An Output Control of a Class of Discrete Second-order Repetitive Processes

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Abstract— **This paper presents new computationally efficient LMI results on stability and feedback stabilization for a class of ill-conditioned discrete, linear second-order repetitive processes, including the uncertain case. The results are derived via transformation of the second-order system to an equivalent first-order descriptor system, thus avoiding the necessity of inversion of an ill-conditioned leading coefficient matrix of the system, which is allowed also to be uncertain. This last feature, as frequently occurs, is of a great significance but the known approaches do not provide the easy way to solve this problem.**

I. Introduction

Second order linear control systems arise in a wide variety of practical applications involving vibrating structures, power systems, economics, computer networks, etc.

An usual way to solve a control problem for a second order model is to transform it to a standard first order state space form and then use the excellent computational methods now available for control problems for such systems (see [6]). Unfortunately, such reduction requires explicit computation of the inverse of the leading coefficient matrix, which could be numerically problematic due to a possible ill-conditioning of this matrix. For example, in vibration control analysis, this matrix, called the mass matrix, is often diagonal and therefore, can be ill-conditioned, whenever some diagonal entries are small (see [10]). To overcome such difficulties research has been focussed in recent years on developing methods for second-order control problems that do not require such explicit computation of the inverse. As a result, such methods have been developed for a few important problems including stability, feedback stabilization, partial pole placement, robust pole placement, model reduction, and others. These methods work either with the equivalent descriptor systems or directly with the coefficient matrices of the second-order systems without making any a priori transformation to the standard state-space or descriptor form [7], [8], [3], [9], [11]. The latter approach has the further advantage that the special structures such as the sparsity, bandness, and positive definiteness of the coefficient matrices, very often offered by large practical problems and considered as assets for large-scale computing for storage and computational costs, can be preserved and be exploited in a computational setting. These papers have so far considered control problems for second order systems in the continuous case only.

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The essential unique characteristic of a repetitive, or multipass [14], process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

Physical examples of repetitive processes include long-wall coal cutting and metal rolling operations [14], [13]. Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications include classes of iterative learning control schemes [1] and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle [12]. In this last case, for example, use of the repetitive process setting provides the basis for the development of highly reliable and efficient solution algorithms and in the former it provides a stability theory which, unlike alternatives, provides information concerning an absolutely critical problem in this application area, i.e. the trade-off between convergence and the learnt dynamics.

Attempts to control these processes using standard (or 1D) systems theory/algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2D systems structure, i.e. information propagation occurs from pass-to-pass and along a given pass and also the initial conditions are reset before the start of each new pass.

In this paper we develop a new computationally efficient approach to the stability analysis for the extended class of repetitive processes, which are second order in the along the pass direction and moreover are written in the descriptor form with ill-conditioning occurrence, i.e. the respective left hand side matrix although is nonsingular possesses at least one very small, close to zero eigenvalue, which makes very difficult the matrix inversion. Our aim is to develop a new approach, which is based on the Lyapunov theory and an extensive use of Linear Matrix Inequalities (LMI) [2], and moreover does not require this troublesome matrix inversion.

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Previous work [5] has lead to the results providing a major step towards a comprehensive and implementable solution to the general problem of control law design. However, the methods developed there had still some computational disadvantages and allowed only the very limited uncertainty analysis. Also, the state controller was used whose implementation would require the direct state measurements or the use of the state observer. In this paper, we extend the results to the more computationally efficient case, which also allows a more general uncertainty analysis, and what is more, we use the output only controller.

Throughout this paper, the null matrix and the identity matrix with the required dimensions are denoted by 0 and I, respectively. Moreover, $M > 0$ (< 0) denotes a real symmetric positive (negative) definite matrix.

II. PRELIMINARIES AND THE NEW MODEL

The most basic discrete linear repetitive process state-space model [14], [13] has the following form over $0 \leq p \leq \alpha - 1$, $k \geq 0$

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p) \end{aligned} \tag{1}$$

Here on pass k, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the pass profile vector, and $u_k(p) \in \mathbb{R}^r$ is the vector of control inputs, and $\alpha \in \mathcal{N}$ denotes the constant and fixed pass length. The boundary conditions (i.e. the pass state initial vector sequence and the initial pass profile) are

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \geq 0 \\ y_0(p) &= f(p), \quad 0 \leq p \leq \alpha - 1 \end{aligned} \tag{2}$$

where the $n \times 1$ vector d_{k+1} has known constant entries and $f(p)$ is an $m \times 1$ vector whose entries are known functions of p.

