On the preservation of co-positive Lyapunov functions under Padé discretization for positive systems

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Abstract— In this paper the discretization of switched and non-switched linear positive systems using Padé approximations is considered. We show:

1) diagonal Padé approximations preserve both linear and quadratic co-positive Lyapunov functions;
2) positivity need not be preserved even for arbitrarily small sampling time for certain Padé approximations.

Sufficient conditions on the Padé approximations are given to preserve positivity of the discrete-time system. Finally, some examples are given to illustrate the efficacy of our results.

I. INTRODUCTION

Switched and non-switched linear positive systems have been the subject of much recent attention [1], [2], [3], [4], [5], [6], [7], [8]. A new problem in the study of such systems concerns how to obtain discrete time approximations to a given continuous time system.

This problem arises when one simulates a given system, and when one approximates a continuous time system for control design. While a complete understanding of this problem exists for LTI systems [9], and while some results exist for switched linear systems [10], the analogous problem for switched positive systems is more challenging since discretization methods must preserve not only the stability properties of the original continuous time system, but also physical properties such as state positivity. To the best of our knowledge, no other authors have yet looked at this problem.

Specifically, in this paper we study diagonal Padé approximations to the matrix exponential, and consider their suitability for discretizing positive systems. Such a study is well motivated as diagonal Padé approximations are a method of choice amongst control engineers.

We deal with two fundamental questions:

(i) under what conditions is the stability of the original positive system preserved;
(ii) under what conditions is positivity itself preserved.

In particular, we establish the following results:

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(i) diagonal Padé approximations preserve both linear and quadratic co-positive Lyapunov functions for all sampling;
(ii) Padé approximations, of any order, do not, in general, preserve positivity, even for an arbitrary small, but positive, sampling time h, hence sufficient conditions on h and p, order of the Padé approximation, are given to preserve it.

This paper is organized as follows: in Section II the notation and preliminary definitions are introduced. In Section III a sufficient condition to find the suitable values of the sampling time h, which map Metzler matrices into positive ones, for the $p - th$ order diagonal Padé transformation is given (it involves a sub-class of the Metzler matrices); some counterexamples are also included; finally Section III presents the special case of the second order systems, for which it is possible to give a complete picture of the solution. In Section IV, the preservation of both quadratic and linear co-positive Lyapunov functions in the discretization process, for each value of the sampling time h, is presented; a result on the discretization of switched positive systems follows. Section V concludes the paper.

II. MATHEMATICAL PRELIMINARIES

A. Notation

Capital letters denote matrices, small letters denote vectors. For matrices or vectors, $(')$ indicates transpose and $(^*)$ the complex conjugate transpose. For matrices X or vectors x, the notation X or $x \succ 0$ ($\succeq 0$) indicates that X, or x, has all positive (nonnegative) inputs and it will be called a positive (non-negative) matrix or vector. The notation $X > 0$ ($X < 0$) or $X \geq 0$ ($X \leq 0$) indicates that the matrix X is positive (negative) definite or positive (negative) semi-definite. The sets of complex, real and natural numbers are denoted by $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{N}$, respectively.

A square matrix A_c is said to be Hurwitz stable if all its eigenvalues lie in the open left-half of the complex plane. A square matrix A_d is said to be Schur stable if all its eigenvalues lie inside the unit disc. A matrix A is said to be Metzler if all its extra-diagonal elements are non-negative, moreover we ask that the others are non-positive, with at least one non-zero element. B is an M-Matrix if

$B = -A$, where A is both Metzler and Hurwitz; if an M-matrix is invertible, then its inverse is nonnegative [11], [12]. The matrix I will be the identity matrix of appropriate dimensions.

B. Definitions

Positive systems [1], [13] have the peculiar property that any nonnegative input and nonnegative initial state generate a nonnegative state trajectory and output for all times. We recall here the well-known definition of positive systems for both continuous and discrete time [1]:

Definition 1 *An LTI autonomous continuous-time system*

$$\dot{x}(t) = A_c x(t), \quad x(0) = x_0 \qquad (1)$$

is positive iff A_c is a Metzler matrix.

Definition 2 *An LTI autonomous discrete-time system*

$$x(k+1) = A_d x(k), \quad x(0) = x_0 \qquad (2)$$

is positive iff A_d is a nonnegative matrix.

In the following we will tackle the discretization problem of positive systems, looking for conditions on the sampling time h in order to preserve positivity in addition to stability. Here we recall the definition of the Padé approximation to the exponential function.

Definition 3 *[14] The $[L/M]$ order Padé approximation to the exponential function e^s is the rational function C_p defined by*

$$C_p(s) = Q_L(s)Q_M^{-1}(-s)$$

where

$$Q_L(s) = \sum_{k=0}^{L} l_k s^k, \ Q_M(s) = \sum_{k=0}^{M} m_k s^k, \quad (3)$$
$$l_k = \frac{L!(L+M-k)!}{(L+M)!k!(L-k)!} \ and \ m_k = \frac{M!(L+M-k)!}{(L+M)!k!(M-k)!}. (4)$$

Thus, the diagonal Padé approximation to $e^{A_c h}$, the matrix exponential with sampling time h, is given by

$$L = M = p \, ,$$
$$C_p(A_c h) = Q_p(A_c h)Q_p^{-1}(-A_c h) \qquad (5)$$

where $Q_p(A_c h) = \sum_{k=0}^{p} c_k (A_c h)^k$ and $c_k = \frac{p!(2p-k)!}{(2p)!k!(p-k)!}$.

