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Dissipativity of Pseudorational Behaviors

Masaki Ogura and Yutaka Yamamoto
Department of Applied Analysis and Complex Dynamical Systems,
Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan
msk.ogura@gmail.com, yy@i.kyoto-u.ac.jp

Abstract— **This paper studies dissipativity for a class of infinite-dimensional systems, called pseudorational, in the behavioral context. First a basic equivalence condition for average nonnegativity of quadratic differential forms induced by distributions is established as a generalization of the finite-dimensional counterpart. For its proof, we derive a new necessary and sufficient condition for entire functions of exponential type (in the Paley-Wiener class) to be symmetrically factorizable. Utilizing these results we then study dissipativity of pseudorational behaviors. An example is given to illustrate results.**

I. Introduction

The notion of dissipativity [9], [10] is one of the most important property in system theory. It can be viewed as a natural generalization of Lyapunov stability to open systems and most of robust stability conditions can be deduced from this property [11].

It is well known that quadratic differential forms induced by two-variable polynomial matrices [12] play an important role in describing dissipativity for linear time-invariant finite-dimensional systems. For example, the theory of analysis and synthesis of dissipative systems are developed in [7], [13] using the quadratic differential forms. Small gain theorems or the celebrated Popov criterion can be deduced using such forms [11].

Extending the existing quadratic differential forms, the latter author and Willems [15] have introduced new quadratic differential forms induced by *distributions* having compact support. The quadratic differential forms can deal with, for example, time delays which cannot be expressed by the existing quadratic differential forms. Based on the new quadratic differential forms, they have also studied Lyapunov stability of a class of behavior, called *pseudorational* [14].

The aim of this paper is to study dissipativity of pseudorational behaviors with respect to the quadratic differential forms induced by distributions. First, as an extension of the finite-dimensional counterpart [12, Proposition 5.4], we establish a basic characterization of *average nonnegativity* for the quadratic differential forms in terms of storage or dissipation functions. For its proof, we derive a new necessary and sufficient condition for entire functions of exponential type (in the Paley-Wiener class) to be symmetrically factorizable. Utilizing these results we then study dissipativity of pseudorational behaviors. An example is given to illustrate results.

A. Notation and Convention

$\mathbb{R}$ and $\mathbb{C}$ denote real and complex fields, respectively. Let $\mathbb{C}_+ := \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$ and $\mathbb{C}_- := \{s \in \mathbb{C} : \operatorname{Re} s < 0\}$. When X is a vector space, X^n and $X^{n\times m}$ denote, respectively, the space of n products of X and that of $n \times m$ matrices with entries in X. When a specific dimension is immaterial, we will simply write $X^\bullet$, $X^{n\times\bullet}$, $X^{\bullet\times m}$, and so forth.

$\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ (often abbreviated as $(\mathscr{C}^\infty)^q$) denotes the space of $\mathbb{R}^q$-valued C^∞ functions on $\mathbb{R}$. The space of those functions with compact support is denoted by $\mathscr{D}(\mathbb{R}, \mathbb{R}^q)$ ($\mathscr{D}^q$ for short). By $\mathscr{E}'(\mathbb{R})$ we denote the space of distributions having compact support in $\mathbb{R}$. $\mathscr{E}'(\mathbb{R})$ is a convolution algebra and every $p \in \mathscr{E}'(\mathbb{R})$ acts on $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$ by the action $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}) \to \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}) : w \mapsto p * w$. The image and kernel of the mapping are denoted by $\operatorname{im} p$ and $\ker p$, respectively. For $\tau \in \mathbb{R}$, δ_τ denotes the Dirac's delta placed on τ. The subscript τ is omitted when $\tau = 0$. Finally $\mathscr{E}'(\mathbb{R}^2)$ denotes the space of distributions in two variables having compact support in $\mathbb{R}^2$.

The Laplace transform of $p \in \mathscr{E}'(\mathbb{R})$ is defined by

$$\mathscr{L}[p](\zeta) = \hat{p}(\zeta) := \langle p, e^{-\zeta t}\rangle_t$$

where the distribution action is taken with respect to t. Similarly, for $p \in \mathscr{E}'(\mathbb{R}^2)$, its Laplace transform is defined by

$$\hat{p}(\zeta, \eta) := \langle p, e^{-(\zeta s+\eta t)}\rangle_{s,t}$$

where the action is taken with respect to two variables s and t. For example, $\mathscr{L}[\delta'_s \otimes \delta''_t] = \zeta \cdot \eta^2$.