The discrete linear repetitive process second order in the along the pass direction can be modelled as

$$\begin{aligned} A_0 x_{k+1}(p+2) &= A_1 x_{k+1}(p+1) + A_2 x_{k+1}(p) \\ &+ B_1 u_{k+1}(p+1) + B_{00} y_k(p+1) \\ y_{k+1}(p+1) &= C_1 x_{k+1}(p+1) + C_2 x_{k+1}(p) \\ &+ D u_{k+1}(p+1) + D_0 y_k(p+1) \end{aligned} \tag{3}$$

Here, the notations are the same as for the model of (1) with only the necessity to extend the boundary conditions of (2) with the additional

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}(0), \quad & k &= 0,1,\ldots \\ x_{k+1}(1) &= e_{k+1}(1), \quad & k &= 0,1,\ldots \\ y_0(p) &= f_0(p), \quad & p &= 0,1,\ldots,(\alpha-2) \end{aligned} \tag{4}$$

where $f_0(p)$ similarly as a vector $f(p)$ is an $m \times 1$ vector whose entries are known functions of p and $d_{k+1}(0)$, $e_{k+1}(1)$ are an $n \times 1$ vectors. The model (3) can be transformed to the general 1st order form

$$\begin{aligned} \Psi_0 X_{k+1}(p+1) &= \hat{A} X_{k+1}(p) + \hat{B} U_{k+1}(p) + \hat{B}_0 Y_k(p) \\ Y_{k+1}(p) &= C X_{k+1}(p) + D U_{k+1}(p) \\ &+ D_0 Y_k(p) \end{aligned} \tag{5}$$

where
$X_{k+1}(p) = \begin{bmatrix} x_{k+1}(p+1) \\ x_{k+1}(p) \end{bmatrix}$, $Y_k(p) = y_k(p+1)$, $U_k(p) = u_k(p+1)$
and
$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$, $\hat{B}_0 = \begin{bmatrix} B_{00} \\ 0 \end{bmatrix}$,
$\hat{A} = \begin{bmatrix} A_1 & A_2 \\ I_n & 0 \end{bmatrix}$, $\Psi_0 = \begin{bmatrix} A_0 & 0 \\ 0 & I_n \end{bmatrix}$.

Obviously left multiplying the second equation of (5) by the matrix Ψ_0^{-1} yields the repetitive process model of the most common form of (1). However, due to that this matrix is ill-conditioned such an approach is not computationally efficient.

III. STABILITY

The unique control problem for repetitive processes is that the output sequence of pass profiles can contain oscillations that increase in amplitude in the pass-to-pass direction, and the stability theory [13] for linear repetitive processes is based on an abstract model in a Banach space setting. This is of the form $y_{k+1} = L_\alpha y_k$, where $y_k \in E_\alpha$ (E_α a Banach space with norm $||\cdot||$) and L_α is a bounded linear operator mapping E_α into itself. In the case of examples described by (1) and (2), L_α is the convolution operator for a 1D discrete linear system with (state, input, output and direct feedthrough respectively) state-space model matrices $\{A, B_0, C, D_0\}$.

The natural approach to a definition of stability for these processes is to ask that given any initial profile y_0 and any disturbance b_{k+1} that converges strongly to b_∞ as $k \to \infty$, the sequence of pass profiles converges to y_∞ as $k \to \infty$. This is termed asymptotic stability in the repetitive process setting [13] and it can be shown to be equivalent to the existence of real scalars $M_\alpha > 0$ and $\lambda_\alpha \in (0,1)$ such that $||L_\alpha^k|| \leq M_\alpha \lambda_\alpha^k$, $k \geq 0$, where $||\cdot||$ also denotes the induced norm. This requires that a bounded initial pass profile produces a bounded sequence of pass profiles (BIBO stability), and the necessary and sufficient condition for this property is that $r(L_\alpha) < 1$ where $r(\cdot)$ denotes the spectral radius. In the case of processes described by (1) and (2) the necessary and sufficient condition is $r(\hat{D}_0) < 1$, i.e., all eigenvalues of $\hat{D}_0$ must lie in the open unit circle in the complex plane.

