Distributed stabilization of spatially invariant systems: positive polynomial approach

Petr Augusta and Zdeněk Hurák

Abstract— **The paper gives a computationally feasible characterisation of spatially distributed discrete-time controllers stabilising a spatially invariant system. This gives a building block for convex optimisation based control design for these systems. Mathematically, such systems are described by partial differential equations with coefficients independent on time and location. In this paper, a situation with one spatial and one temporal variable is considered. Models of such systems can take a form of a 2-D transfer function. Stabilising distributed feedback controllers are then parametrised as a solution to the Diophantine equation $ax + by = c$ for a given stable bivariate polynomial c. This paper brings a computational characterisation of all such stable 2-D polynomials exploiting the relationship between a stability of a 2-D polynomial and positiveness of a related polynomial matrix on the unit circle. Such matrices are usually bilinear in the coefficients of the original polynomials. It is shown that a factorisation of the Schur-Cohn matrix enables linearisation of the problem, at least in a special instance of first-order systems. Then the computational framework of linear matrix inequalities and semidefinite programming can be used to describe the stability regions in the parameter space using a convex constraint.**

I. INTRODUCTION

Control of *spatially distributed systems* has always been a very active research topic with engineering applications in many areas. Such systems can be mathematically described by *partial differential equations* (PDEs). A vast amount of research has been conducted on control design for PDEs. One group of such design methods relies on the possibility to affect the behaviour of the system (that is, a solution in the domain) by controlling the boundary conditions, so-called *boundary control*, nicely documented in [24]. In contrast, this paper focuses on systems featuring a dense and regular array of sensors and actuators stretching all over the domain. This grid then enforces a natural spatial discretisation of the system. Provided the parameters of the system do not depend on location, the resulting mathematical model is *shift invariant*. Of course, shift invariance assumes the domain is infinite, that is, the boundaries are at infinite distance, which is not realistic. Nonetheless, the assumption of shift invariance seems a reasonable simplification for design. Moreover, it is assumed that exciting the system at any location, response can only be observed in the nearest neighbourhood, in other words, the dynamics of the system is *localised*.

It turned out as early as in the late 1960s and early 1970s that this type of systems can be studied within a broader class of systems whose coefficients are functions of parameters. The right mathematical concept appeared to be that of linear systems over rings, because the coefficients in the state-space matrices and the coefficients in the transfer functions are elements of a ring. This broader class of systems also includes systems with delays or systems over integers. Among the pioneers in the area of linear systems over (commutative) rings were Kalman and his doctoral student Rouchaleau [30] and Kamen [20]. Readable surveys were given by Sontag [35] and Kamen [21]. Specialisation of these general results to spatially distributed systems was given in another survey paper by Kamen [4]. Multidimensional systems coming from discretisation of linear PDEs were studied by Brockett and Willems [5]

A few papers followed in the early 1980s such as [22] and [23], but the interest of the community into this field faded away towards the end of 1980s and throughout 1990s. Surprisingly, the field was revived around the beginning of the new millennium, through the papers by Bamieh [2], D'Andrea [6], Gorinevsky [12], [13], Stewart [37], [17] and Stein [36], to name just a few. A related field of repetitive systems [29] also comes along with this revival. Major incentives for this revival came from the availability of both new technologies (MEMS, adaptive optics, networked systems, low cost UAVs or mobile robots, fabrication techniques in silicon industry, ...) and new theoretical and computational tools such as powerful convex and non-convex optimisation techniques, namely linear (LMI) and bilinear matrix inequalities and semidefinite programming.

Distinguished feature of this paper is that while majority of the mentioned papers rely on state-space formalism, here the preference is given to input-output description, that is, models are given in the form of a fraction of two multivariate polynomials. Major justification for this preference is availability of some promising results from multidimensional system theory, such as [31], which extend the intuitive 1-D polynomial framework pioneered (among a few others) by Kučera [25]. Combined with recent advances in the theory and computation with positive polynomials surveyed in [8] and [14], the polynomial approach appears promising to study the presented class of problems.

