Explicit Parameterization of All Solutions of Linear Periodic Systems with Real-Valued Coefficients

Ichiro Jikuya and Ichijo Hodaka

Abstract— **An extension is introduced to the recently introduced representation for linear periodic systems with real-valued coefficients. In order to parameterize all state transition matrices, a state transition matrix is factored as a multiplication of a T-periodic real-valued factor and two real-valued matrix exponential functions. By solving the matrix equations which give an implicit parameterization, we study the block structure of those factors and propose an explicit parameterization. We also study the corresponding block structure in the coefficient matrices of the systems.**

NOTATIONS

$\mathbb{R}$	the set of all real numbers
$\mathbb{C}$	the set of all complex numbers
$\mathbb{R}^{n\times m}$	the set of all real matrices with n rows and m columns
$\mathbb{C}^{n\times m}$	the set of all complex matrices with n rows and m columns
0	the zero number or the zero matrix
I	the identity matrix
$\det X$	the determinant of a matrix X
X^{-1}	the inverse of a matrix X
e^X	the matrix exponential of a matrix X
$:=$	$X := Y$ denotes that X is defined by Y

The set of all C^k-functions, i.e., k-times continuously differentiable functions, from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $C^k(\mathcal{X},\mathcal{Y})$. If the function $P(t)$ is periodic with a period $T > 0$, i.e., $P(t+T) = P(t)$ for all $t \in \mathbb{R}$, it is called T-periodic. The set of all T-periodic functions in $C^k(\mathbb{R},\mathcal{Y})$ is denoted by $C^k_T(\mathbb{R},\mathcal{Y})$. If the square matrix-valued function $P(t)$ is invertible for all $t \in \mathbb{R}$, it is simply called invertible. The set of all invertible functions in $C^k(\mathbb{R},\mathcal{Y})$ (or $C^k_T(\mathbb{R},\mathcal{Y})$) is denoted by $C^k_{\rm inv}(\mathbb{R},\mathcal{Y})$ (or $C^k_{T,{\rm inv}}(\mathbb{R},\mathcal{Y})$), where $\mathcal{Y} = \mathbb{R}^{n\times n}$ or $\mathcal{Y} = \mathbb{C}^{n\times n}$.

I. INTRODUCTION

Floquet theory plays a fundamental role in the analysis and control of linear periodic continuous-times systems. The theorem consists of two main parts: the Floquet representation theorem in [1] and the Lyapunov reducibility theorem in [2].

The Floquet representation theorem provides a representation of the state transition matrix Φ. It is typically written in a simple statement, "The state transition matrix is factored as the product of a periodic matrix function and a matrix exponential function.":

$$\Phi(t,0) = P(t)e^{Ft},$$

where Φ denotes a state transition matrix, P denotes a periodic factor and F denotes a coefficient of a matrix exponential factor. Although this representation gives a fundamental insight into the structure of the solutions of linear periodic systems, it gives a parametrization of the set of all state transition matrices only in the case of complex-valued coefficients. In practice, it is important to discuss real-valued coefficients. Yakubovich and Montagnier et al. independently obtained a parametrization in the case of real-valued coefficients (see [5] and [3]). However, the period of the periodic factor is not preserved in general. When we apply their result as the Floquet Lyapunov transformation, it is not directly applicable to designing a periodic controller with the same period of the given periodic system.

In order to avoid such a drawback, we have proposed a novel representation which can be written in a simple statement, "The state transition matrix is factored as the product of a periodic matrix function and two matrix exponential functions.":

$$\Phi(t,0) = P(t)e^{Gt}e^{Ft},$$

where Φ denotes a state transition matrix, P denotes a periodic factor, and G and F denote coefficients of matrix exponential factors in [6]. In this framework, we obtained a parametrization of the set of all state transition matrices to linear periodic systems with real-valued coefficients. We also derived a kind of standard form for the coefficients of linear periodic systems with real-valued systems; namely, any linear periodic system with real-valued coefficients can be transformed to linear periodic system of the form

$$\begin{aligned}
&\dot{\xi} = H(t)\xi,\\
&H(t) = H_1 + H_2\cos\left(\frac{2\pi t}{T}\right) + H_3\sin\left(\frac{2\pi t}{T}\right),\\
&H_1, H_2, H_3 \in \mathbb{R}^{n\times n},
\end{aligned}$$

where coefficients are sum of constants and trigonometric functions with fundamental frequency.