If asymptotic stability holds, and the input sequence applied $\{u_{k+1}\}_k$ converges strongly as $k \to \infty$ to u_∞ the sequence of pass profiles produced converges strongly in k to the so-called limit profile $y_\infty = \lim_{k\to\infty} y_k$ with state-space model

$$\begin{aligned} x_\infty(p+1) =& (\hat{A} + \hat{B}_0(I - \hat{D}_0)^{-1}\hat{C}) x_\infty(p) \\ &+ (\hat{B} + \hat{B}_0(I - \hat{D}_0)^{-1}\hat{D}) u_\infty(p), \\ y_\infty(p) =& (I - \hat{D}_0)^{-1}\hat{C} x_\infty(t) + (I - \hat{D}_0)^{-1}\hat{D} u_\infty(p) \\ x_\infty(0) =& d_\infty \end{aligned} \tag{6}$$

where (again a strong limit) $d_\infty := \lim_{k\to\infty} d_k$ and the matrix $I - \hat{D}_0$ is invertible since $r(\hat{D}_0) < 1$ by asymptotic stability. In physical terms, this result states that under asymptotic stability the repetitive dynamics can, after a "sufficiently large" number of passes have elapsed, be replaced by those of a 1D discrete linear system.

Asymptotic stability does not guarantee that the limit profile has acceptable along the pass dynamics since it can be unstable in the 1D linear systems sense. As an example, consider the case when $\hat{A} = -0.5$, $\hat{B} = 1$, $\hat{B}_0 = 0.5+\gamma$, $\hat{C} = 1$, $\hat{D} = 0$, $\hat{D}_0 = 0$, where γ is a real scalar. Asymptotic stability holds in this case with resulting limit profile

$$y_\infty(p+1) = \gamma y_\infty(p) + u_\infty(p).$$

Hence for $|\gamma| > 1$, the sequence of pass profiles converge (in the pass-to-pass direction (k)) to an unstable 1D discrete linear system. Note also that this occurs even though the state matrix $\hat{A}$ is stable in the 1D sense.

To prevent cases such as the above example from arising, the BIBO property is imposed for any possible value of the pass length (mathematically this can be analyzed by letting $\alpha \to \infty$). This is the stability along the pass property which requires the existence of finite real scalars $M_\infty > 0$ and $\lambda_\infty \in (0,1)$, which are independent of α, such that $||L_\alpha^k|| \le M_\infty \lambda_\infty^k$, $k \ge 0$. For the processes considered here this requires that (i) $r(\hat{D}_0) < 1$ (asymptotic stability), (ii) $r(\hat{A}) < 1$ and (iii) all eigenvalues of the transfer-function matrix $G(z) = \hat{C}(sI - \hat{A})^{-1}\hat{B}_0 + \hat{D}_0$ must lie inside the unit circle in the complex plane for all $|z| = 1$. Stability along the pass for linear repetitive processes demands that the signals involved are uniformly bounded when both independent variables (k and p) are of unbounded duration and hence is equivalent to the asymptotic stability of the 2D system written in the Roesser model form.

The LMI techniques provide the most effective approach to the stability analysis for repetitive processes, as other based e.g. on the polynomial theory are numerically very difficult, see for details [13]. We have the following results, which can be the base for further investigations for the case considered in this paper.

Theorem 1: [13] A discrete linear repetitive process (1) is stable along the pass if $\exists\, W = W^T > 0$, such that

$$\Phi^T W \Phi - W < 0 \tag{7}$$

or by using the Schur complement

$$\begin{bmatrix} -W & \Phi^T W \\ W\Phi & -W \end{bmatrix} < 0 \tag{8}$$

where $\Phi = \begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix}$ and $W = diag(W_1, W_2)$ with dimensions corresponding to those of A and D_0 respectively.

This result however cannot be directly applied to the second order process of (3), or (5) due to that the matrix A_0 and hence Ψ_0 are ill-conditioned and their inverses are difficult to reach or even impossible.

In [5] we have achieved the following computationally tractable result.