The basic idea presented in this paper is that stabilisation of a system modelled by a 2-D spatio-temporal transfer function can be studied by "positivisation" of a related

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symmetric polynomial matrix on the unit circle. Even though computationally feasible convex LMI conditions are available for testing positiveness of a given polynomial matrix, these tools may become unuseful when the same formalism is used for controller synthesis. The reason is that the polynomial matrix to be tested for positiveness depends bilinearly on the parameters of the system and the controller. However, it will be shown in this paper that thanks to a linearising factorisation of the Schur-Cohn matrix, simple LMI condition can be obtained for special instances of the problem, which gives computationally feasible constraints on stability region in the parameter (or coefficient) space of the characteristic polynomial.

We shall restrict to linear spatially-distributed time-invariant systems described by a *parabolic* PDE, which can be written in the form

$$d\frac{\partial u(x,t)}{\partial t} - c\,\Delta u(x,t) - b\,\nabla u(x,t) - a\,u(x,t) = f(x,t), \tag{1}$$

where $a > 0, b > 0, c > 0, d > 0$ be constants, u be a solution and f right-hand side. Parabolic equations contain first derivation with respect to time. A heat conduction, diffusion, chemical reactions and others irreversible processes are described by a parabolic PDE.

II. Stability of a 2-D transfer function

A. Discrete-time case

In the paper we shall concentrate on the linear spatially-distributed time-invariant systems described by constant coefficient PDEs. Assuming infinite spatial domain, a sequence of two z-transforms can be performed to obtain a transfer function: one unilateral, corresponding to the temporal variable t and the other bilateral, corresponding to the spatial variable x. Of course, this assumption is never valid for a physical system, but it simplifies the design a lot, allowing to neglect the influence of boundary conditions. Validity of controllers designed with such approximate models must be checked aposteriori, mostly using simulations.

Using one of the common discretisation schemes for both the temporal and the spatial variables, the system can be described by the transfer function

$$P(z,w) = \frac{b(z,w)}{a(z,w)}, \tag{2}$$

where the variable z corresponds to time shift and the variable w corresponds to a shift along the spatial coordinate axes. Since the system is causal in time and non-causal in space, the polynomial a is *one-sided* in z and *two-sided* in w. For physical systems, it is reasonable to assume their spatial symmetry: the polynomial a can be assumed in the form

$$a(z,w) = \sum_{k=0}^{n}\sum_{l=0}^{m} a_{k,l}\, z^k \left(w^l + w^{-l}\right) \tag{3}$$

and similarly for the polynomial b. Following the systems-over-rings concept, the notation

$$a[w](z) = a_n(w)\, z^n + a_{n-1}(w)\, z^{n-1} + \ldots + a_0(w) \tag{4}$$

can be used to emphasise that the polynomial a can be viewed as a polynomial in z with coefficients being functions of w.

Stability of such systems can be studied by analysing roots of their denominator polynomials, with the first stability criterion given in [19]. This is similar to the lumped (1-D) case, but having two variables, the values of the denominator polynomial $a(z,w)$ must be studied on the unit bidisc, bicircle or a combination of the unit disc and circle. It was shown in [11] that Shank's theorem loses its validity in the case when the the system has a nonessential singularity of the second kind, in which case the numerator affects the stability too. This is also discussed in [18] or [7]. Disregarding these uncommon situations, examining denominator polynomials usually suffices. Specialising the general Shank's theorem to systems with spatio-temporal transfer functions, the classical results on stability follows.

Theorem 1: [4](Theorem 4.3, pp. 126) Spatially distributed system described by the transfer function (2) with the polynomials free of a common factor is BIBO stable if

$$a(z,w) \neq 0 \qquad \text{for all} \quad \{|w| = 1\} \cap \{|z| \leq 1\}\,.$$

An immediate reformulation of this test goes in the spirit of the concept of systems over rings.

Corollary 1: Spatially distributed system described by the transfer function (2) with the polynomials free of a common factor is BIBO stable if $a[w](z) = a_n(w)\, z^n + a_{n-1}(w)\, z^{n-1} + \ldots + a_0(w)$ is stable (has its roots outside the open unit circle) for all $|w| = 1$.

