Nonnegativity of descriptor systems of index 1

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***Abstract*— In [S.K. Jain, J. Tynan, Nonnegative matrices A with $AA^{\#} \geq O$. Linear Algebra & Appl. 379, 2004], the authors gave a characterization of nonnegative matrices A such that $AA^{\#}$ is a nonnegative matrix, where $A^{\#}$ denotes the group inverse of the square matrix A. The product $AA^{\#}$ will be called the group-projector of the matrix A. Later, a slightly simplified characterization of the above result was presented in [A. Herrero, F. J. Ramírez, and N. Thome, Characterization of matrices with nonnegative group-projector. Lecture Notes in Control and Information Sciences 389, 2009]. In this paper, we present an application of the group-projectors to obtain the nonnegativity of control descriptor systems of index 1. This work improves some previous results in the literature in the sense that the nonnegativity of the coefficient matrix of the descriptor system is removed. So, we can apply this result to a wider class of systems in order to study its nonnegativity.**

I. INTRODUCTION

The establishment of mathematical models represents a very important part of engineering work. A descriptor system allows to describe the dynamic of the state variables and its algebraic behavior, giving a mathematical description for many practical dynamic systems [1]. There is a wide range of topics from several disciplines involving descriptor systems. In particular, a state-space positive system has received an increasing interest attracting the attention of many authors [2], [3], [4].

A dynamical system is said to be nonnegative if it leaves the first orthant of $\mathbb{R}^n$ invariant for future times when initiated in this orthant. Over the past two decades, these systems have gained much attention appearing in a wide variety of applied areas such as biology, chemistry, and sociology [5], [6], [7].

The nonnegativity of the matrices plays an important role in control theory. In particular, the nonnegativity of the coefficient matrices characterizes the nonnegative discrete standard systems. However, it is not the case for the singular systems because this characterization is made by means of products of the Drazin inverses of the state matrices. In the literature, descriptor systems are also called singular systems, implicit systems or generalized state-space systems. Some properties of them can be found in [1], [8]. In some situations these systems have the symmetry properties and then its solution involves the group inverse of certain matrices [9]. Descriptor systems appear when modelling some physical phenomena and interconnected systems such as electrical, mechanical, and chemical processes [10], [11]. Concretely, they arise when solving computational problems in the analysis and design of standard linear systems.

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For a given matrix $F \in \mathbb{R}^{n\times n}$, a matrix $G \in \mathbb{R}^{n\times n}$ is called its Drazin inverse if the properties $GFG = G$, $FG = GF$ and $F^{k+1}G = F^k$ hold, where $k = \text{ind}(F)$ is the index of F, that is, the smallest nonnegative integer such that $\text{rank}(F^{k+1}) = \text{rank}(F^k)$. This matrix will be denoted by F^D and it is unique. If $k = 1$, this matrix is called group inverse and is denoted by $F^{\#}$ [12]. In this case, the properties are $FGF = F$, $GFG = G$, and $FG = GF$. Moreover, we will stand $F \geq O$ for a matrix F with nonnegative elements. It is well-known that the product $FF^{\#}$ is a projector on the range of F along the null space of $F^{\#}$. In order to distinguish this projector among others defined using other generalized inverses, we will call group-projector to $FF^{\#}$.

The group inverse has been widely studied in the literature and applied to solve real problems. For instance, it is applied in model electric networks, Markov chains, symmetric singular control systems, etc. [9], [10], [12], [13]. On the other hand, a characterization of matrices with nonnegative group-projector has been presented in [14]. In this work, this characterization will be used to analyze more in depth the positivity of a block descriptor system.

The following results will be useful in this paper.

Theorem 1.1 (Theorem 7.7.1, [13]): Let

$$M = \begin{bmatrix} A & B \\ O & C \end{bmatrix} \in \mathbb{R}^{n\times n},$$

where A and C are square matrices, and $k = \text{ind}(A)$, $l = \text{ind}(C)$. Then

$$M^D = \begin{bmatrix} A^D & X \\ O & C^D \end{bmatrix} \in \mathbb{R}^{n\times n},$$

where

$$\begin{aligned} X &= (A^D)^2 \left[\sum_{i=0}^{l-1} (A^D)^i B C^i\right] (I - CC^D) \\ &\quad + (I - AA^D) \left[\sum_{i=0}^{k-1} A^i B (C^D)^i\right] (C^D)^2 \qquad (1) \\ &\quad - A^D B C^D \end{aligned}$$

being $O^0 = I$ by convention.

Theorem 1.2 (Corollary 7.7.1, [13]): Let

$$L = \begin{bmatrix} C & O \\ B & A \end{bmatrix} \in \mathbb{R}^{n\times n},$$

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where A and C are square matrices, and $k = \text{ind}(A)$, $l = \text{ind}(C)$. Then

$$L^D = \begin{bmatrix} C^D & O \\ X & A^D \end{bmatrix} \in \mathbb{R}^{n\times n},$$

where X is the matrix given in (1).