By the well-known Paley-Wiener theorem [5], a distribution p belongs to $\mathscr{E}'(\mathbb{R})$ if and only if its Laplace transform $\hat{p}$ is an entire function of exponential type satisfying the *Paley-Wiener estimate*

$$|\hat{p}(\xi)| \le C(1 + |\xi|)^m e^{a|\operatorname{Re}\xi|} \tag{1}$$

for some $C \ge 0$, $a \ge 0$, and a nonnegative integer m. We denote by $\mathscr{PW}$ the class of entire functions satisfying the estimate above. In other words, $\mathscr{PW} = \mathscr{L}[\mathscr{E}'(\mathbb{R})]$.

Likewise, for $p \in \mathscr{E}'(\mathbb{R}^2)$, there exist $C \ge 0$, $a \ge 0$, and a nonnegative integer m such that its Laplace transform satisfies

$$|\hat{p}(\zeta, \eta)| \le C(1 + |\zeta| + |\eta|)^m e^{a(|\operatorname{Re}\zeta|+|\operatorname{Re}\eta|)}. \tag{2}$$

This is also a sufficient condition for a function $\hat{p}(\cdot,\cdot)$ to be the Laplace transform of a distribution in $\mathscr{E}'(\mathbb{R}^2)$.

Let $\Phi \in \mathscr{E}'(\mathbb{R}^2)^{n\times m}$. In the Laplace transform domain, define $\Phi^* \in \mathscr{E}'(\mathbb{R}^2)^{m\times n}$ by $(\Phi^*)\hat{}(\zeta, \eta) := \hat{\Phi}(\eta, \zeta)^\top$ [15]. Φ

is said to be symmetric if $\Phi = \Phi^*$. The quadratic differential form [15] induced by symmetric Φ is denoted by Q_Φ.

Let F be a $\mathbb{C}^{\bullet\times\bullet}$-valued function defined on a subset of $\mathbb{C}$. Its para-Hermitian conjugate $F^{\sim}$ is given by $F^{\sim}(\xi) := \overline{F(-\bar{\xi})}^{\top}$. F is said to be entire if its each entry is entire. If F is entire, it is said to be of exponential type if its each entry is of exponential type.

For $x \geq 0$ let $\log^+(x) := \max\{0, \log x\}$. For real-valued functions f and g defined on $\mathbb{R}$, we write $f \geq g$ ($f \leq g$) if, for all $t \in \mathbb{R}$, $f(t) \geq g(t)$ ($f(t) \leq g(t)$, respectively). Let A be a complex matrix. Its maximal singular value is denoted by $\|A\|$. When A is nonnegative (positive) definite we write $A \geq 0$ ($A > 0$, respectively).

II. Characterization of Average Nonnegativity

In this section, following [12], we study average nonnegativity of the quadratic differential forms defined by distributions with compact support [15]. First we introduce the notion of average nonnegativity:

Definition 2.1: Let $\Phi \in \mathscr{E}'(\mathbb{R}^2)^{q\times q}$ be symmetric. The quadratic differential form Q_Φ is said to be *average nonnegative* if

$$\int_{-\infty}^{\infty} Q_\Phi(w)\, dt \geq 0 \tag{3}$$

for all $w \in \mathscr{D}^q$.

Then we define storage functions and dissipation functions as follows:

Definition 2.2: Let Φ, Ψ, and Δ belong to $\mathscr{E}'(\mathbb{R}^2)^{q\times q}$ and be symmetric.

- Q_Ψ is said to be a *storage function* for Q_Φ if
$$\frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w) \tag{4}$$
for all $w \in (\mathscr{C}^\infty)^q$
- Q_Δ is said to be a *dissipation function* for Q_Φ if
$$Q_\Delta(w) \geq 0 \tag{5}$$
for all $w \in (\mathscr{C}^\infty)^q$ and
$$\int_{-\infty}^{\infty} Q_\Phi(w)\, dt = \int_{-\infty}^{\infty} Q_\Delta(w)\, dt \tag{6}$$
for all $w \in \mathscr{D}^q$.

The purpose of this section is to show a basic equivalence condition for average nonnegativity, as a generalization of the finite-dimensional counterpart [12, Proposition 5.4]:

Theorem 2.3: Let $\Phi \in \mathscr{E}'(\mathbb{R}^2)^{q\times q}$ be symmetric. The following conditions are equivalent:

(i) Q_Φ is average nonnegative;
(ii) $\hat{\Phi}(-j\omega, j\omega) \geq 0$ for all $\omega \in \mathbb{R}$;
(iii) Q_Φ admits a storage function;
(iv) Q_Φ admits a dissipation function.