A vast number of extensions and simplifications have been proposed in the last decades, such as [39], [27], [28], [16] and [32], to name just a few. The key trick used in this paper is described in [34]. It consists in establishing an equivalence of stability of a 2-D polynomial and positiveness of a certain symmetric polynomial matrix (Schur-Cohn matrix for discrete-time systems) on the unit circle. Apart from algebraic criteria like [33], the advanced toolset of linear matrix inequalities can be used to test positiveness of a polynomial matrix on a unit circle, see [8], [14], [9] and [10]. The LMI formalism offers easy extension from analysis to constructive synthesis, which is the topic for the next section. The Schur-Cohn matrix H for a polynomial $a(z,w)$ has the form

$$H(w) = S_1^{\mathrm{T}} S_1 - S_2^{\mathrm{T}} S_2, \tag{5}$$

where

$$S_1 = \begin{pmatrix} a_0(w) & a_1(w) & \cdots & a_{n-1}(w) \\ 0 & a_0(w) & \cdots & a_{n-2}(w) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_0(w) \end{pmatrix},$$

$$S_2 = \begin{pmatrix} a_n(w) & a_{n-1}(w) & \cdots & a_1(w) \\ 0 & a_n(w) & \cdots & a_2(w) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_n(w) \end{pmatrix}.$$

See [3] for comprehensive overview. The following lemma formally states the key tool for this paper.

Lemma 1: A polynomial $a[w](z)$ of the form (3) is stable if and only if its Schur-Cohn matrix $H(w)$ is positive definite on the unit circle, that is, $H(w) \succ 0$ for all $|w| = 1$.

The Schur-Cohn matrix is a symmetric polynomial matrix $H(w) = H_0 + H_1(w + w^{-1}) + \cdots + H_{2\,m}(w^{2\,m} + w^{-2\,m})$ of size n. Using the result stated in [9] the matrix is positive definite for all $|w| = 1$ if and only if there exists a symmetric matrix M of size $2\,n\,m$ such that

$$
\begin{aligned}
L(M) = &\left(\begin{array}{c|ccc} H_0 & H_1 & \cdots & H_{2m} \\ \hline H_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{2m} & 0 & \cdots & 0 \end{array}\right) \\
&+ \left(\begin{array}{ccc} I & & \\ \hline & \ddots & \\ & & I \\ 0 & \cdots & 0 \end{array}\right) M \left(\begin{array}{c|ccc} I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{array}\right) \\
&- \left(\begin{array}{ccc} 0 & \cdots & 0 \\ \hline I & & \\ & \ddots & \\ & & I \end{array}\right) M \left(\begin{array}{c|ccc} 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \end{array}\right) \succ 0.
\end{aligned}
\tag{6}
$$

B. Continuous-time case

In the continuous-time case, the situation is similar. Transfer function can be obtain performing unilateral Laplace transform, corresponding to the temporal variable t, and bilateral z-transform, corresponding to the spatial variable x. Using one of the common discretisation schemes for the spatial variable, the system can be described by the transfer function

$$P(s, w) = \frac{b(s, w)}{a(s, w)}, \tag{7}$$

where the variable s corresponds to time and the variable w corresponds to shift along the spatial coordinate axes. Since the system is causal in time and non-causal in space, the polynomial a is *one-sided* in s and *two-sided* in w. For physical systems, it is reasonable to assume their spatial symmetry: the polynomial a can be assumed in the form

$$a(s, w) = \sum_{k=0}^{n} \sum_{l=0}^{m} a_{k,l} s^k \left(w^l + w^{-l}\right) \tag{8}$$

and similarly for the polynomial b. Following the systems-over-rings concept, the notation

$$a[w](s) = a_n(w)s^n + a_{n-1}(w)s^{n-1} + \ldots + a_0(w) \tag{9}$$

can be used to emphasise that the polynomial a can be viewed as a polynomial in s with coefficients being functions of w.

Theorem 2: [4](Theorem 4.3, pp. 126) Spatially distributed system described by the transfer function (7) with the polynomials free of a common factor is BIBO stable if

$$a(s, w) \neq 0 \qquad \text{for all} \quad \{|w| = 1\} \cap \{\Re\{s\} \geq 0\}.$$

An immediate reformulation of this test goes in the spirit of the concept of systems over rings.

Corollary 2: Spatially distributed system described by the transfer function (7) with the polynomials free of a common factor is BIBO stable if $a[w](s) = a_n(w)\,s^n + a_{n-1}(w)\,s^{n-1} + \ldots + a_0(w)$ is stable (has its roots in the left half-plane) for all $|w| = 1$.