Much is known about the Padé maps in the context of LTI systems, particularly it is known that diagonal Padé transformations of any order preserve stability [15].

Remark 1 *The $(0, 1)$ Padé transformation preserves both stability and positivity for each choice of the sampling time h. Stability preservation is straightforward from [15]. While, since A_c is Metzler and Hurwitz and $A_d(h) = (I - h A_c)^{-1}$, nonnegativity of $A_d(h)$ follows for every value of h.*

III. Sufficient condition for diagonal Padé of the $p - th$ order

In this Section a sufficient condition to find the candidate values of h, which preserve the system's positivity, is presented for diagonal Padé with both one real root (first order) and two complex conjugate roots (second order). The decomposition in roots has been used in Section III-C in order to generalize the theory to the $p - th$ order diagonal Padé.

A. One real root - first order diagonal Padé

According to (3) and (5) the first order diagonal Padè approximation to $e^{A_c h}$ with sampling time h is given by

$$A_{d1} = \left(I + \frac{h}{2}A_c\right)\left(I - \frac{h}{2}A_c\right)^{-1}. \qquad (6)$$

It has one real root, $\alpha(h)$, and it can be expressed as a function of it:

$$A_{d1} = \left(\alpha(h)I + A_c\right)\left(\alpha(h)I - A_c\right)^{-1}, \qquad (7)$$

where $\alpha(h) = \frac{\bar{\alpha}}{h}$ and $\bar{\alpha} > 0$.

Theorem 1 *Let $A_c = \{a_{ij}\}$ be the Metzler and Hurwitz stable matrix of system (1) and A_{d1} the matrix achieved through the transformation (7). If*

$$h \leq \min_i \frac{\bar{\alpha}}{|a_{ii}|} \qquad (8)$$

then A_{d1} is nonnegative and Schur stable.

Proof: The stability proof follows from [15]. Recall that since A_c is a Metzler matrix then $(\alpha(h)I - A_c)^{-1} \succeq 0$. If we also have $\theta = \alpha(h)I + A_c \succeq 0$ then A_{d1} is non-negative. All the elements on the main diagonal of θ are $\alpha(h) - |a_{ii}|$. In order to have a nonnegative matrix $\alpha(h) \geq \max_i |a_{ii}|$, hence $h \leq \min_i \frac{\bar{\alpha}}{|a_{ii}|}$.

This concludes the proof. ∎

Remark 2 *Theorem 1 is a general condition for the generic real root of a $p - th$ order diagonal Padé. For the first order one it becomes $h \leq 2 \min_i \frac{1}{|a_{ii}|}$, since $\bar{\alpha} = 2$.*

B. Two complex conjugate roots - second order Padé

According to (3) and (5) the second order diagonal Padè approximation to $e^{A_c h}$ with sampling time h is given by

$$A_{d2} = \left(I + \frac{h}{2}A_c + \frac{h^2}{12}A_c^2\right)\left(I - \frac{h}{2}A_c + \frac{h^2}{12}A_c^2\right)^{-1}. \qquad (9)$$

It has two complex conjugate roots, $\lambda(h)$ and $\lambda^*(h)$, and it can be expressed as a function of them:

$$
\begin{aligned}
A_{d2} &= \left(|\lambda(h)|^2 I + 2Re(\lambda(h))A_c + A_c^2\right) \times \\
&\times \left(|\lambda(h)|^2 I - 2Re(\lambda(h))A_c + A_c^2\right)^{-1} = \theta_1 \theta_2^{-1}
\end{aligned}
\tag{10}
$$

where $\lambda(h) = \frac{\bar{\lambda}}{h}$ and $Re(\bar{\lambda}) > 0$, $\lambda^*(h) = \frac{\bar{\lambda}^*}{h}$ and $Re(\bar{\lambda}^*) > 0$. For the second order diagonal Padè approximation $\bar{\lambda} = 3 + 1.7321i$.

Define $A_c = \{a_{ij}\}$ and $A_c^2 = \{b_{ij}\}$ then let $\mathcal{P}$ be the set of indexes i, j, $i \neq j$, such that the ratio $\frac{a_{ij}}{b_{ij}}$ is well-defined.

Theorem 2 *Let* $A_c = \{a_{ij}\}$ *be the Metzler and Hurwitz stable matrix of system* (1) *and* A_{d2} *the matrix achieved through the transformation* (10). *If*

$$
h \leq 2Re(\bar{\lambda}) \min_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|},
\tag{11}
$$

where b_{ij} *are the elements of* A_c^2, *then* A_{d2} *is nonnegative and Schur stable.*

Proof: The stability proof follows from [15].

Recall that the sufficient condition aims to make θ_1 nonnegative and θ_2 an M-matrix.

The elements on the diagonal of θ_2 are positive, since $b_{ii} > 0$ and $a_{ii} \leq 0$, while the extra-diagonal elements are: $-2Re(\lambda(h))a_{ij} + b_{ij}$. They are all non-positive if $b_{ij} \leq 0$, $\forall i \neq j$, since $a_{ij} \geq 0$, $\forall i \neq j$; otherwise the following condition is needed:

$$
2Re(\lambda(h)) \geq \frac{b_{ij}}{a_{ij}}.
\tag{12}
$$

Similarly, consider now the extra-diagonal elements of θ_1: $2Re(\lambda(h))a_{ij} + b_{ij}$, which are nonnegative if $b_{ij} \geq 0$, $\forall i \neq j$, since $a_{ij} \geq 0$, $\forall i \neq j$; otherwise the following condition is needed:

$$
2Re(\lambda(h)) \geq \frac{|b_{ij}|}{a_{ij}}.
\tag{13}
$$

Finally, all the elements on the main diagonal of θ_1 are:

$$
|\lambda(h)^2| - 2Re(\lambda(h))|a_{ii}| + b_{ii}
\tag{14}
$$

where $b_{ii} > 0$. Recalling that $b_{ii} = a_{ii}^2 + \sum_{\substack{j=0 \\ j \neq i}}^{n} a_{ij}a_{ji}$ is easy to verify that the expression (14) is always positive for all $h > 0$.