The main aim of this paper is to present a method to check the nonnegativity property of a descriptor system using information about blocks of the coefficient matrices of the descriptor system instead of conditions on the whole matrices. Opposite to the known results on nonnegativity of descriptor systems where the nonnegativity of the singular matrix E is required, here this condition is removed. Then the class of systems where nonnegativity can be analyzed is more general.

II. PREVIOUS RESULTS

In this paper we consider singular discrete-time control systems

$$\begin{cases} Ex(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{cases} \tag{2}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $x(k) \in \mathbb{R}^{n\times 1}$, $u(k) \in \mathbb{R}^{m\times 1}$, and $y(k) \in \mathbb{R}^{p\times 1}$. In general, this system is denoted by (E, A, B, C) and when $C = I$, it is denoted by (E, A, B). We will suppose that this system satisfy the regularity condition, that is, there exists a scalar α such that $\det(\alpha E + A) \neq 0$. In this case, the system (2) can be transformed into the equivalent system

$$\begin{cases} \widehat{E}x(k+1) &= \widehat{A}x(k) + \widehat{B}u(k) \\ y(k) &= \widehat{C}x(k) \end{cases} \tag{3}$$

where $\widehat{E} = (\alpha E + A)^{-1}E$, $\widehat{A} = (\alpha E + A)^{-1}A$, $\widehat{B} = (\alpha E + A)^{-1}B$, and $\widehat{C} = C$. This new system (3) satisfies the conditions

(i) $\widehat{E}\widehat{A} = \widehat{A}\widehat{E}$,
(ii) $\text{Ker}(\widehat{E}) \cap \text{Ker}(\widehat{A}) = \{0\}$, where $\text{Ker}(\cdot)$ denotes the null space of $(\cdot)$,
(iii) $\widehat{A} = I - \alpha\widehat{E}$.

The regularity condition assures that the system (3) has solution and it is given by $y(k) = \widehat{C}x(k)$ where

$$\begin{aligned} x(k) &= (\widehat{E}^D\widehat{A})^k\widehat{E}^D\widehat{E}x(0) \\ &\quad + \sum_{i=0}^{k-1}\widehat{E}^D(\widehat{E}^D\widehat{A})^{k-i-1}\widehat{B}u(i) \\ &\quad - (I - \widehat{E}^D\widehat{E})\sum_{i=0}^{q-1}(\widehat{E}\widehat{A}^D)^i\widehat{A}^D\widehat{B}u(k+i), \end{aligned} \tag{4}$$

with $q = \text{ind}(\widehat{E})$ and $x(0)$ an initial admissible condition [8]. The set of admissible initial conditions is given by $\text{Im}\begin{bmatrix} \widehat{E}^D\widehat{E} & H_0 & \dots & H_{q-1} \end{bmatrix}$, where $H_i = (I - \widehat{E}^D\widehat{E})(\widehat{E}\widehat{A}^D)^i\widehat{A}^D$, $i = 0, \dots, q-1$ and $\text{Im}[\cdot]$ denotes the range of $[\cdot]$.

Next, a relation between the index of the involved matrices E and $\widehat{E}$ is obtained. In general, neither $\text{ind}(E) = 1$ imply $\text{ind}(\widehat{E}) = 1$ nor $\text{ind}(\widehat{E}) = 1$ imply $\text{ind}(E) = 1$. For instance, given the matrices

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and setting $\alpha = 0$, one has that

$$\widehat{E} = (\alpha E + A)^{-1}E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

with $\text{ind}(\widehat{E}) = 2$ while $\text{ind}(E) = 1 \neq 2$. Interchanging the roles of E and $\widehat{E}$ we can show that $\text{ind}(\widehat{E}) = 1$ does not imply that $\text{ind}(E) = 1$.

However, we can prove the following result using the Corollary 7.2.2 in [13].

Lemma 2.1: Let $E \in \mathbb{R}^{n\times n}$ be a matrix with $\text{ind}(E) = 1$, that is,

$$E = Q\begin{bmatrix} C & O \\ O & O \end{bmatrix}Q^{-1}$$

where C is a nonsingular matrix. Let $A \in \mathbb{R}^{n\times n}$ such that there exists $(\alpha E + A)^{-1}$ for some scalar α and $Q^{-1}(\alpha E + A)^{-1}Q$ is a block upper-triangular matrix. Then $\text{ind}((\alpha E + A)^{-1}E) = 1$.