Theorem 2: A discrete linear repetitive process (1) is stable along the pass if $\exists\, V = V^T > 0$, $V = diag(V_1, V_2)$, such that

$$\begin{bmatrix} -V & V\hat{\Phi}^T \\ \hat{\Phi}V & -\Psi V \Psi^T \end{bmatrix} < 0 \tag{9}$$

where $\hat{\Phi} = \begin{bmatrix} \hat{A} & \hat{B}_0 \\ C & D_0 \end{bmatrix}$ and $\Psi = \begin{bmatrix} \Psi_0 & 0 \\ 0 & I_m \end{bmatrix}$.

This result however still is not easy to implement as the matrix A_0 is ill-conditioned and hence it possesses at least one very small, close to zero eigenvalue. Then, the term $\Psi V \Psi^T$ can contain some very, very small entries, which cannot be omitted and hence a great computation accuracy is required.

The following result, equivalent from the mathematical point of view, allows us to lighten this problem.

Theorem 3: Discrete linear repetitive process described by (1) is stable along the pass if $\exists\, V = V^T > 0$, $V = diag(V_1, V_2)$, an some G such that the following LMI is feasible

$$\begin{bmatrix} -V & \hat{\Phi}^T G \\ G\hat{\Phi} & V - G\Psi - \Psi^T G^T \end{bmatrix} < 0 \tag{10}$$

Proof Left multiply (10) by $\begin{bmatrix} I & \hat{\Phi}^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \Psi^{-T} \end{bmatrix}$ and right-multiply the result by its transpose to achieve

$$(\Psi^{-1}\hat{\Phi})^T V (\Psi^{-1}\hat{\Phi}) - V < 0$$

which gives the required result and the proof is completed. ■

Note, that non-singularity of the matrix A_0 plays a significant role. When this matrix is singular, both these approach cannot be used. Instead, the methods valid for singular systems have to be adapted.

IV. Second order process control

If the along the pass stability property is not present for a given example then in control and signal processing applications it will clearly be necessary to introduce regulation action to guarantee it. One way of doing this is to use the stabilization law (for discrete processes)

$$U_{k+1}(p) = \tilde{K}_1 X_{k+1}(p) + \hat{K}_2 Y_k(p) \tag{11}$$

However, the problem appears that the state control law is very difficult to implement. The widely known its disadvantages are that it requires a direct state measurement or, at least the state observer is necessary to implement. What is more, the state vector that appears on the left hand side of (3) is left-multiplied by an ill-conditioned matrix A_0 and due to this it will be much more difficult to have the proper state observer. In such a case an output control law is much more recommended. Hence we suggest the following control law

$$U_{k+1}(p) = \hat{K}_1 Y_{k+1}(p) + \hat{K}_2 Y_k(p) \tag{12}$$

To consider the effect of a controller of the form (12) on process dynamics, first substitute the pass profile equation of (5) into (12) to obtain (assuming the required matrix inverse exists):

$$\begin{aligned} U_{k+1}(p) &= (I - \hat{K}_1 D)^{-1}\hat{K}_1 C X_{k+1}(p) \\ &+ (I - \hat{K}_1 D)^{-1}(\hat{K}_2 + \hat{K}_1 D_0) Y_k(p) \end{aligned} \tag{13}$$

Introduce now the following matrices K_1 and K_2

$$\begin{aligned} K_1 &= (I - \hat{K}_1 D)^{-1}\hat{K}_1 \\ K_2 &= (I - \hat{K}_1 D)^{-1}(\hat{K}_2 + \hat{K}_1 D_0) \end{aligned} \tag{14}$$

to have the control law in the following form

$$U_{k+1}(p) = K_1 C X_{k+1}(p) + K_2 Y_k(p) \quad (15)$$

After simple mathematical operations we have immediately

$$\begin{aligned} \hat{K}_1 &= K_1(I + DK_1)^{-1} \\ \hat{K}_2 &= (I - \hat{K}_1 D)K_2 - \hat{K}_1 D_0 \end{aligned} \quad (16)$$

Application of the stabilization law (15) to the process of (5) now gives the closed loop process state-space model

$$\begin{aligned} \Psi_0 X_{k+1}(p+1) &= (\hat{A} + \hat{B}K_1 C)X_{k+1}(p) \\ &+ (\hat{B}_0 + \hat{B}K_2)Y_k(p) \\ Y_{k+1}(p) &= (C + DK_1 C)X_{k+1}(p) \\ &+ (D_0 + DK_2)Y_k(p) \end{aligned} \quad (17)$$

In the following, we extend the results of Crusius and Trofino [4], known for first order 1D systems.