Now we can consider control problems utilizing the above standard form. However, the conditions in the previous parametrization are implicitly characterized and redundant. When we utilize the proposed representation for control problems, it would be effective to find an explicit parametrization of the set of all state transition matrices

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to linear periodic systems with real-valued coefficients. The purpose of this note is to give a solution to this problem.

The subsequent of this note is organized as follows: The Floquet representation theorem and the novel representation proposed by the authors are summarized in Section 2. The structure of the novel parametrization is intensively investigated in terms of explicit parametrization in Section 3. Its state-space counterpart is studied in Section 4. The conclusions are summarized in Section 5.

II. Background

We consider the linear periodic system

$$\dot{x} = A(t)x, \quad \dot{x} := \frac{dx}{dt} \tag{1}$$

where $t \in \mathbb{R}$ is a time, $x(t) \in C^1(\mathbb{R}, \mathbb{R}^n)$ is a state vector, $A(t) \in C^0_T(\mathbb{R}, \mathbb{R}^{n\times n})$ is a coefficient matrix. Let $\Phi(s,t) \in C^1(\mathbb{R}\times\mathbb{R}, \mathbb{R}^{n\times n})$ denotes the state transition matrix of (1), i.e., Φ is the unique solution of

$$\frac{\partial}{\partial s}\Phi(s,t) = A(s)\Phi(s,t) \tag{2}$$

$$\Phi(t,t) = I. \tag{3}$$

The Floquet representation theorem provides a representation of the state transition matrix Φ by

$$\Phi(t,0) = P(t)e^{Ft},$$

where $P(t)$ denotes a periodic factor and F denotes the coefficient of the matrix exponential factor. In general, this representation is applicable for complex-valued $A(t)$. When we focus on real-valued $A(t)$, the classes of $P(t)$ and F are classified as follows:

Theorem 1: (See e.g., [3]) Let Φ denotes the state transition matrix of (1). Then, (i) there exist $P_c(t) \in C^1_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{C}^{n\times n})$ and $F_c \in \mathbb{C}^{n\times n}$ satisfying

$$\Phi(t,0) = P_c(t)e^{F_c t}. \tag{4}$$

(ii) there exist $P_d(t) \in C^1_{2T,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ and $F_d \in \mathbb{R}^{n\times n}$ satisfying

$$\Phi(t,0) = P_d(t)e^{F_d t}, \tag{5}$$

(iii) there exist $P_r(t) \in C^1_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ and $F_r \in \mathbb{R}^{n\times n}$ satisfying

$$\Phi(t,0) = P_r(t)e^{F_r t} \tag{6}$$

if and only if $\Phi(T,0)$ has a real logarithm, i.e. there exists a real matrix $\tilde{F}_r \in \mathbb{R}^{n\times n}$ satisfying $\Phi(T,0) = e^{\tilde{F}_r}$. □

We note that the representation of the form (4), (5) or (6) does not give a parameterization of a set of all state transition matrices generated by T-periodic systems with real-valued coefficients. Namely, it is always possible to factor $\Phi(t,0)$ as (4) or (5); but, it is not always possible to factor $\Phi(t,0)$ as (6). On the contrary, $\Phi(t,0)$ defined by any $P_r(t)$ and F_r becomes a state transition matrix of a linear periodic system with T-periodic real-valued $A(t)$; but, $\Phi(t,0)$ defined by $P_c(t)$ and F_c (or $P_d(t)$ and F_d) might not be a state transition matrix of a linear periodic system with a T-periodic real-valued $A(t)$ Hence, the representation of the form (4), (5) or (6) does not give a parameterization of a set of all state transition matrices generated by T-periodic systems with real-valued coefficients.

Yakubovich obtained a parameterization of a set of all state transition matrices generated by T-periodic systems with real-valued coefficients [5]. Montagnier et al. also obtained another parameterization [3]. In both parameterizations, $\Phi(t,0)$ is factored in the framework of (5); therefore, the periodic factor becomes $2T$-periodic.

In practice, it is important to factorize $\Phi(t,0)$ as the product of a T-periodic real-valued matrix function and the other components. For example, when we apply the Floquet-Lyapunov transformation to a T-periodic real-valued system

$$\dot{x} = A(t)x + B(t)u,$$

it can be simplified to

$$\dot{x} = \tilde{A}x + \tilde{B}(t)u$$

without loss of generality (see e.g., [4]). In order to guarantee the transformed system to be T-periodic and real-valued, we need to reconsider the form of factorization so that we can always extract a T-periodic real-valued factor from $\Phi(t,0)$.