Proof. Since $Q^{-1}(\alpha E + A)^{-1}Q$ is a block upper-triangular matrix,

$$Q^{-1}(\alpha E + A)^{-1}Q = \begin{bmatrix} R_1 & R_2 \\ O & R_3 \end{bmatrix}.$$

Then,

$$(\alpha E + A)^{-1}E = Q\begin{bmatrix} R_1C & O \\ O & O \end{bmatrix}Q^{-1}.$$

The nonsingularity of $(\alpha E + A)^{-1}$ implies that $Q^{-1}(\alpha E + A)^{-1}Q$ is nonsingular. So, R_1 is nonsingular and, by Corollary 7.2.2 in [13], $\text{ind}((\alpha E + A)^{-1}E) = 1$ because C is nonsingular. □

A similar result is valid interchanging the roles of E and $(\alpha E + A)^{-1}E$.

It is well-known that a system (E, A, B, C) is called nonnegative if, for every admissible initial state $x(0) \geq 0$ and for every nonnegative control sequence $u(i)$, $i = 0, 1, \dots, k - 1 + \text{ind}(E)$, the states $x(k)$ are nonnegative and the outputs $y(k)$ are nonnegative, $\forall k \in \mathbb{Z}^+$. The following result gives necessary and sufficient conditions on the matrices E, A, B, and C such that the system (E, A, B, C) is nonnegative.

Theorem 2.2: Let (E, A, B, C) be a discrete-time singular system such that $E^DE \geq O$, $EA = AE$, and $\text{Ker}(E) \cap \text{Ker}(A) = \{0\}$. The system (E, A, B, C) is nonnegative if and only if for each $i = 0, 1, \dots, \text{ind}(E) - 1$, the following conditions hold:

(a) $E^DA \geq O$,
(b) $E^DB \geq O$,
(c) $CE^DE \geq O$,
(d) $-(I - EE^D)(EA^D)^iA^DB \geq O$, and
(e) $-C(I - EE^D)(EA^D)^iA^DB \geq O$.

Proof. From Proposition 1 of [15], it is clear that the nonnegativity of the system (E, A, B) implies (a), (b), and (d).

Since $E^D E \geq 0$ and for each $j = 1, \ldots, n$, the vectors $x(0) = E^D E e_j \geq 0$ are admissible initial conditions (where e_j is the canonical vector with 1 in the *j*-th component and 0's otherwise). Then, $y(0) = CE^D E e_j \geq 0$, for each $j = 1, \ldots, n$ and for every nonnegative control sequence $u(i)$. Hence $CE^D E = CE^D E \begin{bmatrix} e_1 & e_2 & \ldots & e_n \end{bmatrix}$ and then

$$\begin{bmatrix} CE^D E e_1 & CE^D E e_2 & \ldots & CE^D E e_n \end{bmatrix} \geq O,$$

so (c) holds.

For each $j = 1, \ldots, n$, if we choose $x(0) = 0$, $u(k+h) = e_j$, and $u(i) = 0$, $i \in \{0, 1, \ldots, k-1+\text{ind}(E)\} - \{k+h\}$, we have that

$$y(k) = Cx(k) = -C(I - EE^D)(EA^D)^h A^D B e_i \geq 0,$$

then $-C(I - EE^D)(EA^D)^h A^D B \geq O$ for all $h = 0, 1, \ldots \text{ind}(E) - 1$ and (e) holds.

Conversely, from $E^D A = AE^D$ and $E^D E E^D = E^D$, it is easy to check that (a), (b), (c), (d), and (e) imply that $x(k) \geq 0$ and $y(k) \geq 0$ for every $u(i) \geq 0$, $i = 1, \ldots, k-1+\text{ind}(E)$ and every $x(0)$ nonnegative admissible initial condition. □

III. CHARACTERIZATION OF INDEX 1 DESCRIPTOR SYSTEMS

We start this section with a result that gives a canonical form for matrices of index 1 having nonnegative group-projector.

Theorem 3.1: Let $\mathcal{E} \in \mathbb{R}^{n \times n}$ be a matrix of rank $r > 0$. Then $\mathcal{E}\mathcal{E}^\# \geq O$ if and only if there is a permutation matrix P such that

$$\mathcal{E} = P^t \begin{bmatrix} XTY & XTYM & O \\ O & O & O \\ NXTY & NXTYM & O \end{bmatrix} P \tag{5}$$

where M, N are nonnegative matrices of appropriate size, $T \in \mathbb{R}^{r \times r}$ is nonsingular, and $X = diag(x_1, x_2, \ldots, x_r)$, $Y = diag(y_1^t, y_2^t, \ldots, y_r^t)$ being x_i and y_i positive column vectors with $i, j \in \{1, \ldots, r\}$ such that $YX = I$.

In this case,

$$\mathcal{E}^\# = P^t \begin{bmatrix} XT^{-1}Y & XT^{-1}YM & O \\ O & O & O \\ NXT^{-1}Y & NXT^{-1}YM & O \end{bmatrix} P. \tag{6}$$

Remark 1: Note that in the previous theorem the matrix $\mathcal{E}$ has nonnegative group-projector but it is not necessarily nonnegative. In fact, the additional condition $\mathcal{E} \geq O$ in the previous theorem implies that the matrix T in (5) must be nonnegative and conversely.