From (12) and (13), the condition $h \leq 2Re(\bar{\lambda}) \min_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|}$ is straightforward.

This concludes the proof. ∎

Remark 3 *Theorem 2 is a general condition for all the complex conjugate roots of a* $p - th$ *order diagonal Padé. For the second order one it becomes* $h \leq 6 \min_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|}$.

C. $p - th$ order diagonal Padé transformation

Finally, we can write the conclusion of the decomposition issue, introducing a sufficient criterion to find a sampling time interval of values, which preserves both positivity and stability, for the $p - th$ order diagonal Padé transformation.

Theorem 3 *Let* A_c *be a Metzler and Hurwitz stable matrix and* $A_d(h) = C_p(A_c h)$ *be the* $p - th$ *order diagonal Padé approximation to* $e^{A_c h}$. *Let* $\bar{\alpha}_\theta$ *and* $\bar{\lambda}_\delta$, $\bar{\lambda}_\delta^*$ *the real and complex conjugate roots of the Padé, respectively. Let* $\hat{\alpha} = \min_\theta \bar{\alpha}_\theta$ *and* $\hat{\lambda} = \min_\delta Re(\bar{\lambda}_\delta)$. *Then* $A_d(h)$ *is nonnegative and Schur stable for every* $h \leq h^*$, *where*

$$
h^* = \min \left\{ \min_i \frac{\hat{\alpha}}{|a_{ii}|}, 2Re(\hat{\lambda}) \min_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|} \right\},
\tag{15}
$$

with a_{ij} *and* b_{ij} *which are the* i, j *element of* A_c *and* A_c^2 *respectively.*

Proof: Decomposing the $p - th$ order diagonal Padé approximation into real and complex conjugate roots we achieve:

$$
\begin{aligned}
A_d(h) &= \prod_{\theta=1}^{m} \left(\alpha_\theta(h)I + A_c\right) \times \\
&\times \prod_{\delta=1}^{n/2} \left(|\lambda_\delta(h)|^2 I + 2Re(\lambda_\delta(h)A_c) + A_c^2\right) \times \\
&\times \prod_{\theta=1}^{m} \left(\alpha_\theta(h)I - A_c\right)^{-1} \times \\
&\times \prod_{\delta=1}^{n/2} \left(|\lambda_\delta(h)|^2 I - 2Re(\lambda_\delta(h)A_c) + A_c^2\right)^{-1}
\end{aligned}
\tag{16}
$$

where $m + n = p$, $\alpha_\theta(h) = \frac{\bar{\alpha}_\theta}{h}$ and $\lambda_\delta(h) = \frac{\bar{\lambda}_\delta}{h}$, $\lambda_\delta^*(h) = \frac{\bar{\lambda}_\delta^*}{h}$ and $\bar{\alpha}_\theta$, $\bar{\lambda}_\delta$ and $\bar{\lambda}_\delta^*$ are the real and complex conjugate roots of the $p - th$ order diagonal Padé transformation respectively.

Every real and complex conjugate root can be analyzed as in Theorems 1 and 2, respectively, finding, for each of them, a relation like $h_\theta \leq \min_i \frac{\bar{\alpha}_\theta}{|a_{ii}|}$ and $h_\delta \leq \min_{i,j \in \mathcal{P}} 2Re(\bar{\lambda}_\delta)\frac{a_{ij}}{|b_{ij}|}$, $i \neq j$. Hence, it is sufficient choosing the minimum between $\min_\theta h_\theta$ and $\min_\delta h_\delta$, which are achieved taking into account only $\hat{\alpha}$ and $\hat{\lambda}$. The minimum between them is h^* and this completes the proof. ∎

Remark 4 *There may be cases in which the condition on the complex conjugate roots gives* $h = 0$ *as sufficient condition:* $a_{ij} = 0$ *and the corresponding* $b_{ij} \neq 0$. *However,*

there exists a subset of Metzler matrices A_c for which it is always possible to find $h^* > 0$:

$$\mathcal{A} = \left\{ A_c : \min_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|} \neq 0 \right\}, \tag{17}$$

where, as already stated previously, a_{ij} and b_{ij} are the $i, j-$th element of A_c and A_c^2 respectively.

Example 1 *Let*

$A_c = \{a_{ij}\} =$

$$= \begin{bmatrix} -17.0936 & 6.551 & 9.5974 & 7.5127 \\ 7.5469 & -11.6261 & 3.4039 & 2.551 \\ 2.7603 & 1.19 & -15.8527 & 5.0596 \\ 6.797 & 4.9836 & 2.2381 & -16.9908 \end{bmatrix},$$

that is Metzler and Hurwitz. Instead, $A_c^2 = \{b_{ij}\}$ is not Metzler. It is trivial to verify that $A_c \in \mathcal{A}$. We use, for example, the $5-th$ order Padé transformation, whose roots are $\alpha = 7.2935$, $\lambda_1 = 4.6493 + 7.1420i$, $\lambda_1^* = 4.6493 - 7.1420i$, $\lambda_2 = 6.7039 + 3.4853i$ and $\lambda_2^* = 6.7039 - 3.4853i$. Applying Theorem 3, $h \leq 0.3221$, which gives as result, with $h = 0.3221$:

$$A_d = \begin{bmatrix} 0.2911 & 0.3107 & 0.2856 & 0.2644 \\ 0.2855 & 0.3077 & 0.2762 & 0.2550 \\ 0.1446 & 0.1527 & 0.1431 & 0.1327 \\ 0.2226 & 0.2372 & 0.2149 & 0.2007 \end{bmatrix},$$

which is positive and Schur stable.