To this end, we have proposed a novel representation of the state transition matrix Φ by

$$\Phi(t,0) = P(t)e^{Gt}e^{Ft},$$

where $P(t)$ denotes the T-periodic factor, and G and F denotes the coefficient of matrix exponential factors. The set $\{P(t), G, F\}$ is called the coefficient set.

In this framework, we obtained characterizations of parameter sets such that the proposed representation gives a parametrization of all state transition matrices with real-valued coefficients as follows:

Theorem 2: (Implicit Parametrization; See [6]) Let Φ denotes the state transition matrix of (1). Then there exist a matrix-valued function $P(t) \in C^1_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ and real matrices $F, G \in \mathbb{R}^{n\times n}$ satisfying

$$\Phi(t,0) = P(t)e^{Gt}e^{Ft}, \tag{7}$$

where $P(t)$, F and G satisfy the following conditions:

$$P(0) = I \tag{8}$$

$$e^{GT}F = Fe^{GT} \tag{9}$$

$$e^{2GT} = I. \tag{10}$$

Conversely, let $P(t) \in C^1_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ be any matrix-valued function satisfying (8) and let $F, G \in \mathbb{R}^{n\times n}$ be any matrices satisfying (9) and (10). Then, a matrix-valued function

$$\Phi(s,t) := P(s)e^{Gs}e^{F(s-t)}e^{-Gt}P(t)^{-1} \tag{11}$$

is a state transition matrix for some equation of the form (1) with a matrix-valued function $A(t) \in C^0_T(\mathbb{R}, \mathbb{R}^{n\times n})$. □

III. Explicit Parameterization

As shown in the converse direction of Theorem 2, the representation in (7) gives a parameterization of the all set of state transition matrices generated by T-periodic systems with real-valued coefficients. The conditions (9) and (10) with respect to F and G are implicit and inconvenient to parametrize the coefficients of the transformed system. Hence, by solving the implicit conditions in (8), (9), and (10), we investigate another parametrization in which G and F have a certain block structure; See (13) and (14).

Theorem 3: (Explicit Parametrization) Let Φ denotes the state transition matrix of (1). Then, there exists a matrix-valued function $P(t) \in C^1_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$, an even number n_1, an invertible matrix $V \in \mathbb{R}^{n\times n}$, and matrices $F_1 \in \mathbb{R}^{n_1\times n_1}, F_2 \in \mathbb{R}^{(n-n_1)\times(n-n_1)}$ satisfying

$$\Phi(t,0) = P(t)e^{Gt}e^{Ft} \tag{12}$$

$$G = V\begin{bmatrix} \frac{1}{T}\Pi_{n_1} & \\ & 0 \end{bmatrix}V^{-1} \tag{13}$$

$$F = V\begin{bmatrix} F_1 & \\ & F_2 \end{bmatrix}V^{-1}, \tag{14}$$

where $P(t)$ satisfies (8) and $\Pi_{n_1} \in \mathbb{R}^{n_1\times n_1}$ is defined by

$$\Pi_{n_1} := \begin{bmatrix} \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix} \end{bmatrix}$$

with the even subscript n_1.

Conversely, let $n_1 \geq 0$ denotes any even number, $V \in \mathbb{R}^{n\times n}$ denotes any invertible matrix, $P(t) \in C^1_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$ denotes any matrix-valued function satisfying (8), $F_1 \in \mathbb{R}^{n_1\times n_1}, F_2 \in \mathbb{R}^{(n-n_1)\times(n-n_1)}$ denotes any constant matrices. Then, $\Phi(t,0)$ given by (12) is the state transition matrix for some equation of the form (1) with a matrix-valued function $A(t) \in C^0_T(\mathbb{R}, \mathbb{R}^{n\times n})$. □

Proof: Consider the real Jordan decomposition of the monodromy matrix $\Phi(T,0)$. Then, $\Phi(T,0)$ is factored as

$$\Phi(T,0) =: V\begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix}V^{-1}, \tag{15}$$

where $V \in \mathbb{R}^{n\times n}$ is invertible, all negative eigenvalues of $\Phi(T,0)$ are contained in $\Lambda_1 \in \mathbb{R}^{n_1\times n_1}$, and the other eigenvalues are contained in $\Lambda_2 \in \mathbb{R}^{(n-n_1)\times(n-n_1)}$. Since Φ is the state transition matrix, we have $\det\Phi(T,0) > 0$. Since the eigenvalues of Λ_2 are positive real numbers or complex conjugate pairs, we have $\det\Lambda_2 > 0$. It follows that $\det\Lambda_1 > 0$, and therefore, n_1 is even. Then, we have the relation