From now on, we will consider the descriptor control system (3) in which $\widehat{E}$ is a matrix with index equals 1 and with group-projector $\widehat{E}^\# \widehat{E} \geq O$. Thus, using Theorem 3.1, the matrix $\widehat{E}$ has the form (5). Then, we can use the orthogonal matrix P appearing in (5) to transform the system (3) into a equivalent system with matrices in a simpler form given as follows:

$$\left\{ \begin{array}{rcl} \widetilde{E}z(k+1) & = & \widetilde{A}z(k) + \widetilde{B}u(k) \\ y(k) & = & \widetilde{C}z(k) \end{array} \right. , \tag{7}$$

where $z(k) = Px(k)$,

$$\widetilde{E} = P\widehat{E}P^t = \begin{bmatrix} XTY & XTYM & O \\ O & O & O \\ NXTY & NXTYM & O \end{bmatrix}, \tag{8}$$

$$\begin{aligned} \widetilde{A} &= P\widehat{A}P^t = I - \alpha\widetilde{E} \\ &= \begin{bmatrix} I - \alpha XTY & -\alpha XTYM & O \\ O & I & O \\ -\alpha NXTY & -\alpha NXTYM & I \end{bmatrix}, \end{aligned} \tag{9}$$

$$\widetilde{B} = P\widehat{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}, \tag{10}$$

and

$$\widetilde{C} = \widehat{C}P^t = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}. \tag{11}$$

An easy computation allows to show that $\text{ind}(\widehat{E}) = 1$ implies $\text{ind}(\widetilde{E}) = 1$. Actually, the equivalence between $\text{ind}(\widehat{E}) = 1$ and $\text{ind}(\widetilde{E}) = 1$ is valid. Moreover, the system $(\widehat{E}, \widehat{A}, \widehat{B}, \widehat{C}) \geq O$ if and only if the system $(\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}) \geq O$ because clearly we have that the following equivalences hold:

$$\begin{array}{rcl} \widehat{E}^\# \widehat{E} \geq O & \Leftrightarrow & \widetilde{E}^\# \widetilde{E} \geq O \\ (\widehat{E}^\# \widehat{A})^k \geq O & \Leftrightarrow & (\widetilde{E}^\# \widetilde{A})^k \geq O \\ (\widehat{E}^\# \widehat{A})^k \widehat{E}^\# \widehat{E} \geq O & \Leftrightarrow & (\widetilde{E}^\# \widetilde{A})^k \widetilde{E}^\# \widetilde{E} \geq O \\ \widehat{E}^\# \widehat{B} \geq O & \Leftrightarrow & \widetilde{E}^\# \widetilde{B} \geq O \\ -(I - \widehat{E}\widehat{E}^\#)\widehat{A}^D \widehat{B} \geq O & \Leftrightarrow & -(I - \widetilde{E}\widetilde{E}^\#)\widetilde{A}^D \widetilde{B} \geq O \\ \widehat{C}\widehat{E}^\# \widehat{E} \geq O & \Leftrightarrow & \widetilde{C}\widetilde{E}^\# \widetilde{E} \geq O \\ -\widehat{C}(I - \widehat{E}\widehat{E}^\#)\widehat{A}^D \widehat{B} \geq O & \Leftrightarrow & -\widetilde{C}(I - \widetilde{E}\widetilde{E}^\#)\widetilde{A}^D \widetilde{B} \geq O \end{array}$$

since $(P\widehat{A}P^{-1})^D = P\widehat{A}^D P^{-1}$.

We are interested on finding conditions on the matrices $\widetilde{E}$, $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ such that $\widetilde{E}^\# \widetilde{A} \geq O$, $(\widetilde{E}^\# \widetilde{A})^k \widetilde{E}^\# \widetilde{E} \geq O$, $\widetilde{E}^\# \widetilde{B} \geq O$, $-(I - \widetilde{E}\widetilde{E}^\#)\widetilde{A}^D \widetilde{B} \geq O$, $\widetilde{C}\widetilde{E}^\# \widetilde{E} \geq O$, and $-\widetilde{C}(I - \widetilde{E}\widetilde{E}^\#)\widetilde{A}^D \widetilde{B} \geq O$, under the assumptions $\text{ind}(\widetilde{E}) = 1$, $\widetilde{E}^\# \widetilde{E} \geq O$, and $\widetilde{A} = I - \alpha\widetilde{E}$.

Firstly we study the case $\widetilde{C} = I$, that is, when the output vector is equal to the state vector. The result is given in the following lemma, which is a technical result previous to the main one of this section.