D. Counterexample for the sufficient condition on h

When $A_c \notin \mathcal{A}$ the sufficient condition introduced in Theorem 3 returns $h = 0$ for every $p \geq 2$, because of the condition on the complex conjugate roots. It could be interesting, therefore, to understand if the sufficient condition found with the first order Padé transformation, $h_1 \leq h_1^*$, can be used with higher orders Padé, $h_p^\nu \geq h_1^*$; where h_p^* is the value found applying the sufficient condition of Theorem 3 to the $p-th$ order Padé transformation and h_p^ν is the real maximum suitable value of the sampling time h which makes the $p-th$ order Padé transformation mapping Metzler into nonnegative matrices. Such a value can be found by simulation.

Two scenarios are possible:

1) $p-th$ diagonal Padé transformation, with $p > 1$, maps Metzler and Hurwitz matrices into nonnegative and Schur ones when h belongs to an interval $[0, h_p^\nu]$, with $h_p^\nu > h_1^*$;

2) there is a gap into the interval of suitable sampling time values for some values of p.

Therefore, in case 1, see Example 2, it is possible to use h_1^*, i.e., the sufficient condition found for the first order diagonal Padé transformation, with higher order diagonal Padé. In the second case, instead, it is not possible to use it *apriori*, since the interval found is not included in the suitable h values for higher orders transformations, even if the intersection can be not empty, see Example 3.

Example 2 *Let*

$A_c = \{a_{ij}\} =$

$$= \begin{bmatrix} -0.5369 & 0.2920 & 0 & 0 & 0 \\ 0.5269 & -0.3175 & 0.3724 & 0 & 0 \\ 0 & 0.0155 & -0.8721 & 0.0527 & 0 \\ 0 & 0 & 0.4897 & -0.3318 & 0.4177 \\ 0 & 0 & 0 & 0.2691 & -0.4277 \end{bmatrix},$$

that is Metzler and Hurwitz, while $A_c^2 = \{b_{ij}\}$ is not Metzler. Applying Theorem 3 with $p = 1$ and, for example, $p = 2$ we find $h_1^* = 2.2933$ and $h_2^* = 0$, because of the zero elements in A_c and the corresponding nonzero elements in A_c^2. Still, it is possible to find, numerically, the real sampling time intervals that achieve a nonnegative discretization for both the first and the second order diagonal Padé and they are: $[0, h_1^\nu] = [0, 2.37]$ and $[0, h_2^\nu] = [0, 17.37]$.

Hence, in this case, it was possible to use h_1^* also for the second order Padé transformation and, according to simulations, it works for higher orders too.

Example 3 *Let*

$A_c = \{a_{ij}\} =$

$$= \begin{bmatrix} -10^{-3} & 1 & 0 & \dots & 0 & 0 \\ 0 & -10^{-3} & 1 & \dots & 0 & 0 \\ 0 & 0 & -10^{-3} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -10^{-3} & 1 \\ 0 & 0 & 0 & \dots & 0 & -10^{-3} \end{bmatrix}$$

that is a 10×10 Metzler and Hurwitz matrix, while $A_c^2 = \{b_{ij}\}$ is not Metzler. Applying Theorem 3 with $p = 1$ and, for example, $p = 2$ we find $h_1^* = 2000$ and $h_2^* = 0$, because of the zero elements in A_c and the corresponding nonzero elements in A_c^2. In this case, if we look numerically for the real sampling time intervals, which achieve a nonnegative discretization for both the first and the second order diagonal Padé, we find: $[0, h_1^\nu] = [0, 2000]$ and $\{0\} \bigcup [h_2^-, h_2^\nu] = \{0\} \bigcup [1512.2, 3464]$. The former is not a subset of the latter, hence it is not possible to use a-priori a value of the first interval with the second order Padé transformation, even if it is still possible to find an interval of values of h that satisfies both the transformations.

The not trivial issue is to distinguish a-priori between cases 1 and 2.

E. Second order systems

The two cases shown in Section III-D cannot happen when A_c is a second order matrix. In this case A_c always belongs to $\mathcal{A}$, since $a_{ij} = 0$ implies that $|b_{ij}| = 0$ too, therefore the sufficient condition formulated in Theorem 3 always succeeds in finding a solution.

For the second order systems, $n = 2$, the sampling time h that satisfies the sufficient condition for the first order Padé achieves a nonnegative discretization using the second order Padé too.