$$e^{\Pi_{n_1}} = -I.$$

Since $-\Lambda_1$ is invertible and does not have negative real eigenvalues, there exists $F_1 \in \mathbb{R}^{n_1\times n_1}$ such that

$$e^{F_1T} = -\Lambda_1.$$

It then follows that

$$e^{\Pi_{n_1}}e^{F_1T} = \Lambda_1. \tag{16}$$

Since Λ_2 is also invertible and does not have negative real eigenvalues, there exists $F_2 \in \mathbb{R}^{(n-n_1)\times(n-n_1)}$ such that

$$e^{F_2T} = \Lambda_2. \tag{17}$$

By utilizing (16) and (17), (15) is rewritten as

$$\Phi(T,0) = e^{GT}e^{FT}. \tag{18}$$

Define $P(t)$ by

$$P(t) := \Phi(t,0)e^{-Ft}e^{-Gt}. \tag{19}$$

Then, $P(t)$ satisfies (8) and $P(t) \in C^1_{T,\mathrm{inv}}(\mathbb{R}, \mathbb{R}^{n\times n})$, e.g.,

$$\begin{aligned} P(t+T) &= \Phi(t+T,0)e^{-F(t+T)}e^{-G(t+T)} \\ &= \Phi(t+T,T)\Phi(T,0)e^{-FT}e^{-Ft}e^{-GT}e^{-Gt} \\ &= \Phi(t,0)\Phi(T,0)e^{-FT}e^{-GT}e^{-Ft}e^{-Gt} \\ &= \Phi(t,0)e^{-Ft}e^{-Gt} \\ &= P(t), \end{aligned}$$

where we have used the identity

$$e^{GT}e^{Ft} = e^{Ft}e^{GT}.$$

Now, (19) is rewritten as (12). Conversely, let $n_1 \geq 0$ denotes any even number, $V \in \mathbb{R}^{n\times n}$ denotes any invertible matrix, $P(t) \in C^1_T(\mathbb{R}, \mathbb{R}^{n\times n})$ denotes any invertible matrix-valued function satisfying (8), $F_1 \in \mathbb{R}^{n_1\times n_1}, F_2 \in \mathbb{R}^{(n-n_1)\times(n-n_1)}$ denotes any constant matrices. Then, F and G satisfy (9) and (10). It follows from Theorem 2 that $\Phi(t,0)$ is the state transition matrix for some equation of the form (1) with a T-periodic real-valued function $A(t) \in \mathbb{R}^{n\times n}$. □

Let us compare Theorem 2 and Theorem 3. In Theorem 2, F and G are characterized by implicit conditions (9) and (10). In contrast, F and G are characterized by explicit conditions (13) and (14) with respect to F_1, F_2, and n_1 in Theorem 3, as we have expected.

In Theorem 3, we still have a condition $P(0) = I$. We imposed this condition in order to represent $\Phi(t,0)$ in the form of (12). When this condition is not satisfied, we can reselect the coefficient set as follows:

$$\begin{aligned} \tilde{P}(t) &= P(t)P(0)^{-1} \\ \tilde{F} &= P(0)FP(0)^{-1} \\ \tilde{G} &= P(0)GP(0)^{-1}. \end{aligned}$$

Then, the new coefficient set generates the same state transition matrix by

$$\Phi(s,t) = \tilde{P}(s)e^{\tilde{G}s}e^{\tilde{F}(s-t)}e^{-\tilde{G}s}\tilde{P}(t)^{-1},$$

and $\tilde{P}(t)$ satisfies $\tilde{P}(0) = I$.

The former part of the proof in Theorem 3 is constructive; i.e., n_1, F_1, and F_2 are constructed from the real Jordan decomposition of the monodromy matrix in (15). However, the parameter set n_1, F_1, and F_2 is not uniquely determined from the above constructive proof. Suppose, for instance, the monodromy matrix $\Phi(T,0)$ has the real logarithm, the even number n_1 can be chosen to be 0. In this case, (12) coincides with (6). In contrast, the even number n_1 might not be 0 according to the above constructive proof.