Theorem 4: A discrete linear repetitive process presented in the form of (5), subject to the control law (15) is stable along the pass if $\exists$ appropriately dimensioned matrices $Q > 0$, $Q = diag(Q_1, Q_2)$, M, S and $F = [F_1 \quad F_2]$, such that

$$\begin{bmatrix} -Q & (\hat{\Phi}S^T + \hat{\Pi}F\widetilde{C})^T \\ \hat{\Phi}S^T + \hat{\Pi}F\widetilde{C} & Q - \Psi S^T - S\Psi^T \end{bmatrix} < 0 \quad (18)$$

$$M\widetilde{C} = \widetilde{C}S^T \quad (19)$$

and then the controller matrix K is given by

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} = FM^{-1} \quad (20)$$

The required output controller matrices $\hat{K}_1, \hat{K}_2$ can be calculated from (16).

Proof: Make use of Theorem 3 with $\hat{\Phi}_{new} = \hat{\Phi} + \hat{\Pi}K\widetilde{C}$ and next left- and right- multiply the result by $\begin{bmatrix} G^{-1} & 0 \\ 0 & G^{-1} \end{bmatrix}$ and its transpose respectively. Finally, introduce the new variables $S = G^{-1}$, $Q = G^{-1}VG^{-T}$ and note that $F = KM$ completes the proof. ■

To reduce the possible method conservativeness, and for further extension of the robust controller design, we show the following complementary result, where another equation constraint is introduced.

Theorem 5: A discrete linear repetitive process presented in the form of (5), subject to the control law (15) is stable along the pass if $\exists$ appropriately dimensioned matrices $V > 0$, $V = diag(V_1, V_2)$, N, G and $F = [F_1 \quad F_2]$, such that

$$\begin{bmatrix} -V & (G\hat{\Phi} + \hat{\Pi}F\hat{C})^T \\ G\hat{\Phi} + \hat{\Pi}F\hat{C} & V - G\Psi - \Psi^T G^T \end{bmatrix} < 0 \quad (21)$$

$$G\hat{\Pi} = \hat{\Pi}N \quad (22)$$

and then the controller matrix K is given by

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} = N^{-1}F \quad (23)$$

The required output controller matrices $\hat{K}_1, \hat{K}_2$ are again to be calculated from (16).

Proof: Make use of Theorem 3 with $\hat{\Phi}_{new} = \hat{\Phi} + \hat{\Pi}K\widetilde{C}$ and note that $F = NK$ completes the proof. ■

V. Uncertain process stability and robust stabilization

Often an exact model of the process dynamics is not available due to the presence of uncertainty. In this section, the previous results are generalized to the case when the process uncertainty is modelled in the polytopic form. Note here that due to the occurrence of the quadratic term $\Psi V \Psi^T$ in Theorem 2 it is very difficult to handle the uncertainty of the left hand side model matrix A_0. When using the results of Theorem 3, and for control – Theorem 4, no such disadvantages. Hence, the model (3) matrices are assumed to be within the matrix polytope

$$\begin{bmatrix} A_0 & A_1 & A_2 & B_1 & B_{00} \\ I & C_1 & C_2 & D & D_0 \end{bmatrix} \in Co\left(\begin{bmatrix} A_0^i & A_1^i & A_2^i & B_1^i & B_{00}^i \\ I & C_1^i & C_2^i & D^i & D_0^i \end{bmatrix}\right) \quad (24)$$

where $i = 1, 2, \ldots, h$ and

$$Co(\mathscr{X}_i) := \left\{ X := \textstyle\sum_{i=1}^h \alpha_i \mathscr{X}_i, \quad \alpha_i \geq 0, \quad \displaystyle\sum_{i=1}^h \alpha_i = 1 \right\}$$

Note that here we are able also to allow the left hand model (3) matrix A_0 to be uncertain, which is a very trouble some to handle when using the result of Theorem 2. It must be stressed however that the sub-polytope $Co(A_0^i)$ contains no singular matrices.