In this theorem, the proofs of **(iv)** $\Rightarrow$ **(iii)**, **(iii)** $\Rightarrow$ **(i)**, and **(i)** $\Rightarrow$ **(ii)** can be done in the same way as in the finite-dimensional case [12]. However, to show the implication **(ii)** $\Rightarrow$ **(iv)**, we need a special type of factorization of the matrix-valued function $\hat{\Phi}(-\xi, \xi)$, called *symmetric factorization*.

It is well-known [2] that, when $\hat{\Phi}(-\xi, \xi)$ is a polynomial, the condition **(ii)** ensures the existence of such a factorization. However, in Theorem 2.3, $\hat{\Phi}(-\xi, \xi)$ is not a polynomial but an *entire function*. In the next section we derive a new necessary and sufficient condition for matrix-valued entire functions having entries in $\mathscr{PW}$ to admit a symmetric factorization.

III. Symmetric Factorization over $\mathscr{PW}$

We can naturally introduce the symmetric factorization over $\mathscr{PW}^{q\times q}$ as follows:

Definition 3.1: Let $\Gamma \in \mathscr{PW}^{q\times q}$. We say that $F \in \mathscr{PW}^{q\times q}$ induces a *symmetric factorization* of Γ if

$$\Gamma(\xi) = F^{\sim}(\xi) F(\xi). \tag{7}$$

The aim of this section is to prove the next theorem:

Theorem 3.2: Let $\Gamma \in \mathscr{PW}^{q\times q}$. Γ allows a symmetric factorization if and only if

$$\Gamma(j\omega) \geq 0,\ \forall \omega \in \mathbb{R}. \tag{8}$$

The necessity is trivial in this theorem. For the sufficiency, we begin by quoting a basic result from the factorization theory of operator valued entire functions [4]:

Proposition 3.3 ([4, Theorem 3.6]): Let Γ be a $\mathbb{C}^{q\times q}$-valued entire function of exponential type. Suppose that (8) holds and the integral

$$\int_{-\infty}^{\infty} \frac{\log^+ \|\Gamma(j\omega)\|}{1+\omega^2}\, d\omega \tag{9}$$

is finite. Then there exists a $\mathbb{C}^{q\times q}$-valued entire function F of exponential type such that (7) holds and the determinant of F has no zeros in $\mathbb{C}_+$.

To make use of this proposition in proving Theorem 3.2, we additionally need to show that

(i) the integral (9) always exists for every $\Gamma \in \mathscr{PW}^{q\times q}$;
(ii) in Proposition 3.3, if $\Gamma \in \mathscr{PW}^{q\times q}$ then $F \in \mathscr{PW}^{q\times q}$.

First, using the Paley-Wiener estimate (1), we can show the existence of the integral (9) for every $\Gamma \in \mathscr{PW}^{q\times q}$:

Proposition 3.4: The integral (9) is finite if Γ belongs to $\mathscr{PW}^{q\times q}$.

Proof: Let Γ belong to $\mathscr{PW}^{q\times q}$. Then each entry of Γ satisfies the Paley-Wiener estimate (1). From this we can easily check that $\|\Gamma(\xi)\|$ as a function of ξ also satisfies the Paley-Wiener estimate; i.e., there exist $C > 0$, $a > 0$, and a nonnegative integer m such that

$$\|\Gamma(\xi)\| \leq C(1+|\xi|)^m e^{a|\operatorname{Re}\xi|}.$$

Substituting $j\omega$ into ξ we have, since $\operatorname{Re}(j\omega) = 0$,

$$\|\Gamma(j\omega)\| \leq C(1+|\omega|)^m,\ \forall \omega \in \mathbb{R}. \tag{10}$$

Therefore the integral (9) can be bounded from above by a constant as

$$\begin{aligned}(9) &\leq \int_{-\infty}^{\infty} \frac{\log^+ C}{1+\omega^2}\, d\omega + m \int_{-\infty}^{\infty} \frac{\log(1+|\omega|)}{1+\omega^2}\, d\omega \\ &\leq \pi \log^+ C + 3m.\end{aligned}$$

This completes the proof. ∎

We then show that, in Proposition 3.3, if Γ belongs to $\mathscr{PW}^{q\times q}$ then the function F also belongs to $\mathscr{PW}^{q\times q}$:

Proposition 3.5: Let Γ and F be $\mathbb{C}^{q\times q}$-valued entire functions of exponential type. Suppose that (7) holds. If Γ belongs to $\mathscr{PW}^{q\times q}$, then F also belongs to $\mathscr{PW}^{q\times q}$.

For the proof of this proposition we need the next lemma, which states that the growth rate of holomorphic functions of exponential type can be governed by those on the real and imaginary axises:

Lemma 3.6 ([1, Theorem 6.2.4]): Let f be a complex-valued function defined on a subset of $\mathbb{C}$. Assume that f satisfies the following conditions:

(i) f is holomorphic on $\overline{\mathbb{C}}_+$.