Lemma 3.2: Let $(\widetilde{E}, \widetilde{A}, \widetilde{B})$ be a singular system where $\widetilde{E}, \widetilde{A} \in \mathbb{R}^{n \times n}$, $\widetilde{B} \in \mathbb{R}^{n \times m}$ are matrices such that $\text{ind}(\widetilde{E}) = 1$, $\text{rank}(\widetilde{E}) = r > 0$, $\widetilde{E}\widetilde{E}^\# \geq O$, $\widetilde{A} = I - \alpha\widetilde{E}$, for some scalar α, and $\widetilde{B}$ is partitioned as in (10). Then $(\widetilde{E}, \widetilde{A}, \widetilde{B})$ is nonnegative if and only if the following conditions hold

(a) $T^{-1} - \alpha I \geq O$,
(b) $T^{-1}Y(B_1 + MB_2) \geq O$,
(c) $-B_2 \geq O$,
(d) $(XY - I)(H^D B_1 + SB_2) + XYMB_2 \geq O$,

(e) $NXY(H^DB_1+SB_2+MB_2)-UB_1-VB_2-B_3 \geq O$, where X, Y, M, N, and T are the matrices defined in (5), $H = I - \alpha XTY$,

$$S = (I-HH^D)\left[\sum_{i=0}^{l-1} H^i J\right] - H^D J, \tag{12}$$

$$U := -\alpha NXTY\left(\sum_{j=0}^{l-1} H^j(I-HH^D) - H^D\right), \tag{13}$$

$$V := \alpha NXTY\left(\sum_{j=0}^{l-1} H^j(HS+J) + S\right), \tag{14}$$

and $J := -\alpha XTYM$.

Proof. Assuming that $\widetilde{E}^{\#}\widetilde{E} \geq O$, from Theorem 2.2, the system $(\widetilde{E},\widetilde{A},\widetilde{B})$ is nonnegative if and only if $(\widetilde{E}^{\#}\widetilde{A})^k \geq O$, $(\widetilde{E}^{\#}\widetilde{A})^k\widetilde{E}^{\#}\widetilde{E} \geq O$, $\widetilde{E}^{\#}\widetilde{B} \geq O$, and $-(I-\widetilde{E}\widetilde{E}^{\#})\widetilde{A}^D\widetilde{B} \geq O$.

From the expressions of the matrices $\widetilde{E}$ and $\widetilde{A}$ given in (8) and (9), denoting $\mathcal{T} = T^{-1} - \alpha I$, we have that

$$\begin{aligned}\widetilde{E}^{\#}\widetilde{A} &= \widetilde{E}^{\#} - \alpha\widetilde{E}^{\#}\widetilde{E}\\ &= \begin{bmatrix} X\mathcal{T}Y & X\mathcal{T}YM & O\\ O & O & O\\ NX\mathcal{T}Y & NX\mathcal{T}YM & O\end{bmatrix}.\end{aligned}$$

By induction one has

$$(\widetilde{E}^{\#}\widetilde{A})^k = \begin{bmatrix} (X\mathcal{T}Y)^k & (X\mathcal{T}Y)^kM & O\\ O & O & O\\ N(X\mathcal{T}Y)^k & N(X\mathcal{T}Y)^kM & O\end{bmatrix} \tag{15}$$

for every positive integer k. Given that $YX = I$, it follows $(X\mathcal{T}Y)^k = X\mathcal{T}^kY$ and then the nonnegativity of $(E^{\#}A)^k$ holds if and only if the following conditions are fulfilled:

(i) $X\mathcal{T}^kY \geq O$,
(ii) $X\mathcal{T}^kYM \geq O$,
(iii) $NX\mathcal{T}^kY \geq O$, and
(iv) $NX\mathcal{T}^kYM \geq O$,

which can be reduced to the only condition $X\mathcal{T}^kY \geq O$, because M and N are nonnegative matrices. Now, since $X \geq O$, $Y \geq O$, and $YX = I$, one has that $X\mathcal{T}^kY \geq O$ is equivalent to $\mathcal{T}^k = (T^{-1}-\alpha I)^k \geq O$, for all positive integer k, and in particular for $k = 1$.

Now we analyze the product $(\widetilde{E}^{\#}\widetilde{A})^k(\widetilde{E}^{\#}\widetilde{E})$. From expressions (8) and (15), we get

$$(\widetilde{E}^{\#}\widetilde{A})^k\widetilde{E}^{\#}\widetilde{E} = \begin{bmatrix} X\mathcal{T}^kY & X\mathcal{T}^kYM & O\\ O & O & O\\ NX\mathcal{T}^kY & NX\mathcal{T}^kYM & O\end{bmatrix}.$$

So, analogously, $(\widetilde{E}^{\#}\widetilde{A})^k\widetilde{E}^{\#}\widetilde{E} \geq O$ if and only if $T^{-1} - \alpha I \geq O$.

