \\ \frac{3}{2}+y & 4+z \end{bmatrix},$$
$$a(w^1) = \begin{bmatrix} 2-x & \left(\frac{1}{2}-i\right)-y \\ \left(\frac{1}{2}+i\right)-y & 2-z \end{bmatrix} = a(w^3)^\top,$$
$$a(w^2) = \begin{bmatrix} x & -\frac{1}{2}+y \\ -\frac{1}{2}+y & z \end{bmatrix}, \tag{9}$$

whose eigenvalues are

$$\text{eig}\left\{a(w^0)\right\} = 4 + \frac{x}{2} + \frac{z}{2} \pm \sqrt{\frac{9+(x-z)^2+4y(3+y)}{4}},$$
$$\text{eig}\left\{a(w^1)\right\} = 2 - \frac{x}{2} - \frac{z}{2} \pm \sqrt{\frac{5+(x-z)^2-4y(1-y)}{4}},$$
$$\text{eig}\left\{a(w^2)\right\} = \frac{x}{2} + \frac{z}{2} \pm \sqrt{\frac{1+(x-z)^2-4y(1-y)}{4}}.$$

These are positive on the feasible set shown in Figure 1.

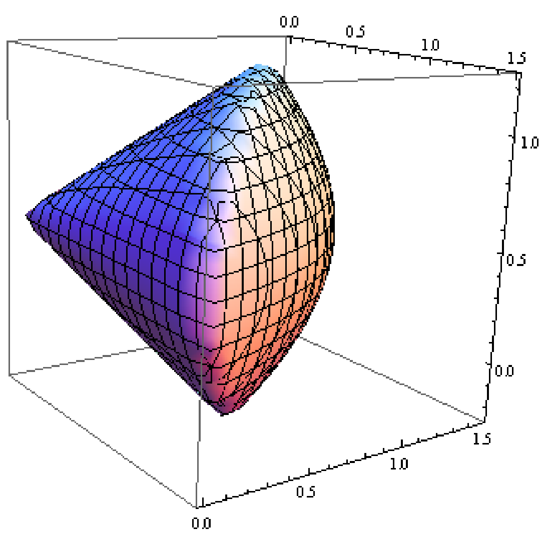

Fig. 1. Feasible set $\{(x, y, z) \mid a(S) \geq 0\}$.

Example 2: Let now $m = 1$ and $a(S) = \sum_{k=0}^{6} a_k S^k \in \mathcal{C}_7$, with $a_0 = 2$, $a_1 = a_6 = 1$, while the coefficients $a_2 = a_5 = x$ and $a_3 = a_4 = y$ are left unspecified. The eigenvalues of $a(S)$ are

$$a(w^0) = 2(2+x+y) \tag{10a}$$
$$a(w^1) = 2 - 2y\cos\frac{\pi}{7} - 2x\sin\frac{\pi}{14} + 2\sin\frac{3\pi}{14} \tag{10b}$$
$$a(w^2) = -2\left(-1 + x\cos\frac{\pi}{7} + \sin\frac{\pi}{14} - y\sin\frac{3\pi}{14}\right) \tag{10c}$$
$$a(w^3) = -2\left(-1 + \cos\frac{\pi}{7} + y\sin\frac{\pi}{14} - x\sin\frac{3\pi}{14}\right) \tag{10d}$$

and its determinant

$$\begin{aligned}\det(a(S)) = {} & 4 + 42x + 56x^2 - 294x^3 + 140x^4 + 84x^5 \\ & - 28x^6 + 2x^7 - 14y - 28xy + 350x^2y \\ & - 196x^3y - 112x^4y - 84x^5y + 14x^6y \\ & - 168xy^2 + 56x^2y^2 + 238x^3y^2 + 112x^4y^2 \\ & + 14x^5y^2 + 28y^3 - 238x^2y^3 - 28x^3y^3 \\ & - 42x^4y^3 + 98xy^4 - 14y^5 + 28x^2y^5 \\ & - 14xy^6 + 2y^7. \end{aligned} \tag{11}$$

Thus, the determinant is a polynomial of degree 7 in x and y. Factorization of the determinant over the ring of polynomials with rational coefficients (.e.g., using Matlab or Mathematica) gives

$$\begin{aligned}\det a(S) = {} & 2(2+x+y)\Big(1 + 5x - 8x^2 + x^3 - 2y \\ & + 5xy + 3x^2y - y^2 - 4xy^2 + y^3\Big)^2\end{aligned}$$

However, in view of (10a-10d), we already know that

$$\begin{aligned}\det a(S) = {} & 2(2+x+y) \\ & \left[2 - 2y\cos\frac{\pi}{7} - 2x\sin\frac{\pi}{14} + 2\sin\frac{3\pi}{14}\right]^2 \\ & \left[-2\left(-1 + x\cos\frac{\pi}{7} + \sin\frac{\pi}{14} - y\sin\frac{3\pi}{14}\right)\right]^2 \\ & \left[-2\left(-1 + \cos\frac{\pi}{7} + y\sin\frac{\pi}{14} - x\sin\frac{3\pi}{14}\right)\right]^2. \end{aligned}$$

I.e. $\det a(S)$ readily factors, whereas this is impossible using standard methods [1] without the prior knowledge of a suitable field extension of $\mathbb{Q}$ containing the coefficients of the factors (such as, $\mathbb{Q}[\cos\left(\frac{\pi}{7}\right), \sin\left(\frac{\pi}{14}\right), \text{etc.}]$). Finding such an extension, in general, is a challenging problem. Of course, on the other hand, expressing a given rational polynomial as the determinant of a circulant matrix with rational coefficients may, in general, be an equally challenging one. The point of the example is not to suggest a general procedure, unless the multivariable polynomial is easily seen to be, or originates as, the determinant of a circulant matrix.

V. Concluding Remarks

The main contribution in this paper is the proof and insight that has been gained into the problem of completion of circulant covariance matrices by Theorem 6 and Remark 2. Although such matrices have been widely used in the signal processing literature [5], [11], the case for completion problems has only been brought forth in [2]. The present work builds on [2] and exposes the finer structure of the feasible set of such completions in a general setting (see Remark 2).

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