(ii) f is of exponential type on $\overline{\mathbb{C}}_+$; i.e., there exist $K \geq 0$ and $\tau \geq 0$ such that $|f(\xi)| \leq Ke^{\tau|\xi|}$ for all $\xi \in \overline{\mathbb{C}}_+$.

(iii) There exist $M > 0$ such that

$$|f(j\omega)| \leq M, \ \forall\omega \in \mathbb{R}.$$

Then $|f(\xi)| \leq Me^{\tau \operatorname{Re}\xi}$ for all $\xi \in \overline{\mathbb{C}}_+$.

Using this lemma we prove the next lemma that enables us to judge whether or not a given entire function of exponential type belongs to $\mathscr{PW}$, from the growth rate of the function on the imaginary axis:

Lemma 3.7: Assume that an entire function f satisfies the following conditions:

(i) f is of exponential type;

(ii) There exist $M > 0$ and a nonnegative integer m such that

$$|f(j\omega)| \leq M(1 + |\omega|)^m, \ \forall\omega \in \mathbb{R}. \tag{11}$$

Then f belongs to $\mathscr{PW}$.

Proof: Let f be an entire function that satisfies the two conditions above. Define a meromorphic function f_0 by

$$f_0(\xi) := \frac{f(\xi)}{(\xi+1)^m}.$$

Since f is an entire function of exponential type, clearly f_0 is holomorphic and of exponential type on $\overline{\mathbb{C}}_+$. From (11) we can check that f_0 is bounded on the imaginary axis. Then by Lemma 3.6 there exist $M > 0$ and $\tau > 0$ such that $|f_0(\xi)| \leq Me^{\tau \operatorname{Re}\xi}$ for all $\xi \in \overline{\mathbb{C}}_+$. Therefore

$$|f(\xi)| = |f_0(\xi)| \cdot |(1+\xi)^m| < M(1 + |\xi|)^m e^{\tau|\operatorname{Re}\xi|}$$

for all $\xi \in \overline{\mathbb{C}}_+$. Note that this inequality is nothing but the Paley-Wiener estimate (1) on the closed right half plane. In the similar way, we can show the Paley-Wiener estimate on the closed left half plane using the left half plane version of Lemma 3.6. Combining these estimates we obtain a Paley-Wiener estimate of f on the entire complex plane. Hence f belongs to $\mathscr{PW}$. ■

Then we can prove Proposition 3.5:

Proof of Proposition 3.5: Let Γ and F be $\mathbb{C}^{q\times q}$-valued entire functions of exponential type satisfying (7). Suppose that Γ belongs to $\mathscr{PW}^{q\times q}$. Let f be any entry of F. We show that f belongs to $\mathscr{PW}$.

Since F is of exponential type, f is also of exponential type. Hence, by Lemma 3.7, it is sufficient to show that there exist $C > 0$ and a nonnegative integer m satisfying (11).

From the definition of maximal singular values there exists a constant $M > 0$ such that

$$|f(j\omega)| \leq M\|F(j\omega)\|, \ \forall\omega \in \mathbb{R}. \tag{12}$$

Since $\Gamma(j\omega) = F(j\omega)^* F(j\omega)$ by (7), we have

$$\|F(j\omega)\|^2 = \|\Gamma(j\omega)\|, \ \forall\omega \in \mathbb{R}. \tag{13}$$

From inequalities (12), (13), and (10), we can obtain the estimate of type (11) as

$$|f(j\omega)| \leq MC^{1/2}(1 + |\omega|)^{m/2}, \ \forall\omega \in \mathbb{R}.$$

This completes the proof. ■

We are now ready to prove Theorem 3.2:

Proof of Theorem 3.2: The necessity is obvious. We prove the sufficiency. Let $\Gamma \in \mathscr{PW}^{q\times q}$ satisfy the inequality (8). Because the integral (9) is finite by Proposition 3.4, Proposition 3.3 ensures the existence of a $\mathbb{C}^{q\times q}$-valued entire function F of exponential type that satisfies (7). By Proposition 3.5 this function F actually belongs to $\mathscr{PW}^{q\times q}$. ■

Before closing this section, we refer to a more special type of symmetric factorizations, called *symmetric (anti-)Hurwitz factorization*. These factorizations play a key role in, for example, examining the existence of positive storage functions for finite-dimensional systems [12].

Definition 3.8: Suppose that $F \in \mathscr{PW}^{q\times q}$ induces a symmetric factorization for an element in $\mathscr{PW}^{q\times q}$. The factorization is said to be a symmetric (anti-)Hurwitz factorization if all the zeros of the determinant of F belong to $\mathbb{C}_-$ ($\mathbb{C}_+$, respectively).