Lemma 1 *Let A_c be a 2×2 Metzler and Hurwitz stable matrix. If $A_{d1} = (\alpha(h)I + A_c)(\alpha(h)I - A_c)^{-1} \succeq 0$ then $\tilde{A}_c = 2Re(\lambda(h))\frac{\alpha(h)}{|\lambda(h)|}\frac{A_c}{|\lambda(h)|}\left(I + \frac{A_c^2}{|\lambda(h)|^2}\right)^{-1}$ is a Metzler and Hurwitz matrix. Here $\alpha(h)$ and $\lambda(h)$ are the roots of the first order and second order Padé respectively, and both functions of the sampling time h:*

$$\alpha(h) = \frac{\bar{\alpha}}{h}, \quad \lambda(h) = \frac{\bar{\lambda}}{h}.$$

Proof: Define

$$A_{d1} = (\alpha(h)I + A_c)(\alpha(h)I - A_c)^{-1}$$
$$A_{d2} = (|\lambda(h)|^2 I + 2Re(\lambda(h))A_c + A_c^2) \times \qquad (18)$$
$$\times (|\lambda(h)|^2 I - 2Re(\lambda(h))A_c + A_c^2)^{-1}$$

where A_{d1} is the matrix achieved using the first order Padé, which has a real root $\alpha(h)$, and A_{d2} is the one achieved through the second order Padé, whose roots are complex conjugate: $\lambda(h)$ and $\lambda^*(h)$. $\tilde{A}_c$ is the matrix achieved applying the inverse Padé of the first order to A_{d2}:

$$\tilde{A}_c = (A_{d2} - I)(A_{d2} + I)^{-1}\alpha(h) =$$
$$= 2Re(\lambda(h))\frac{\alpha(h)}{|\lambda(h)|}\frac{A_c}{|\lambda(h)|}\left(I + \frac{A_c^2}{|\lambda(h)|^2}\right)^{-1}. \qquad (19)$$

Let A_c be a 2×2 Metzler and Hurwitz matrix:

$$A_c = \begin{bmatrix} -a & b \\ c & -d \end{bmatrix},$$

where $a, b, c, d \geq 0$, $(a + d) > 0$ and $ad - bc > 0$.

$$A_{d1} = (\alpha(h)I + A_c)(\alpha(h)I - A_c)^{-1} =$$
$$= \frac{1}{\triangle_1}\begin{bmatrix} \beta_1 & 2b\alpha(h) \\ 2c\alpha(h) & \beta_2 \end{bmatrix}, \qquad (20)$$

where $\beta_1 = \alpha^2(h) - (a-d)\alpha(h) - ad + bc$, $\beta_2 = \alpha^2(h) + (a-d)\alpha(h) - ad + bc$ and $\triangle_1 = \alpha^2(h) + (a+d)\alpha(h) + ad - bc > 0$ since A_c Hurwitz and the Padé roots have positive real part.
A_{d1} is nonnegative if $ad - bc \leq \alpha^2$ and this condition is fulfilled if the sampling time h is such that $\alpha(h) \geq \max\{a, d\}$, i.e. $\alpha(h) \geq \max_i |a_{ii}|$, hence $h \leq \min_i \frac{\bar{\alpha}}{|a_{ii}|}$.

From (19) it follows:

$$\tilde{A}_c = 2Re(\lambda(h))\frac{\alpha(h)}{|\lambda(h)|}\frac{A_c}{|\lambda(h)|}\left(I + \frac{A_c^2}{|\lambda(h)|^2}\right)^{-1} =$$
$$= \frac{2Re(\lambda(h))}{\triangle_2}\frac{\alpha(h)}{|\lambda(h)|}\begin{bmatrix} \phi_1(h) & \phi_2(h) \\ \phi_3(h) & \phi_4(h) \end{bmatrix}, \qquad (21)$$

where

$$\phi_1(h) = -\frac{a}{|\lambda(h)|} - \frac{d}{|\lambda(h)|}\frac{ad-bc}{|\lambda(h)|^2}$$
$$\phi_2(h) = \frac{b}{|\lambda(h)|}\left(1 - \frac{ad-bc}{|\lambda(h)|^2}\right)$$
$$\phi_3(h) = \frac{c}{|\lambda(h)|}\left(1 - \frac{ad-bc}{|\lambda(h)|^2}\right) \qquad (22)$$
$$\phi_4(h) = -\frac{d}{|\lambda(h)|} - \frac{a}{|\lambda(h)|}\frac{ad-bc}{|\lambda(h)|^2}$$
$$\triangle_2 = \frac{(ad-bc)^2}{|\lambda|^4} + \frac{a^2 + d^2 + 2bc}{|\lambda|^2} + 1.$$

The elements on the main diagonal of $\tilde{A}_c$ are always negative, while the extra-diagonal elements are positive if $1 - \frac{ad-bc}{|\lambda(h)|^2} > 0$, that is true if $\alpha(h)^2 < |\lambda(h)|^2$. This last inequality is always verified since the root of the first order Padé $\alpha(h)$ is greater than 1 and less than the absolute value of the root $\lambda(h)$ of the second order Padé.
$\tilde{A}_c$ is also Hurwitz. Indeed, consider a complex number $x + iy$ with $x < 0$ and apply the transformation $(x+iy)(I + (x+iy)^2)^{-1}$:

$$\frac{x+iy}{1+(x+iy)^2} = \frac{x(1+x^2+y^2) + iy(1-x^2-y^2)}{(1+x^2-y^2)^2 + 4x^2y^2} \qquad (23)$$
$$Re\left(\frac{x+iy}{1+(x+iy)^2}\right) = \frac{x(1+x^2+y^2)}{(1+x^2-y^2)^2 + 4x^2y^2} < 0, \ \forall x < 0. \qquad (24)$$