Next, we study the nonnegativity of $E^{\#}B$. Again, partitioning B as in (10), we get

$$\widetilde{E}^{\#}\widetilde{B} = \begin{bmatrix} XT^{-1}YB_1 + XT^{-1}YMB_2\\ O\\ NXT^{-1}YB_1 + NXT^{-1}YMB_2\\ O\end{bmatrix}.$$

Hence, $\widetilde{E}^{\#}\widetilde{B} \geq O$ if and only if $XT^{-1}Y(B_1+MB_2) \geq O$ (since $N \geq O$). Premultipliying this last inequality by Y and using that $YX = I$ and that Y is a nonnegative matrix, we get that $\widetilde{E}^{\#}\widetilde{B} \geq O$ if and only if $T^{-1}Y(B_1+MB_2) \geq O$.

Finally, we study the product $(I-\widetilde{E}\widetilde{E}^{\#})\widetilde{A}^D\widetilde{B}$. For that, we have to perform $\widetilde{A}^D$. In fact, denoting $H = I - \alpha XTY$ and $J = -\alpha XTYM$,

$$\widetilde{A} = \left[\begin{array}{cc|c} H & J & O\\ O & I & O\\ \hline -\alpha NXTY & NJ & I\end{array}\right] =: \left[\begin{array}{c|c} F & O\\ \hline G & I\end{array}\right], \tag{16}$$

where F and G have the adequate sizes, applying the Theorem 1.2 we obtain that

$$\widetilde{A}^D = \begin{bmatrix} F^D & O\\ \widetilde{G} & I\end{bmatrix},$$

where

$$\widetilde{G} \;=\; \left[\sum_{i=0}^{l-1} GF^i\right](I-FF^D) - GF^D \tag{17}$$

being l the index of F.

Note that $\widetilde{A}$, F and H have the same index. In fact, from (16) it is easy to see that $\operatorname{rank}(\widetilde{A}^k) = \operatorname{rank}(F^k) + \operatorname{rank}(I)$ and $\operatorname{rank}(\widetilde{A}^{k-1}) = \operatorname{rank}(F^{k-1}) + \operatorname{rank}(I)$. So, $\operatorname{rank}(\widetilde{A}^k) - \operatorname{rank}(\widetilde{A}^{k-1}) = \operatorname{rank}(F^k) - \operatorname{rank}(F^{k-1})$, which implies that $\operatorname{ind}(\widetilde{A}) = \operatorname{ind}(F)$. Similarly, one gets that $\operatorname{rank}(F^k) = \operatorname{rank}(H^k) + \operatorname{rank}(I)$, for all nonnegative integer k, and thus $\operatorname{ind}(F) = \operatorname{ind}(H)$. This shows that the relevant information of $\widetilde{A}$ is in its $(1,1)$ block.

From Theorem 1.1 one has that the Drazin inverse of F is

$$F^D = \begin{bmatrix} H^D & S\\ O & I\end{bmatrix},$$

being

$$S = (I-HH^D)\left[\sum_{i=0}^{l-1} H^i J\right] - H^D J. \tag{18}$$

Relative to the matrix $\widetilde{G}$ appearing in (17), we have that

$$F^i = \begin{bmatrix} H^i & \sum_{j=0}^{i-1} H^j J\\ O & I\end{bmatrix},$$

and then

$$\sum_{i=0}^{l-1} F^i = I + \begin{bmatrix} \sum_{j=1}^{l-1} H^j & \sum_{p=1}^{l-1} pH^{l-p-1}J\\ O & (l-1)I\end{bmatrix}.$$

So,

$$\widetilde{G} = \begin{bmatrix} U & V\\ O & O\end{bmatrix},$$

where U and V are defined in (13) and (14), respectively.

Then, the Drazin inverse of $\widetilde{A}$ is

$$\widetilde{A}^D = \begin{bmatrix} H^D & S & O\\ O & I & O\\ U & V & I\end{bmatrix}. \tag{19}$$

Observe that the matrices $\widetilde{A}$ and $\widetilde{A}^D$ have the same block structure.

Then computing the product $(I-\widetilde{E}^{\#}\widetilde{E})\widetilde{A}^D\widetilde{B}$, it results

$$\begin{bmatrix} (I-XY)\mathcal{B}-XYMB_2 \\ B_2 \\ -NXY\mathcal{B}-NXYMB_2+UB_1+VB_2+B_3 \end{bmatrix}, \quad (20)$$

where $\mathcal{B}=H^DB_1+SB_2$. So, $-(I-E^{\#}E)A^DB\geq O$ if and only if $-B_2\geq O$, $(XY-I)(H^DB_1+SB_2)+XYMB_2\geq O$, and $NXY(H^DB_1+SB_2+MB_2)-UB_1-VB_2-B_3\geq O$. This ends the proof. □

Remark 2: From the proof of the previous lemma, note that a necessary condition to obtain $\widetilde{E}^{\#}\widetilde{B}\geq O$ is $Y(B_1+MB_2)\geq O$ since $TT^{-1}=I$ and $T\geq O$.