The next theorem extends the result given in [2]:

Theorem 3.9: Let $\Gamma \in \mathscr{PW}^{q\times q}$. Γ allows both a symmetric Hurwitz factorization and a symmetric anti-Hurwitz factorization if and only if

$$\Gamma(j\omega) > 0, \ \forall\omega \in \mathbb{R}.$$

Proof: The statement on the symmetric Hurwitz factorization is trivial because, in Proposition 3.3, the determinant of F already has no zeros in $\mathbb{C}_+$. A symmetric anti-Hurwitz factorization can be then obtained from a symmetric Hurwitz factorization of $\Gamma^\top$. ■

IV. Proof of the Main Result

Having established Theorem 3.2, we can proceed to the proof of the main result Theorem 2.3.

Proof of Theorem 2.3: Let $\Phi \in \mathscr{E}'(\mathbb{R}^2)$ be symmetric. We run the cycle (**iv**) ⇒ (**iii**) ⇒ (**i**) ⇒ (**ii**).

(**iv**) ⇒ (**iii**): Suppose that Q_Φ admits a dissipation function Q_Δ with symmetric $\Delta \in \mathscr{E}'(\mathbb{R}^2)^{q\times q}$. From the definition of the dissipation function (5) we have

$$\int_{-\infty}^{\infty} Q_{\Phi-\Delta}(w)\,dt = 0, \ \forall w \in \mathscr{D}^q.$$

Then, by [15, Theorem 6.2], there exists a symmetric $\Psi \in \mathscr{E}'(\mathbb{R}^2)^{q\times q}$ such that

$$\frac{d}{dt}Q_\Psi(w) = Q_{\Phi-\Delta}(w), \ \forall w \in (\mathscr{C}^\infty)^q.$$

Since $Q_\Delta(w) \geq 0$ by (6) we obtain $\frac{d}{dt}Q_\Psi(w) \leq Q_\Phi(w)$ for all $w \in (\mathscr{C}^\infty)^q$. Hence Q_Ψ is a storage function for Q_Φ.

(**iii**) $\Rightarrow$ (**i**): Let $\Psi = \Psi^* \in \mathscr{E}'(\mathbb{R}^2)^{q\times q}$ induce a storage function for Q_Φ. Then the integration of (4) for $w \in \mathscr{D}^q$ readily yields (3) and hence Q_Φ is average nonnegative.

(**i**) $\Rightarrow$ (**ii**): We show the contraposition. Suppose that there exists $\omega_0 \in \mathbb{R}$ such that $\hat{\Phi}(-j\omega_0, j\omega_0) \ngeq 0$. Then there exists $v \in \mathbb{C}^q$ satisfying

$$v^*\hat{\Phi}(-j\omega_0, j\omega_0)v < 0. \tag{14}$$

Take any $\rho \in \mathscr{D}$ such that

$$\hat{\rho}(0) \neq 0. \tag{15}$$

For a positive integer N, define $w_N \in \mathscr{D}$ by

$$w_N := \rho * \frac{(e^{j\omega_0 t})|_{[-N,N]}}{\sqrt{2N}} v.$$

Using Parseval's identity

$$\int_{-\infty}^{\infty} Q_\Phi(w)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{w}(-j\omega)^*\hat{\Phi}(-j\omega, j\omega)\hat{w}(j\omega)\, d\omega \tag{16}$$

that holds for every $w \in \mathscr{D}$, we can obtain

$$\int_{-\infty}^{\infty} Q_\Phi(w_N)\, dt = \int_{-\infty}^{\infty} f(\omega)\cdot N\pi^{-1}\operatorname{sinc}^2(N\omega)\, d\omega, \tag{17}$$

where $\operatorname{sinc}\omega := \sin(\omega)/\omega$ and

$$f(\omega) := v^*\hat{\rho}(j\omega)^*\hat{\Phi}\left(-j(\omega+\omega_0), j(\omega+\omega_0)\right)\hat{\rho}(j\omega)v.$$

We show that f belongs to the space $\mathscr{S}$ of testing functions of rapid descent. First $\hat{\rho}$ belongs to $\mathscr{S}$ because $\mathscr{S}$ is invariant under the Fourier transform [8] and ρ belongs to $\mathscr{S}$. Second, the growth rate of $\hat{\Phi}(-j\omega, j\omega)$ as a function of ω is at most that of polynomials because $\hat{\Phi}(\zeta,\eta)$ satisfies the Paley-Wiener estimate (2). Therefore f belongs to $\mathscr{S}$.