Example 1: Consider a 1-periodic system given by (1) with

$$A(t) = \begin{bmatrix} \pi(\sin(2\pi t) & \frac{\pi}{2}(4\cos(2\pi t)+\cos(4\pi t)-3) \\ \pi & -\pi\sin(2\pi t) \end{bmatrix}. \tag{20}$$

The state transition matrix is given by

$$\Phi(t,0) = \begin{bmatrix} \frac{1}{2}(3\cos(\pi t)-\cos(3\pi t)) & \frac{1}{2}(\sin(3\pi t)-\sin(\pi t)) \\ \sin(\pi t) & \cos(\pi t) \end{bmatrix}.$$

The monodromy matrix

$$\Phi(1,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

has two real Jordan blocks belonging to a negative eigenvalue, it follows Theorem 1 that there is a factorization of the form (6). By taking the real logarithm matrix of $\Phi(t,0)$,

$$\Phi(1,0) = \exp\begin{bmatrix} 0 & -\pi \\ \pi & 0 \end{bmatrix},$$

$\Phi(t,0)$ is factored as $\Phi(t,0) = P(t)e^{Gt}e^{Ft}$ by

$$P(t) = \begin{bmatrix} 1 & \sin(2\pi t) \\ 0 & 1 \end{bmatrix} \tag{21}$$

$$G = 0 \tag{22}$$

$$F = \begin{bmatrix} 0 & -\pi \\ \pi & 0 \end{bmatrix}, \tag{23}$$

where n_1 corresponds to $n_1 = 0$. In contrast, according to the constructive proof in Theorem 3, $\Phi(1,0)$ is factored as

$$\Phi(1,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, by taking the real logarithms

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \exp\begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \exp\begin{bmatrix} 0 & -2\pi \\ 2\pi & 0 \end{bmatrix},$$

$\Phi(t,0)$ is factored as $\Phi(t,0) = \tilde{P}(t)e^{\tilde{G}t}e^{\tilde{F}t}$ by

$$\tilde{P}(t) = \begin{bmatrix} 1 & \sin(2\pi t) \\ 0 & 1 \end{bmatrix} \tag{24}$$

$$\tilde{G} = \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix} \tag{25}$$

$$\tilde{F} = \begin{bmatrix} 0 & -2\pi \\ 2\pi & 0 \end{bmatrix}, \tag{26}$$

where n_1 corresponds to $n_1 = 2$. □

In addition to the difference among implicit or explicit characterizations, the classes of coefficient matrices G and F in Theorem 2 and Theorem 3 are different. As shown in the latte part of the above proof, G and F of the forms (13) and (14) satisfy the conditions (9) and (10). In contrast, the converse statement is not true, i.e., even if G and F satisfy the conditions (9) and (10), G and F does not always have the block structure of the forms (13) and (14).

Example 2: Consider a 1-periodic system given by (1) with

$$A(t) = \begin{bmatrix} -1-\frac{1}{2}\sin(6\pi t)+\frac{\dot{p}(t)}{p(t)} & -\frac{1}{2}+3\pi-\frac{1}{2}\cos(6\pi t) \\ \frac{1}{2}-3\pi-\frac{1}{2}\cos(6\pi t) & -1+\frac{1}{2}\sin(6\pi t)+\frac{\dot{p}(t)}{p(t)} \end{bmatrix}, \tag{27}$$

where a function $p(t) \in C_1^1(\mathbb{R},\mathbb{R})$ satisfies $p(0) = 1$ and $p(t) \neq 0,\ \forall t \in \mathbb{R}$. The state transition matrix is given by

$$\Phi(t,0) = \begin{bmatrix} e^{-t}p(t)\cos(3\pi t) & e^{-t}p(t)(-t\cos(3\pi t)+\sin(3\pi t)) \\ -e^{-t}p(t)\sin(3\pi t) & e^{-t}p(t)(t\sin(3\pi t)+\cos(3\pi t)) \end{bmatrix}.$$

The monodromy matrix

$$\Phi(t,0) = \begin{bmatrix} -e^{-1} & e^{-1} \\ 0 & -e^{-1} \end{bmatrix}$$

has one real Jordan block belonging to a negative eigenvalue, it follows from Theorem 1 that there is no factorization of the form (6). As shown in Theorem 2, there exists a parameter set $P(t)$, G, F satisfying the conditions (8)-(10). For example, $\Phi(t,0)$ is factored as $\Phi(t,0) = P(t)e^{Gt}e^{Ft}$ by selecting

$$P(t) = \begin{bmatrix} p(t) & 0 \\ 0 & p(t) \end{bmatrix} \in C^1_{1,\mathrm{inv}}(\mathbb{R},\mathbb{R}^{2\times 2}) \tag{28}$$

$$G = \begin{bmatrix} 0 & 3\pi \\ -3\pi & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2} \tag{29}$$

$$F = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \in \mathbb{R}^{2\times 2}, \tag{30}$$

where $P(t)$, G, and F satisfy the conditions (8)-(10) in Theorem 2. However, the condition (13) in Theorem 3 is not satisfied for any even n_1 and invertible $V \in \mathbb{R}^{2\times 2}$. Instead, we choose

$$\tilde{P}(t) = \begin{bmatrix} p(t)\cos(2\pi t) & p(t)\sin(2\pi t) \\ -p(t)\sin(2\pi t) & p(t)\cos(2\pi t) \end{bmatrix} \in C^1_{1,\mathrm{inv}}(\mathbb{R},\mathbb{R}^{2\times 2}) \tag{31}$$

$$\tilde{G} = \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2} \tag{32}$$

$$\tilde{F} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \in \mathbb{R}^{2\times 2}. \tag{33}$$