Now, introducing the following notations, $\Psi_0^i = \begin{bmatrix} A_0^i & 0 \\ 0 & I_n \end{bmatrix}$, $\Psi^i = \begin{bmatrix} \Psi_0^i & 0 \\ 0 & I_m \end{bmatrix}$ and $\Phi^i = \begin{bmatrix} \hat{A}^i & \hat{B}_0^i \\ \hat{C}^i & D_0^i \end{bmatrix}$ for $i = 1, 2, \ldots, h$ allows us to generalize the result of Theorem 3 to this uncertain case.

Theorem 6: Discrete linear repetitive process described by (3) with uncertainty defined by (25) is stable along the pass if $\exists$ appropriately dimensioned matrices $V^i > 0$, $V^i = diag(V_1^i, V_2^i)$, and some G such that the following LMI is feasible

$$\begin{bmatrix} -V^i & \hat{\Phi}^{iT} G \\ G\hat{\Phi}^i & V^i - G\Psi^i - \Psi^{iT} G^T \end{bmatrix} < 0 \quad (25)$$

Proof Apply virtually the same steps as for Theorem 3 for each polytope vertex and make a convex combination as in (25), which gives the required result and the proof is completed. ■

Finally, we can develop the robust stabilization scheme when using the output only control law of (12). Applying the constant output controller to the uncertain case requires due to (16) that the model matrices D, D_0 are known exactly. Note also, that the Theorem 4 result can be extended to the uncertain case when also the matrices C_1 and C_2 are not the subject of uncertainty, i.e. in this case

$$\begin{bmatrix} A_0 & A_1 & A_2 & B_1 & B_{00} \\ I & C_1 & C_2 & D & D_0 \end{bmatrix} \in Co\left(\begin{bmatrix} A_0^i & A_1^i & A_2^i & B_1^i & B_{00}^i \\ I & C_1 & C_2 & D & D_0 \end{bmatrix}\right), \quad (26)$$

where $i = 1, 2, \ldots, h$ which yields that now $\Phi^i = \begin{bmatrix} \hat{A}^i & \hat{B}_0^i \\ \hat{C} & D_0 \end{bmatrix}$, $\hat{\Pi}^i = \begin{bmatrix} \hat{B}^i \\ D \end{bmatrix}$ and hence simply whole the pass profile equation of (3) does not possesses uncertain terms. However the left hand side matrix A_0 can remain uncertain.

Theorem 7: A discrete linear repetitive process presented in the form of (5), subject to the control law (15) and uncertainty defined by (26) is stable along the pass if $\exists$ appropriately dimensioned matrices $Q^i > 0$, $Q^i = diag(Q_1^i, Q_2^i)$, M, S and $F = [F_1 \quad F_2]$, such that

$$\begin{bmatrix} -Q^i & (\hat{\Phi}^i S^T + \hat{\Pi}^i F\widetilde{C})^T \\ \hat{\Phi}^i S^T + \hat{\Pi}^i F\widetilde{C} & Q^i - \Psi^i S^T - S\Psi^{iT} \end{bmatrix} < 0 \qquad (27)$$

$$M\widetilde{C} = \widetilde{C}S^T \qquad (28)$$

and then the controller matrix K is given by

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} = FM^{-1} \qquad (29)$$

The required output controller matrices $\hat{K}_1, \hat{K}_2$ are again to be calculated from (16).

Proof: Let matrix Q be defined as

$$Q = \sum_{i=1}^{h} \alpha_i Q^i$$

then multiplying conditions (27) by α_i and summing up from 1 to h one gets the same conditions as in Theorem 4. ■

Also, the Theorem 5 result can be applied to the modified uncertainty structure, where the matrices C_1 and C_2 can remain uncertain but B_1, and obviously D not, i.e.

$$\begin{bmatrix} A_0 & A_1 & A_2 & B_1 & B_{00} \\ I & C_1 & C_2 & D & D_0 \end{bmatrix} \in Co\left(\begin{bmatrix} A_0^i & A_1^i & A_2^i & B_1 & B_{00}^i \\ I & C_1^i & C_2^i & D & D_0^i \end{bmatrix}\right), \qquad (30)$$

where $i = 1, 2, \ldots, h.$, which is equivalent to that $\Phi^i = \begin{bmatrix} \hat{A}^i & \hat{B}_0^i \\ \hat{C}^i & D_0^i \end{bmatrix}$ and the matrix $\hat{\Pi}$ is not a subject of uncertainty.