Because, in (17), $N\pi^{-1}\operatorname{sinc}^2(N\omega)$ converges to δ with respect to the topology of the dual of $\mathscr{S}$ as N goes to infinity [8], the integral in its right hand side converges to $f(0) = |\hat{\rho}(0)|^2 v^*\hat{\Phi}(-j\omega_0, j\omega_0)v$ that is negative by (14) and (15). Therefore there exists w_N such that $\int_{-\infty}^{\infty} Q_\Phi(w_N)\, dt < 0$ and hence Q_Φ is not average nonnegative.

(**ii**) $\Rightarrow$ (**iv**): By Theorem 3.2, there exists $F \in \mathscr{E}'(\mathbb{R}^2)^{q\times q}$ that induces the symmetric factorization

$$\hat{\Phi}(-\xi,\xi) = \hat{F}^{\sim}(\xi)\hat{F}(\xi).$$

Define $\Delta \in \mathscr{E}'(\mathbb{R})^{q\times q}$ by

$$\hat{\Delta}(\zeta,\eta) := \overline{\hat{F}(\bar{\zeta})}^{\top}\hat{F}(\eta). \tag{18}$$

Then we can see that Δ equals to the tensor product $\bar{F}_s^\top \otimes F_t$, where $\bar{F}$ is defined by the action $\langle \bar{F}, f\rangle := \overline{\langle F, \bar{f}\rangle}$ for $f \in \mathscr{D}$. Hence, for all $w \in (\mathscr{C}^\infty)^q$,

$$\begin{aligned} Q_\Delta(w) &= (w^\top * \bar{F}^\top)\cdot(F * w) \\ &= \overline{(F*w)}^{\top}\cdot(F*w) \\ &= \|(F*w)(\cdot)\|^2 \\ &\geq 0. \end{aligned} \tag{19}$$

Moreover the equality (6) holds from Parseval's identity (16) and the equation $\hat{\Delta}(-j\omega, j\omega) = \hat{\Phi}(-j\omega, j\omega)$ which can be derived by a straightforward calculation. Hence Q_Δ is a dissipation function for Q_Φ. ■

V. Dissipativity of Pseudorational Behaviors

In this section we discuss dissipativity of pseudorational behaviors [14] with respect to the quadratic differential forms induced by distributions. Let $R \in \mathscr{E}'(\mathbb{R})^{\bullet\times q}$ be pseudorational [14] and $\mathscr{B}$ be the behavior defined by R:

$$\mathscr{B} := \{w \in (\mathscr{C}^\infty)^q : R * w = 0\}.$$

Following [6], we introduce the notion of dissipativity for pseudorational behaviors as follows:

Definition 5.1: Let $\mathscr{B}$ be the behavior with pseudorational $R \in \mathscr{E}'(\mathbb{R})^{\bullet\times q}$ and let Φ, Ψ, and $\Delta \in \mathscr{E}'(\mathbb{R}^2)^{q\times q}$ be symmetric.

- The pair $(\mathscr{B}, Q_\Phi)$ is said to be *dissipative* if the inequality (3) holds for all $w \in \mathscr{B}$.
- Q_Ψ is said to be a *storage function* for $(\mathscr{B}, Q_\Phi)$ if (4) holds for all $w \in \mathscr{B}$
- Q_Δ is said to be a *dissipation function* for $(\mathscr{B}, Q_\Phi)$ if (5) holds for all $w \in \mathscr{B}$ and (6) holds for all compactly supported $w \in \mathscr{B}$.

Theorem 3.2 enables us to give a sufficient condition for a pseudorational behavior to be dissipative, to admit a storage function, and to admit a dissipation function:

Proposition 5.2: Let $\mathscr{B}$ be the behavior with pseudorational $R \in \mathscr{E}'(\mathbb{R})^{\bullet\times q}$ and $\Phi \in \mathscr{E}'(\mathbb{R}^2)^{q\times q}$ be symmetric. Suppose that there exists $X \in \mathscr{E}'(\mathbb{R})^{\bullet\times q}$ such that

$$\hat{\Phi}(-j\omega, j\omega) + \hat{X}(-j\omega)^\top\hat{R}(j\omega) + \hat{R}(-j\omega)^\top\hat{X}(j\omega) \geq 0 \tag{20}$$

for all $\omega \in \mathbb{R}$. Then $(\mathscr{B}, Q_\Phi)$ is dissipative, admits a storage function, and admits a dissipation function.