Then, $\Phi(t,0)$ is also factored as $\Phi(t,0) = \tilde{P}(t)e^{\tilde{G}t}e^{\tilde{F}t}$. $\tilde{P}(t)$, $\tilde{G}$, and $\tilde{F}$ also satisfy the conditions (8)-(10) in Theorem 2. In addition, the conditions (13) and (14) in Theorem 3 are satisfied for $n_1 = 2$, $V = I$, $F_1 = \tilde{F}$ □

IV. Extension of the Lyapunov Reducibility Theorem

As an extension of the Lyapunov reducibility theorem, we have shown that any T-periodic system

$$\dot{x} = A(t)x$$

is transformed to a T-periodic system

$$\dot{\xi} = H(t)\xi$$

by a T-periodic coordinate transformation

$$\xi = P(t)^{-1}x,$$

where $H(t)$ includes at most the fundamental frequency components, i.e.

$$H(t) = H_1 + H_2 \cos\left(\frac{2\pi t}{T}\right) + H_3 \sin\left(\frac{2\pi t}{T}\right)$$

for some $H_1, H_2, H_3 \in \mathbb{R}^{n\times n}$ (See [6]). Although the transformed system is not time-invariant, each element of $H(t)$ is consisted of a sum of a constant and trigonometric functions with fundamental frequency. It is important to note that $H(t)$ is not obtained from the first order truncation of the Fourier expansion of $A(t)$.

When we focus on the case where the monodromy matrix $\Phi(T,0)$ has a real logarithm matrix $\tilde{F} \in \mathbb{R}^{n\times n}$ satisfying $\Phi(T,0) = e^{\tilde{F}}$, we can select

$$H_1 = \frac{1}{T}\tilde{F}, \quad H_2 = 0, \quad H_3 = 0,$$

which corresponds to the consequence of the Lyapunov reducibility theorem.

In this section, we consider the general case where the monodromy matrix $\Phi(T,0)$ might not have a real logarithm matrix. Then, we we further investigate the structure of H_1, H_2, and H_3 based on the block structure of G and F given in Theorem 3.

Theorem 4: Let Φ denotes the state transition matrix of (1). Suppose that $\Phi(t,0)$ is factored as (12) with an appropriate even number n_1, an appropriate invertible $P(t) \in C^1_T(\mathbb{R}, \mathbb{R}^{n\times n})$, and appropriate matrices $G, F \in \mathbb{R}^{n\times n}$ in the forms (13) and (14) for certain $F_1 \in \mathbb{R}^{n_1\times n_1}$ and $F_2 \in \mathbb{R}^{(n-n_1)\times(n-n_1)}$, $V \in \mathbb{R}^{n\times n}$. Then, a T-periodic coordinate transformation

$$\eta = P(t)^{-1}x$$

transforms (1) to a T-periodic system of the form

$$\dot{\xi} = H(t)\xi \tag{34}$$

$$H(t) = H_1 + \cos\left(\frac{2\pi t}{T}\right) H_2 + \sin\left(\frac{2\pi t}{T}\right) H_3 \tag{35}$$

where

$$H_1 = G + F - \frac{V}{2}\begin{bmatrix} F_1 + \frac{1}{\pi^2}\Pi_{n_1}F_1\Pi_{n_1} & 0 \\ 0 & 0 \end{bmatrix} V^{-1} \tag{36}$$

$$H_2 = \frac{V}{2}\begin{bmatrix} F_1 + \frac{1}{\pi^2}\Pi_{n_1}F_1\Pi_{n_1} & 0 \\ 0 & 0 \end{bmatrix} V^{-1} \tag{37}$$

$$H_3 = V \begin{bmatrix} e^{\frac{\Pi_{n_1}}{4}} F_1 e^{-\frac{\Pi_{n_1}}{4}} - \frac{F_1}{2} + \frac{1}{2\pi^2}\Pi_{n_1}F_1\Pi_{n_1} & 0 \\ 0 & 0 \end{bmatrix} V^{-1}. \tag{38}$$