This concludes the proof. ∎

Theorem 4 *Let A_c be a 2×2 Metzler and Hurwitz stable matrix. If $A_{d1} = C_1(A_c h)$ is the first order diagonal Padé approximation of $e^{A_c h}$ and is nonnegative and Schur stable for any $h_1 \leq h_1^*$, then $A_{d2} = C_2(A_c h)$, second order diagonal Padé approximation of $e^{A_c h}$, is nonnegative and Schur stable for any $h_2 \leq h_2^*$, with $h_2^* \geq h_1^*$.*

Proof: According to Lemma 1 if A_{d1} is nonnegative and Schur $\forall h_1 \leq h_1^*$, then $\tilde{A}_c$ is Metzler and Hurwitz $\forall h_1 \leq h_1^*$. This implies that, for the same values of the sampling time, A_{d2} is nonnegative and Schur too. Indeed,

$$A_{d2} = \left(I + \frac{\tilde{A}_c}{\alpha(h_1)}\right)\left(I - \frac{\tilde{A}_c}{\alpha(h_1)}\right)^{-1}. \qquad (25)$$

The second matrix of the product in (25) is the inverse of an M-matrix, hence it is nonnegative. Moreover, since $\tilde{A}_c$ is Metzler, only the elements on the main diagonal of the

first matrix need to be checked, if they are nonnegative A_{d2} will be the product of two nonnegative matrices.

$$\left(I + \frac{\tilde{A}_c}{\alpha(h_1)}\right) = I + \frac{2Re(\lambda(h_1))}{\triangle_2} \frac{1}{|\lambda(h_1)|} \begin{bmatrix} \phi_1(h_1) & \phi_2(h_1) \\ \phi_3(h_1) & \phi_4(h_1) \end{bmatrix} \tag{26}$$

where $\phi_i(h_1)$ are defined as in (22). Recalling the definition of $\triangle_2$ in (22) and that $Re(\lambda(h_1)) \leq |\lambda(h_1)|$, it is easy to see that the elements on the main diagonal are positive. This concludes the proof. ∎

Remark 5 *Let α_1 be the real root of the first order Padé transformation and λ_2, λ_2^* be the complex conjugate roots of the second order one. We have proved in Theorem 4 that $h_1^* \leq h_2^*$, i.e. $\min\limits_i \frac{\alpha_1}{|a_{ii}|} < 2Re(\lambda_2) \min\limits_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|}$. Moreover, by simulation it is possible to notice that the minimum absolute value of the real part of the Padé roots increases with the transformation order, even if we have not been able, by now, to prove it analytically. Since $|a_{ii}|$, a_{ij} and $|b_{ij}|$ do not change with the Padé order, a straightforward consequence is that the sufficient condition of Theorem 3 will always result in increasing values of h. Therefore the value of h found with the first order Padé maps Metzler 2×2 matrices into positive ones for every Padé order p.*

IV. Lyapunov function preservation

In the following section we will present some preliminary results on the Lyapunov functions preservation through the Padé approximation [16]. Particularly both the quadratic and linear co-positive Lyapunov function, see [17], [3] and [18], are preserved.

As we have shown in the previous, positivity preservation is not a trivial issue.

Lemma 2 *Let A_c be a Metzler and Hurwitz stable matrix and $A_d = C_1(A_ch)$ be the first order diagonal Padé approximation of e^{A_ch}. Moreover assume $h \leq \min\limits_i \frac{\bar{\alpha}}{|a_{ii}|}$, where a_{ii} are the elements of the main diagonal of the matrix A_c and $\bar{\alpha}$ is the real root of the first order diagonal Padé approximation. Then*

1) $A_d(h)$ is a nonnegative and Schur matrix;

2) if $v(x) = x'Px$, with $P = P' > 0$, is a quadratic Lyapunov function for A_c, that is

$$x'(A_c'P + PA_c)x \prec 0, \ \forall x \succ 0 \tag{27}$$

then $v(x)$ is a quadratic Lyapunov function for $A_d(h)$ too, that is

$$x'(A_d'PA_d - P)x \prec 0, \ \forall x \succ 0; \tag{28}$$

3) if $v(x) = w'x$, $w \succ 0$ is a linear co-positive Lyapunov function for A_c, that is

$$w'A_c \prec 0,$$

then $v(x)$ is a linear co-positive Lyapunov function for $A_d(h)$ too, that is

$$w'A_d \prec w'.$$

Proof: Since A_d is the first order diagonal Padé approximation, it can be decomposed in its real root $\alpha(h) = \frac{\bar{\alpha}}{h}$: $A_d(h) = (\alpha(h)I + A_c)(\alpha(h)I - A_c)^{-1}$, where $\alpha(h) = \frac{\bar{\alpha}}{h}$, $\bar{\alpha} \succ 0$. For the first order Padé $\bar{\alpha} = 2$.

1) $A_d(h)$ is Schur according to [15], since the transformation used is the Padé $(1,1)$. Moreover, let $a_{ij} \geq 0$ be the elements of the Metzler matrix A_c, which has the following structure:

$$\begin{pmatrix} -|a_{11}| & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & -|a_{nn}| \end{pmatrix}; \tag{29}$$

$A_d(h)$ is also nonnegative if $\frac{\bar{\alpha}}{h} \geq \max\limits_i |a_{ii}|$, that is if $\bar{\alpha} \geq h \max\limits_i |a_{ii}|$ or, equivalently, $h \leq \min\limits_i \frac{\bar{\alpha}}{|a_{ii}|}$, since it becomes the product of two nonnegative matrices, recalling that the second one is the inverse of an M-matrix.