Proof: Note that, for the distribution $\Phi_0 \in \mathscr{E}'(\mathbb{R}^2)^{q\times q}$ defined by $\hat{\Phi}_0(\zeta,\eta) := \hat{\Phi}(\zeta,\eta) + \hat{X}(\zeta)^\top\hat{R}(\eta) + \hat{R}(\zeta)^\top\hat{X}(\eta)$, the inequality (20) is equivalent to

$$\hat{\Phi}_0(-j\omega, j\omega) \geq 0. \tag{21}$$

First we show that $(\mathscr{B}, Q_\Phi)$ is dissipative. Take any $w \in \mathscr{B}$. Parseval's identity and (21) yield $\int_{-\infty}^{\infty} Q_{\Phi_0}(w)\, dt \geq 0$. Since we can easily show

$$Q_{\Phi_0}(w) = Q_\Phi(w) \tag{22}$$

using the equation $R * w = 0$, $(\mathscr{B}, Q_\Phi)$ is dissipative.

Then we construct a dissipation function for $(\mathscr{B}, Q_\Phi)$. From Theorem 3.2, there exists $F \in \mathscr{E}'(\mathbb{R})^{q\times q}$ that induces the symmetric factorization $\hat{\Phi}_0(-\xi,\xi) = \hat{F}^{\sim}(\xi)\hat{F}(\xi)$. Define $\Delta \in \mathscr{E}'(\mathbb{R}^2)^{q\times q}$ by (18). Then (5) follows from (19) for all $w \in \mathscr{B}$ and (6) follows from Parseval's identity, (22), and the equation

$$\hat{\Phi}_0(-\xi,\xi) - \hat{\Delta}(-\xi,\xi) = 0 \tag{23}$$

for all compactly supported $w \in \mathscr{B}$. Therefore Q_Δ is a dissipation function for $(\mathscr{B}, Q_\Phi)$.

Finally, to give a storage function for $(\mathscr{B}, Q_\Phi)$, let

$$\hat{\Psi}(\zeta,\eta) := \frac{\hat{\Phi}_0(\zeta,\eta) - \hat{\Delta}(\zeta,\eta)}{\zeta+\eta}.$$

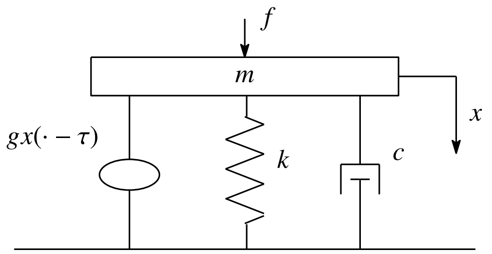

Fig. 1. Delayed resonator

Then Ψ, the inverse Laplace transform of $\hat{\Psi}$, belongs to $\mathscr{E}'(\mathbb{R}^2)^{q\times q}$ by (23) and [15, Lemma 3.3]. Take any $w \in \mathscr{B}$. Then, from [15, Lemma 3.4], we have

$$\frac{d}{dt}Q_\Psi(w) = Q_{\Phi_0-\Delta}(w) \geq Q_{\Phi_0}(w) = Q_\Phi(w)$$

and hence Q_Ψ is a storage function for $(\mathscr{B}, Q_\Phi)$. ∎

The next proposition gives a necessary and sufficient condition for dissipativity of behaviors $\mathscr{B}$ that admit an image representation

$$\operatorname{im} M := \{M * \ell : \ell \in (\mathscr{C}^\infty)^m\}$$

with $M \in \mathscr{E}'(\mathbb{R})^{q\times m}$.

Proposition 5.3: Let $\mathscr{B} = \operatorname{im} M$ with $M \in \mathscr{E}'(\mathbb{R})^{q\times\bullet}$ and $\Phi \in \mathscr{E}'(\mathbb{R}^2)^{q\times q}$ be symmetric. Then $(\mathscr{B}, Q_\Phi)$ is dissipative if and only if

$$\hat{M}(-j\omega)^\top \hat{\Phi}(-j\omega, j\omega)\hat{M}(j\omega) \geq 0 \tag{24}$$

for all $\omega \in \mathbb{R}$.

Proof: This is a direct consequence of Parseval's identity (16) and the image representation $\mathscr{B} = \operatorname{im} M$. ∎

A. Example

Let us consider the mechanical system depicted in Fig. 1. In this figure, $m > 0$ denotes the mass, $k > 0$ the spring constant, and $c > 0$ the damping coefficient. f is the force applied to the mass and x is the relative position of the mass from its equibrium. g and τ are nonnegative constants and $gx(\cdot - \tau)$ represents a delayed feedback. Such a feedback is used in, for example, delayed resonators [3].