□

Proof: A direct calculation gives

$$H(t) = \frac{d(e^{Gt}e^{Ft})}{dt}(e^{Gt}e^{Ft})^{-1} = G + e^{Gt}Fe^{-Gt}. \tag{39}$$

Each element of $e^{\Pi_{n_1}t}$ consists of $\cos(\frac{\pi t}{T})$ or $\sin(\frac{\pi t}{T})$, which are $2T$-periodic. It follows that $H(t)$ has the form

$$H(t) = H_1 + \cos\left(\frac{2\pi t}{T}\right) H_2 + \sin\left(\frac{2\pi t}{T}\right) H_3 + \cos\left(\frac{\pi t}{T}\right) H_4 + \sin\left(\frac{\pi t}{T}\right) H_5$$

for certain constant matrices H_1, H_2, H_3, H_4 and H_5. However, $H(t)$ is shown to be T-periodic

$$\begin{aligned} H(t+T) &= G + e^{G(t+T)}Fe^{-G(t+T)} \\ &= G + e^{Gt}e^{GT}Fe^{-GT}e^{-Gt} \\ &= G + e^{Gt}Fe^{GT}e^{-GT}e^{-Gt} \\ &= G + e^{Gt}Fe^{-Gt} \\ &= H(t), \end{aligned}$$

and therefore, $H(t)$ has the form (35) for certain constant matrices H_1, H_2, and H_3. Taking the average of $H(t)$ over $0 \le t \le T$, we have

$$H_1 = G + \frac{1}{T}\int_0^T e^{Gt}Fe^{-Gt}dt.$$

Substituting (13) and (14) into the above equation, we have

$$\begin{aligned} H_1 &= G + \frac{1}{T}V\left(\int_0^T \begin{bmatrix} e^{\frac{\Pi_{n_1}t}{T}}F_1e^{-\frac{\Pi_{n_1}t}{T}} & 0 \\ 0 & F_2 \end{bmatrix} dt\right) V^{-1} \\ &= G + V\begin{bmatrix} \frac{1}{T}\int_0^T e^{\frac{\Pi_{n_1}t}{T}}F_1e^{-\frac{\Pi_{n_1}t}{T}}dt & 0 \\ 0 & F_2 \end{bmatrix} V^{-1}. \end{aligned}$$

A direct calculation gives

$$\frac{1}{T}\int_0^T e^{\frac{\Pi_{n_1}t}{T}}F_1e^{-\frac{\Pi_{n_1}t}{T}}dt = \frac{1}{2}\left(F_1 - \frac{1}{\pi^2}\Pi_{n_1}F_1\Pi_{n_1}\right).$$

Hence, H_1 is represented by (36). Substituting $t = 0$ into (35) and (39), we have

$$\begin{aligned} &H_2 \\ &= H(0) - H_1 \\ &= G + F - G - F + \frac{V}{2}\begin{bmatrix} F_1 + \frac{1}{\pi^2}\Pi_{n_1}F_1\Pi_{n_1} & 0 \\ 0 & 0 \end{bmatrix} V^{-1} \\ &= \frac{V}{2}\begin{bmatrix} F_1 + \frac{1}{\pi^2}\Pi_{n_1}F_1\Pi_{n_1} & 0 \\ 0 & 0 \end{bmatrix} V^{-1}. \end{aligned}$$

Hence, H_2 is represented by (37). Substituting $t = \frac{\pi}{4}$ into (35) and (39), we have

$$\begin{aligned} & H_3 \\ &= H(\tfrac{\pi}{4}) - H_1 \\ &= e^{\frac{GT}{4}} F e^{-\frac{GT}{4}} - F + \frac{V}{2} \begin{bmatrix} F_1 + \frac{1}{\pi^2}\Pi_{n_1} F_1 \Pi_{n_1} & 0 \\ 0 & 0 \end{bmatrix} V^{-1} \\ &= V \begin{bmatrix} e^{\frac{\Pi_{n_1}}{4}} F_1 e^{-\frac{\Pi_{n_1}}{4}} - \frac{1}{2}F_1 + \frac{1}{2\pi^2}\Pi_{n_1} F_1 \Pi_{n_1} & 0 \\ 0 & 0 \end{bmatrix} V^{-1}. \end{aligned}$$

Hence, H_3 is represented by (38). □

It is immediate to prove that (1) is transformed to a linear time-invariant system when Π_{n_1} and F_1 commute. We note that it is always possible to construct n_1 and F_1 satisfying this commutative condition when the monodromy matrix $\Phi(T, 0)$ has a real logarithm.