Theorem 8: A discrete linear repetitive process presented in the form of (5), subject to the control law (15) and uncertainty defined by (30) is stable along the pass if $\exists$ appropriately dimensioned matrices $V^i > 0$, $V^i = diag(V_1^i, V_2^i)$, N, G and $F = [F_1 \quad F_2]$, such that

$$\begin{bmatrix} -V^i & (G\hat{\Phi}^i + \hat{\Pi}F\hat{C}^i)^T \\ G\hat{\Phi}^i + \hat{\Pi}F\hat{C}^i & V^i - G\Psi^i - \Psi^{iT}G^T \end{bmatrix} < 0 \qquad (31)$$

$$G\hat{\Pi} = \hat{\Pi}N \qquad (32)$$

and then the controller matrix K is given by

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} = N^{-1}F \qquad (33)$$

The required output controller matrices $\hat{K}_1, \hat{K}_2$ are again to be calculated from (16).

Proof: Make use of Theorem 6 with $\hat{\Phi}_{new}^i = \hat{\Phi}^i + \hat{\Pi}K\widetilde{C}^i$ and note that $F = NK$ completes the proof. ■

VI. Illustrative example

Consider the uncertain discrete linear repetitive process defined by matrices belonging to a polytope whose vertices are given by

$$A_0^1 = \begin{bmatrix} 159.1 & 0.99 \\ 0.54 & 0.14 \end{bmatrix},$$

$$A_0^2 = \begin{bmatrix} 1686.46 & 1.19 \\ 0.54 & 0.14 \end{bmatrix},$$

$$A_1^1 = \begin{bmatrix} 1.02 & -0.82 \\ -0.96 & -0.27 \end{bmatrix},$$

$$A_1^2 = \begin{bmatrix} 2.14 & -1.23 \\ -0.96 & -0.51 \end{bmatrix},$$

$$A_2^1 = \begin{bmatrix} -1.23 & -0.13 \\ 0.4 & -0.41 \end{bmatrix},$$

$$A_2^2 = \begin{bmatrix} -1.23 & -0.13 \\ 0.36 & -0.41 \end{bmatrix},$$

$$B_{00}^1 = \begin{bmatrix} 0.01 & 0 \\ -1.76 & -1.01 \end{bmatrix},$$

$$B_{00}^2 = \begin{bmatrix} 0.01 & 0 \\ -1.58 & -1.01 \end{bmatrix},$$

$$B_1^1 = \begin{bmatrix} 0.08 & 0.01 \\ -14.2 & -0.66 \end{bmatrix},$$

$$B_1^2 = \begin{bmatrix} 0.1 & 0.01 \\ -14.2 & -0.66 \end{bmatrix},$$

and matrices D, D_0, C_1 and C_2 are not the subject of uncertainty.

$$D = \begin{bmatrix} 0 & -1.02 \\ -0.35 & -1.42 \end{bmatrix},$$

$$D_0 = \begin{bmatrix} 1.46 & -0.34 \\ -0.94 & -0.98 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.45 & -0.27 \\ 1.23 & 0.66 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} -1.18 & 0.57 \\ 0.04 & 0.31 \end{bmatrix}.$$

Consider the case the pass length $\alpha = 30$ with boundary conditions

$$x_{k+1}(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$$
$$y_0(p) = 6, \qquad 0 \le p \le 30$$

with matrix polytope for $\alpha_1 = 0.3$ and $\alpha_2 = 0.7$.

According to the Theorem 7, the robust output-feedback gain

$$K_1 = \begin{bmatrix} -0.06 & -0.061 \\ 0.755 & 0.252 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.167 & -0.052 \\ 0.984 & -0.408 \end{bmatrix}.$$

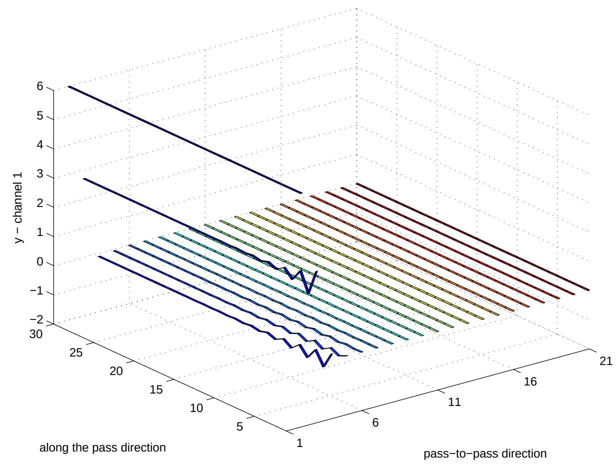

Fig. 1. Output progression for channel 1.