The dynamics of the system can be written by the equation $m\ddot{x}(t) = f(t) - kx(t) - c\dot{x}(t) - gx(t-\tau)$ or, equivalently by $(m\delta'' + c\delta' + k\delta + g\delta_\tau) * x = \delta * f$. Therefore the set of all the trajectories taken by $w := [x \quad f]^\top$ is given by

$$\mathscr{B} = \ker\begin{bmatrix} m\delta'' + c\delta' + k\delta + g\delta_\tau & -\delta \end{bmatrix} =: \ker R.$$

Now let

$$\Phi := \frac{1}{2}\begin{bmatrix} 0 & \delta' \otimes \delta \\ \delta \otimes \delta' & 0 \end{bmatrix}.$$

The quadratic differential form $Q_\Phi(w) = f\dot{x}$ represents the mechanical energy supplied to the mass.

We examine dissipativity of $(\mathscr{B}, Q_\Phi)$ by Proposition 5.2. Let $X := [\delta \quad 0]^\top$. Then a straightforward computation gives that the left hand side of (20) equals

$$\begin{bmatrix} \omega(c\omega - g\sin(\tau\omega)) & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, from Proposition 5.2, a sufficient condition for $(\mathscr{B}, Q_\Phi)$ to be dissipative is

$$\omega\,(c\omega - g\sin(\tau\omega)) \geq 0, \tag{25}$$

which can be shown to be equivalent to the inequality

$$c \geq g\tau. \tag{26}$$

Using Proposition 5.3, we can show that this sufficient condition (26) is also necessary. Note that the behavior $\mathscr{B}$ admits the image representation

$$\operatorname{im}\begin{bmatrix} \delta & m\delta'' + c\delta' + k\delta + g\delta_\tau \end{bmatrix} =: \operatorname{im} M.$$

Then we can easily check that the left hand side of (24) is equal to that of (25). Hence, from Proposition 5.3, the sufficient condition (26) is also necessary.

VI. Conclusion

Dissipativity of pseudorational behaviors is studied. We have established a basic equivalence condition for average nonnegativity of the quadratic differential forms induced by distributions, as a generalization of the finite-dimensional counterpart. For its proof, we have derived a new necessary and sufficient condition for entire functions in the Paley-Wiener class to be symmetrically factorizable. We also has given some conditions for pseudorational behaviors to be dissipative. An example was given to illustrate the results.

References

[1] R. P. Boas Jr., *Entire Functions*. Academic Press, 1954.
[2] F. M. Callier, "On polynomial spectral factorization by symmetric factor extraction," *IEEE Trans. Automat. Control*, vol. 30, pp. 453–464, 1985.
[3] N. Olgac and B. T. Holm-Hansen, "A novel active vibration absorption technique: delayed resonator," *J. Sound and Vibration*, vol. 176, pp. 93–104, 1994.
[4] M. Rosenblum and J. Rovnyak, "The factorization problem for non-negative operator valued functions," *Bull. Amer. Math. Soc.*, vol. 77, pp. 287–318, 1971.
[5] L. Schwartz, *Théorie des Distributions*. Hermann, 1966.
[6] H. L. Trentelman and J. C. Willems, "Every storage function is a state function," *Syst. Control Lett.*, vol. 32, pp. 249–259, 1997.
[7] ——, "Synthesis of dissipative systems using quadratic differential forms: Part II," *IEEE Trans. Automat. Control*, vol. 47, pp. 70–86, 2002.
[8] F. Trèves, *Topological Vector Spaces, Distributions and Kernels*. Academic Press, 1967.
[9] J. C. Willems, "Dissipative dynamical systems - Part I: General theory," *Archive for Rational Mechanics and Analysis*, vol. 45, pp. 321–351, 1972.
[10] ——, "Dissipative dynamical systems - Part II: Linear systems with quadratic supply rates," *Archive for Rational Mechanics and Analysis*, vol. 45, pp. 352–393, 1972.
[11] ——, "Path integrals and stability," in *Mathematical Control Theory, Festschrift on the occasion of 60th birthday of Roger Brockett*, J. Baillieul and J. C. Willems, Eds. Springer Verlag, 1999, pp. 1–32.
[12] J. C. Willems and H. L. Trentelman, "On quadratic differential forms," *SIAM J. Control & Optimiz.*, vol. 36, no. 5, pp. 1703–1749, 1998.
[13] ——, "Synthesis of dissipative systems using quadratic differential forms: Part I," *IEEE Trans. Automat. Control*, vol. 47, pp. 53–69, 2002.
[14] Y. Yamamoto and J. C. Willems, "Behavioral controllability and coprimeness for a class of infinite-dimensional systems," in *47th IEEE CDC*, 2008.
[15] ——, "Path integrals and Bézoutians for pseudorational transfer functions," in *48th IEEE CDC*, 2009, pp. 8113–8118.