2) From (27) follows that (28) becomes

$$x'(A_d'PA_d - P)x =$$
$$= x'[(\alpha(h)I - A_c)^{-'}(\alpha(h)I + A_c)' \times$$
$$\times P(\alpha(h)I + A_c)(\alpha(h)I - A_c)^{-1} - P]x =$$
$$= x'(\alpha(h)I - A_c)^{-'} \times$$
$$\times [(\alpha(h)I + A_c)'P(\alpha(h)I + A_c) +$$
$$- (\alpha(h)I - A_c)'P(\alpha(h)I - A_c)] \times$$
$$\times (\alpha(h)I - A_c)^{-1}x =$$
$$= x'(\alpha(h)I - A_c)^{-'}[2\alpha(h)(A_c'P + PA_c)] \times$$
$$\times (\alpha(h)I - A_c)^{-1}x \prec 0, \ \forall x \succ 0$$

since (27) holds and the matrix $(\alpha(h)I - A_c)$ is an M-matrix, hence its inverse is nonnegative for all h.

3) Following the same rationale of the previous point we can write:

$$w'A_d - w' = 2w'A_c(\alpha(h)I - A_c)^{-1} \prec 0.$$

This completes the proof. ∎

Lemma 3 *Let A_c be a Metzler and Hurwitz stable matrix. Let $\theta_1 = (\lambda(h)I + A_c)(\lambda^*(h)I + A_c)$, $\theta_2 = (\lambda(h)I - A_c)(\lambda^*(h)I - A_c)$ and, finally, define*

$$A_d(h) = \theta_1\theta_2^{-1} \tag{30}$$

and assume that $\lambda = \frac{\bar{\lambda}}{h}$, that is a complex number with $Re(\lambda) > 0$, is such that θ_2 is an M-matrix and θ_1 is a nonnegative one. Then

1) *if $v(x) = x'Px$, with $P = P' > 0$, is a quadratic Lyapunov function for A_c, that is*

$$x'(A_c'P + PA_c)x \prec 0, \ \forall x \succ 0 \qquad (31)$$

then $v(x)$ is a quadratic Lyapunov function for $A_d(h)$ too, that is

$$x'(A_d'PA_d - P)x \prec 0, \ \forall x \succ 0; \qquad (32)$$

2) *if $v(x) = w'x$, $w \succ 0$ is a linear co-positive Lyapunov function for A_c, that is*

$$w'A_c \prec 0,$$

then $v(x)$ is a linear co-positive Lyapunov function for $A_d(h)$ too, that is

$$w'A_d \prec w'.$$

Proof:

1) Recalling (30), we want to show that $x'(A_c'P + PA_c)x < 0$ implies $x'(A_d'PA_d - P)x < 0 \ \forall x > 0$. Indeed,

$$x'(A_d'PA_d - P)x =$$
$$= x'\theta_2^{-'}[(|\lambda(h)|^2 I + 2Re(\lambda(h))A_c + A_c^2)' P \times$$
$$\times (|\lambda(h)|^2 I + 2Re(\lambda(h))A_c + A_c^2) - \theta_2'P\theta_2]\theta_2^{-1}x =$$
$$= x'\theta_2^{-'}[4|\lambda(h)|^2 Re(\lambda(h))(A_c'P + PA_c) +$$
$$+ 4Re(\lambda(h))A_c'(A_c'P + PA_c)A_c]\theta_2^{-1}x$$
$$\qquad (33)$$

that is negative for all $x \succ 0$.

2) Following the same rationale of the previous point we can write:

$$w'A_d - w' = 4Re(\lambda(h))w'A_c (\lambda(h)I - A_c)^{-1} \times$$
$$\times (\lambda^*(h)I - A_c)^{-1} \prec 0.$$

This concludes the proof. ∎

Remark 6 *As shown in Theorem 3, it is possible to find a sufficient condition on the real part of the roots $\lambda(h)$ of the complex Padé transformation in order to achieve a nonnegative matrix A_d. Calling a_{ij} the elements of A_c and b_{ij} the elements of the matrix A_c^2, the condition on $\lambda(h)$ would be $Re(\lambda(h)) \geq \frac{1}{2} \max_{i,j \in \mathcal{P}} \frac{|b_{ij}|}{a_{ij}}$. This condition can be formulated using the sampling time h, recalling that $\lambda(h) = \frac{\bar{\lambda}}{h}$: $h \leq 2Re(\lambda) \min_{i,j \in \mathcal{P}} \frac{a_{ij}}{|b_{ij}|}$. Still, as it has been shown in Section III-D, there can be cases in which the result of this sufficient condition is $h = 0$, because $a_{ij} = 0$ and $b_{ij} \neq 0$, that is unacceptable.*

It is now possible to introduce the discretization of switching positive systems by $p - th$ order diagonal Padé transformation.

Consider a continuous-time switched linear positive systems of the general form

$$\dot{x}(t) = A_{c\sigma(t)}x(t), \quad x(0) = x_0, \qquad (34)$$

defined for all $t \geq 0$, where $x(t) \in \mathbb{R}_+^n$ is the state variable vector, $\sigma(t) \in \{1, 2, \dots, N\}$ is the switching rule, $x_0 \in \mathbb{R}_+^n$ is the initial condition and A_{ci} belongs to the set $\{A_{c1}, \dots, A_{cN}\}$. In order to be a positive system A_{ci} has to be Metzler, i.e. $a_{ij} \geq 0$, $\forall(i,j)$, $i \neq j$ and $a_{ii} \leq 0$, $\forall i$.