Remark 3: Note that the condition $\widetilde{E}\geq O$ added as an additional hypothesis in the previous lemma yields to the same conditions adding that $T\geq O$.

We close this section analyzing the nonnegative of a general system $(\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C})$.

Theorem 3.3: Let $(\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C})$ be a singular system where $\widetilde{E},\widetilde{A}\in\mathbb{R}^{n\times n}$, $\widetilde{B}\in\mathbb{R}^{n\times m}$, $\widetilde{C}\in\mathbb{R}^{p\times n}$ are matrices such that $\text{ind}(\widetilde{E})=1$, $\text{rank}(\widetilde{E})=r>0$, $\widetilde{E}\widetilde{E}^{\#}\geq O$, $\widetilde{A}=I-\alpha\widetilde{E}$, for some scalar α, and $\widetilde{B}$ and $\widetilde{C}$ are partitioned as in (10) and (11), respectively. Then $(\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C})$ is nonnegative if and only if the following conditions hold

(a) $T^{-1}-\alpha I\geq O$,
(b) $T^{-1}Y(B_1+MB_2)\geq O$,
(c) $-B_2\geq O$,
(d) $(XY-I)(H^DB_1+SB_2)+XYMB_2\geq O$,
(e) $NXY(H^DB_1+SB_2+MB_2)-UB_1-VB_2-B_3\geq O$,
(f) $(C_1+C_3N)X\geq O$,
(g) $C_1[(XY-I)(H^DB_1+SB_2)+XYMB_2]-C_2B_2+C_3[NXY(H^DB_1+SB_2+MB_2)-UB_1-VB_2-B_3]\geq O$.

where X, Y, M, N, and T are the matrices defined in (5), $H=I-\alpha XTY$, and S, U, and V are given in (18), (13), and (14), respectively.

Proof. Assuming that $\widetilde{E}^{\#}\widetilde{E}\geq O$, from Theorem 2.2, the system $(\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C})$ is nonnegative if and only if $(\widetilde{E}^{\#}\widetilde{A})^k\geq O$, $(\widetilde{E}^{\#}\widetilde{A})^k\widetilde{E}^{\#}\widetilde{E}\geq O$, $\widetilde{E}^{\#}\widetilde{B}\geq O$, $-(I-\widetilde{E}\widetilde{E}^{\#})\widetilde{A}^D\widetilde{B}\geq O$, $\widetilde{C}\widetilde{E}^{\#}\widetilde{E}\geq O$, and $-\widetilde{C}(I-\widetilde{E}^{\#}\widetilde{E})\widetilde{A}^D\widetilde{B}\geq O$.

An immediate application of Lemma 3.2 shows that the four first inequalities directly lead to the conditions (a)-(d).

Now, we have to analyze the condition $\widetilde{C}\widetilde{E}^{\#}\widetilde{E}\geq O$. In fact, partitioning the matrix $\widetilde{C}$ as in (11), we get

$$\widetilde{C}\widetilde{E}^{\#}\widetilde{E}=\begin{bmatrix} (C_1+C_3N)XY & (C_1+C_3N)XYM & O \end{bmatrix}.$$

Then, the equivalence between $\widetilde{C}\widetilde{E}^{\#}\widetilde{E}\geq O$ and the two conditions $(C_1+C_3N)XY\geq O$ and $(C_1+C_3N)XYM\geq O$ is clear. Since $M\geq O$, $Y\geq O$, and $YX=I$, both conditions can be reduced to $(C_1+C_3N)X\geq O$.

The last condition to be studied is $-\widetilde{C}(I-\widetilde{E}^{\#}\widetilde{E})\widetilde{A}^D\widetilde{B}\geq O$. The analysis of the nonnegativity of the product of the matrices $\widetilde{C}$ and $(I-\widetilde{E}^{\#}\widetilde{E})\widetilde{A}^D\widetilde{B}$, given by the expressions (11) and (20) respectively, allows to show the condition (g). This ends the proof. □

Next corollary gives a particular important situation.

Corollary 3.4: Let $(\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C})$ be a singular system where $\widetilde{E},\widetilde{A}\in\mathbb{R}^{n\times n}$, $\widetilde{B}\in\mathbb{R}^{n\times m}$, $\widetilde{C}\in\mathbb{R}^{p\times n}$ are matrices such that $\widetilde{E}\geq O$, $\text{ind}(\widetilde{E})=1$, $\text{rank}(\widetilde{E})=r>0$, $\widetilde{E}\widetilde{E}^{\#}\geq O$, $\widetilde{A}=I-\alpha\widetilde{E}$, for some scalar α, and $\widetilde{B}$ and $\widetilde{C}$ are partitioned as in (10) and (11), respectively. Then $(\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C})$ is nonnegative if and only if the following conditions hold