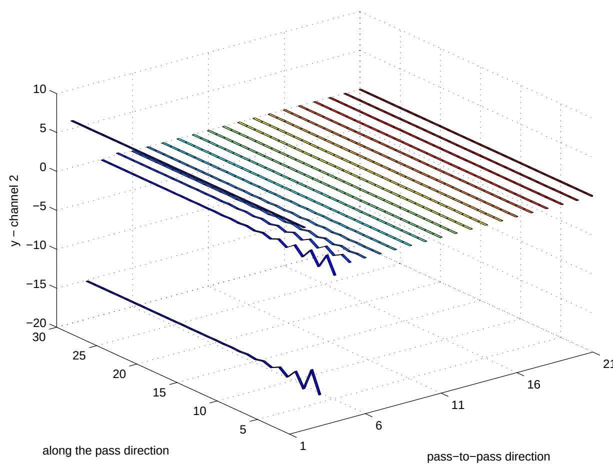

Fig. 2. Output progression for channel 2.

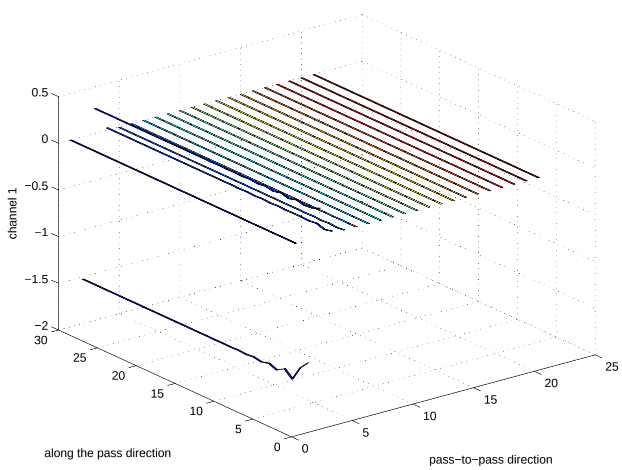

Fig. 3. Input progression for channel 1.

VII. Conclusions

This paper has developed new extended results on stability and stabilization of discrete linear repetitive processes, second order in the along the pass direction, in case when the process is close to singularity, i.e. the respective matrix left multiplying the pass state equation is ill-conditioned. The LMI technique has been employed here as for majority of 2D systems where other methods do not work properly. The core of the achievements is that the proposed methods do not require inverting the ill-conditioned matrix. What is more, the output (pass profile) only control law has been implemented, which does not require the use of the state/descriptor observer, much more complicated than the commonly used one. Also the case with a polytopic uncertainty has been solved. Further work consists, among others, in similar approaching to the differential repetitive processes, and building the Iterative Learning Control (ILC) schemes for the second order ill-conditioned 1D linear systems. Note that all the results can be extended to the case of higher order processes, related in this case to the so-called non-unit memory repetitive processes, which find application in e.g. coal mining processes.

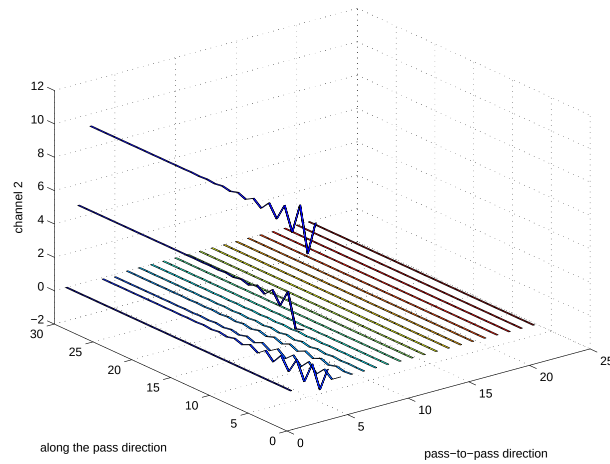

Fig. 4. Input progression for channel 2.

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