Theorem 5 *Consider system (34). Let A_{ci} be a Metzler and Hurwitz stable matrix for all $i = 1, \dots, N$ and $A_{di}(h_i) = C_p(A_{ci}h_i)$ be the $p-th$ order diagonal Padé approximation of $e^{A_{ci}h_i}$, with $h_i \leq h_i^*$, h_i^* achieved using the condition (15) of Theorem 3.*

Let $h^* = \min_i h_i^*$, then $\forall h \leq h^*$ the discretized system

$$x(k+1) = A_{d\mu}(h)x(k) \qquad (35)$$

is positive, where $\mu = \in \{1, 2, \dots, N\}$.

Moreover, if there exists a common quadratic or linear co-positive Lyapunov function for system (34), then the origin $x = 0$ is globally uniformly asymptotically stable for system (35) too under arbitrary switching.

Proof: Positivity of system (35) is straightforward from Theorem 3, since the minimum condition on h, $h \leq h^*$, is taken into account in order to guarantee the positivity of each matrix $A_{di}(h)$; hence the switching system achieved applying the $p - th$ order diagonal Padé transformation is positive under arbitrary switching.

Moreover, following the idea expressed in [16], we recall that $A_{di}(h) = C_p(A_{ci}h) = Q_p(A_{ci}h)Q_p(-A_{ci}h)^{-1}$ and that $A_{di}(h)$ can be factored as in the Proof of Theorem 3.

Since all the constituent matrices commute with each other, we can apply Lemmas 2 and 3; then, if there exists a common quadratic or linear co-positive Lyapunov function for system (34), it is preserved by the $p - th$ order diagonal Padé approximation.

Recalling that if there exists a common Lyapunov function for a switching system, it is globally uniformly asymptotically stable under arbitrary switching [5], [19], the system (35) is globally uniformly asymptotically stable under arbitrary switching and positive for all $h \leq h^*$. ∎

Remark 7 *If $A_{ci} \in \mathcal{A} \ \forall i$, then it is always possible to find $h \neq 0$ for which system (35) is positive.*

V. Conclusions

In this paper we cope with the effect of the $p - th$ Padé transformations on positive systems. We provide sufficient conditions for both positivity and quadratic and linear co-positive Lyapunov function preservation; an analysis of

switched positive systems follows. For second order systems a complete picture of the solution is provided; for higher order systems a few conjectures are included showing the peculiarity of the positivity property under discretization. Many examples and counterexamples are also given to illustrate the efficacy of our results.

REFERENCES

[1] S. Rinaldi and L. Farina, *Positive Linear Systems*. Wiley Interscience Series, 2000.

[2] L. Benvenuti, L. Farina, B. Anderson, and F. De Bruyne, "Minimal positive realizations of transfer functions with positive real poles," *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, vol. 47, no. 9, pp. 1370–1377, Sep 2000.

[3] R. Shorten and O. Mason, "On linear copositive Lyapunov functions and the stability of switched positive linear systems," *IEEE Transactions on Automatic Control*, vol. 52, pp. 1346–1349, 2007.

[4] M. Arcat and E. Sonntag, "Diagonal stability for a class of cyclic systems and applications," 2005, submitted to Automatica.

[5] D. Liberzon, *Switching in Systems and Control*. Birkhäuser, 2003.

[6] R. Shorten, F. Wirth, and D. J. Leith, "A positive systems model of tcp-like congestion control: asymptotic results," *IEEE/ACM Transactions on Networking*, vol. 14, no. 3, pp. 616–629, June 2006.

[7] M. F. Barnsley, S. G. Demko, J. H. Elton, and J. S. Geronimo, "Erratum invariant measures for markov processes arising from iterated function systems with place-dependent probabilities," *Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques*, vol. 25, no. 4, pp. 589–590, 1989.

[8] D. Hartfiel, *Nonhomogeneous matrix products*. World Scientific, 2002.

[9] L. C. Westphal, *Handbook of Control Systems Engineering*. Springer, 2001.

[10] F. Rossi, P. Colaneri, and R. Shorten, "Padé discretization for linear systems with polyhedral Lyapunov functions," *Automatica, submitted, 2009. Preprint available under request*.

[11] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.

[12] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. SIAM, 1994.

[13] L. Benvenuti and L. Farina, "Eigenvalue regions for positive systems," *Systems and Control Letters*, vol. 51, pp. 325–330, 2004.

[14] J. G. A. Baker and P. G. Morris, *Padé Approximants, 2nd edition*. Cambridge University Press, 1996.

[15] J. C. Butcher, "The A-Stability of Methods with Padé and Generalized Padé Stability Functions," *Numerical Algorithms*, vol. 31, pp. 47–58, 2002.

[16] S. Sajja, M. Corless, R. Shorten, and S. Solmaz, "On Padé Approximations and the Preservation of Quadratic Stability for Switched Linear Systems," *Submitted to Automatica, preprint available under request*.

[17] L. Gurvits, R. Shorten, and O. Mason, "On the stability of switched positive linear systems," *IEEE Transactions on Automatic Control*, vol. 52, pp. 1099–1103, 2007.

[18] R. Shorten and O. Mason, "Quadratic and copositive Lyapunov functions and the stability of positive switched linear systems," in *Proceedings of the American Control Conference*, 2007.

[19] A. Zappavigna, P. Colaneri, J. C. Geromel, and R. Shorten, "Dwell time analysis for Continuous-Time Switched Linear Positive Systems," in *Proceedings of the American Control Conference*, 2010.