(a) $T\geq O$
(b) $T^{-1}-\alpha I\geq O$,
(c) $T^{-1}Y(B_1+MB_2)\geq O$,
(d) $-B_2\geq O$,
(e) $(XY-I)(H^DB_1+SB_2)+XYMB_2\geq O$,
(f) $NXY(H^DB_1+SB_2+MB_2)-UB_1-VB_2-B_3\geq O$,
(g) $(C_1+C_3N)X\geq O$,
(h) $C_1[(XY-I)(H^DB_1+SB_2)+XYMB_2]-C_2B_2+C_3[NXY(H^DB_1+SB_2+MB_2)-UB_1-VB_2-B_3]\geq O$.

where X, Y, M, N, and T are the matrices defined in (5), $H=I-\alpha XTY$, and S, U, and V are given in (18), (13), and (14), respectively.

IV. EXAMPLE

Next, we illustrate the obtained results with an example.

Let (E,A,B,C) be the singular system given by the matrices

$$E=\begin{bmatrix} 0 & 3 & 3 \\ 0 & 4 & 4 \\ 0 & -4 & -4 \end{bmatrix}, \qquad A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ -1 & -3 & -9 \end{bmatrix},$$

$$B=\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad C=\begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}.$$

This system is equivalent to the system $(\hat{E},\hat{A},\hat{B},\hat{C})$, where

$$\hat{E}=\begin{bmatrix} 0 & \frac{1}{\alpha+1} & \frac{1}{\alpha+1} \\ 0 & \frac{1}{\alpha+1} & \frac{1}{\alpha+1} \\ 0 & 0 & 0 \end{bmatrix}, \qquad \hat{A}=\begin{bmatrix} 1 & \frac{-\alpha}{\alpha+1} & \frac{-\alpha}{\alpha+1} \\ 0 & \frac{1}{\alpha+1} & \frac{-\alpha}{\alpha+1} \\ 0 & 0 & 1 \end{bmatrix},$$

$$\hat{B}=\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \hat{C}=C,$$

with $\alpha\neq -1$. Now, the matrix $\hat{E}$ has index 1 and $\hat{E}^{\#}=(\alpha+1)^2\hat{E}$. So, $\hat{E}\hat{E}^{\#}\geq O$ and $\hat{E}$ it can be written in the form (5) being

$$P=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$X=Y=M=N=1$, and $T=\frac{1}{\alpha+1}$. Then, we obtain the equivalent system $(\tilde{E},\tilde{A},\tilde{B},\tilde{C})$ where

$$\tilde{E}=\begin{bmatrix} \frac{1}{\alpha+1} & \frac{1}{\alpha+1} & 0 \\ 0 & 0 & 0 \\ \frac{1}{\alpha+1} & \frac{1}{\alpha+1} & 0 \end{bmatrix}, \qquad \tilde{A}=\begin{bmatrix} \frac{1}{\alpha+1} & \frac{-\alpha}{\alpha+1} & 0 \\ 0 & 1 & 0 \\ \frac{-\alpha}{\alpha+1} & \frac{-\alpha}{\alpha+1} & 1 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \qquad \text{and} \qquad \tilde{C} = \begin{bmatrix} C_2 & C_3 & C_1 \end{bmatrix}.$$

Then, Theorem 3.4 assures that the system is nonnegative when $\alpha \neq -1$, $C_1 \geq O$, and $C_1 + C_2 \geq O$.

On the other hand, from expression (4) the state vector is given by

$$\begin{aligned} x(k) &= (\hat{E}^{\#}\hat{A})^k \hat{E}^{\#}\hat{E}x(0) + \sum_{i=0}^{k-1} \hat{E}^{\#}(\hat{E}^{\#}\hat{A})^{k-i-1}\hat{B}u(i) \\ &\quad -(I - \hat{E}^{\#}\hat{E})\hat{A}^D\hat{B}u(k) \\ &= \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(0) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k) \end{aligned}$$

and the output vector is

$$y(k) = \begin{bmatrix} 0 & C_1 + C_2 & C_1 + C_2 \end{bmatrix} x(0) + C_1 u(k),$$

which clearly shows that the conditions found in the previous reasoning are the required to the nonnegativity of the system.

V. Conclusions

In order to decide if a descriptor control system (E, A, B, C) in nonnegative we could firstly apply the Theorem 2.2. The inconvenient of this theorem is due to the fact that the Drazin inverse not always can be computed in an easy way. So, we have designed a technique that involves matrices of smaller sizes partitioning in blocks the original matrices in a suitable way. In this sense, the result given in Theorem 2.2 is improved given that now the whole matrices are not used to check the nonnegativity of the system. Moreover, the step to obtain the mentioned partition only uses permutation matrices, that is, only changes the distribution of the information in the original matrices without making any computation. Finally, note that Theorem 3.4 can be applied to a wider class of matrices than the results previously obtained in the literature.

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