Corollary 1: Let Φ denotes the state transition matrix of (1). Suppose that $\Phi(t, 0)$ is factored as (12) with an appropriate even number n_1, an appropriate invertible $P(t) \in C_T^1(\mathbb{R}, \mathbb{R}^{n\times n})$, and appropriate matrices $G, F \in \mathbb{R}^{n\times n}$ in the forms (13) and (14) for certain $F_1 \in \mathbb{R}^{n_1\times n_1}$ and $F_2 \in \mathbb{R}^{(n-n_1)\times(n-n_1)}$, $V \in \mathbb{R}^{n\times n}$. Moreover, suppose that Π_{n_1} and F_1 commute, i.e., $\Pi_{n_1} F_1 = F_1 \Pi_{n_1}$. Then, a T-periodic coordinate transformation

$$\eta = P(t)^{-1} x$$

transforms (1) to a linear time-invariant system of the form

$$\dot{\xi} = (G + F)\xi.$$

□

Proof: By the assumption $\Pi_{n_1} F_1 = F_1 \Pi_{n_1}$, we have

$$\begin{aligned} \frac{1}{\pi^2}\Pi_{n_1} F_1 \Pi_{n_1} &= -F_1 \\ e^{\frac{\Pi_{n_1}}{4}} F_1 e^{-\frac{\Pi_{n_1}}{4}} &= F_1. \end{aligned}$$

By substituting the above equations into (36)–(38), we have

$$\begin{aligned} H_1 &= G + F \\ H_2 &= 0 \\ H_3 &= 0. \end{aligned}$$

□

Example 3: Consider a 1-periodic system (1) with $A(t)$ given by (20) in Example 1. According to (21)-(23), (1) is transformed to a linear time-invariant system

$$\dot{\xi} = \begin{bmatrix} 0 & -\pi \\ \pi & 0 \end{bmatrix} \xi$$

by $\xi = P(t)^{-1}x$. According to (24)-(26), (1) is also transformed to the same linear time-invariant system by $\xi = \tilde{P}(t)^{-1}x$. We note that $F_1 (= \tilde{F} \neq 0)$ satisfies the condition $\Pi_{n_1} F_1 = F_1 \Pi_{n_1}$ in Corollary 1. □

We note that $H(t)$ given in (35) depends on the choice of G and F in the forms (13) and (14). Suppose, for instance, we select G and F which does not have the forms (13) and (14). Then, $H(t)$ might include higher harmonic components.

Example 4: Consider a 1-periodic system (1) with $A(t)$ given by (27) in Example 2. According to (28)-(30), (1) is transformed to $\dot{\xi} = H(t)\xi$ by $\xi = P(t)^{-1}x$. In this case, (28)-(30) does not satisfy the conditions in Theorem 3. Then, $H(t)$ is given by

$$H(t) = \begin{bmatrix} -1 - \frac{1}{2}\sin(6\pi t) & -\frac{1}{2} + 3\pi - \frac{1}{2}\cos(6\pi t) \\ \frac{1}{2} - 3\pi - \frac{1}{2}\cos(6\pi t) & -1 + \frac{1}{2}\sin(6\pi t) \end{bmatrix}, \tag{40}$$

which includes higher harmonic components $\cos(6\pi t)$ and $\sin(6\pi t)$. Instead, according to (31)-(33), (1) is transformed to $\dot{\xi} = \tilde{H}(t)\xi$ by $\xi = \tilde{P}(t)^{-1}x$. In this case, (31)-(33) satisfy the conditions in Theorem 3. Then, $\tilde{H}(t)$ is given by

$$\tilde{H}(t) = \begin{bmatrix} -1 - \frac{1}{2}\sin(2\pi t) & -\frac{1}{2} + \pi - \frac{1}{2}\cos(2\pi t) \\ \frac{1}{2} - \pi - \frac{1}{2}\cos(2\pi t) & -1 + \frac{1}{2}\sin(2\pi t) \end{bmatrix}, \tag{41}$$

which includes at most fundamental frequency components $\cos(2\pi t)$ and $\sin(2\pi t)$. □

V. Conclusions

In this note, we obtained an explicit parameterization of all state transition matrices of linear periodic systems with real-valued coefficients as the extension of the Floquet representation theorem. We also study the block structure of the new standard form for linear periodic systems in detail as an extension of the Lyapunov reducibility theorem. This new information can be useful for designing a periodic controller.

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