In the continuous-time case, Hermite-Fujiwara matrix plays a role in the stability testing. Let $a^*(s)$ denote $a(-s)$. The Hermite-Fujiwara matrix is defined as $H = (h_{ij})_{i,j=1,\ldots,n}$, where

$$h_{ij} = (-1)^{j-1} \sum_{k=1}^{m_{ij}} a_{j+k-1}\, a^*_{i-k} - a_{i-k}\, a^*_{j+k-1},$$

where $m_{ij} = \min(i, n - j + 1)$.

The following lemma is reformulation of Lemma 1 for the continuous-time case.

Lemma 2: A polynomial $a[w](s)$ of the form (8) is stable if and only if its Hermite-Fujiwara matrix $H(w)$ is positive definite on the unit circle, that is, $H(w) \succ 0$ for all $|w| = 1$.

The Hermite-Fujiwara matrix is a symmetric polynomial matrix $H(w) = H_0 + H_1(w + w^{-1}) + \cdots + H_{2\,m}(w^{2\,m} + w^{-2\,m})$ of size n. The same result [9] that was used for testing positiveness of Schur-Cohn matrix on the unix circle can be used for Hermite-Fujiwara matrix.

The LMI formalism offers easy extension from analysis to constructive synthesis. But the obvious obstacle that prevents us from using the LMIs directly is the bilinear dependence of the coefficients of the Schur-Cohn and the Hermite-Fujiwara matrices on the coefficients of the original polynomials [15]. The next section offers a partial solution.

III. STABILISATION VIA POSITIVITY OF A POLYNOMIAL MATRIX

In this section, we suppose that the denominator polynomial is not fixed but is a function of parameters of the controller. We consider the control scheme of Fig. 1. Since (1) is first-order in the time variable t, we assume the following.

Assumption 1: The system is of first order in the time, hence, polynomials (3) and (8) are of first degree in the variable z and s, respectively, and $n = 1$.

A controller is given by the transfer function

$$R(z, w) = \frac{y(z, w)}{x(z, w)} \tag{10}$$

if the time is considered a discrete variable and

$$R(s, w) = \frac{y(s, w)}{x(s, w)} \tag{11}$$

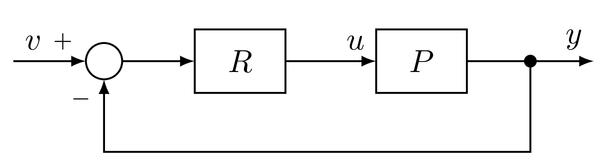

Fig. 1. Standard feedback configuration

in the time is running continuously. The characteristic polynomial determining the stability of the closed loop is then $a\,x + b\,y = c$. Denote this polynomial formally as

$$c = \sum_{k=0}^{\hat{n}} \sum_{l=0}^{\hat{m}} c_{k,l}\, z^k \left(w^l + w^{-l}\right) \tag{12}$$

in the discrete-time case and

$$c = \sum_{k=0}^{\hat{n}} \sum_{l=0}^{\hat{m}} c_{k,l}\, s^k \left(w^l + w^{-l}\right) \tag{13}$$

in the continuous-time one.

Extending the well-known results on solvability of a Diophantine equation in the 1-D setting [25], it is clear that the closed-loop polynomial c has to be of degree $2\,n - 1$ or greater in the variable z and s to design a realisable controller. Since $n = 1$, $\hat{n} \geq 2\,n - 1 = 1$. Thus, let $\hat{n} = 1$. Our task simplifies to finding all stable polynomials with the degree equal to 1 either in z or s. The polynomial can have an arbitrary degree in the variable w corresponding the the spatial shift.

A. Discrete-time case

Due to the above assumptions, S_1 and S_2 are now scalars. However, the matrix notation is kept in the next lines. Without loss of generality we suppose $a_0(w) > 0$, i. e. $S_1 > 0$. A theorem can be stated

Theorem 3: A polynomial c of the form (12) with $\hat{n} = 1$ is stable if and only if

$$\begin{pmatrix} S_1 & S_2 \\ S_2^{\mathrm{T}} & S_1^{\mathrm{T}} \end{pmatrix} \succ 0, \tag{14}$$

where $H(w, w^{-1}) = S_1 S_1^{\mathrm{T}} - S_2 S_2^{\mathrm{T}}$ is the Schur-Cohn matrix corresponding to c.

Proof: It follows from Sylvester's criterion that (14) holds if and only if

$$S_1 > 0 \qquad \text{and} \qquad \det \begin{pmatrix} S_1 & S_2 \\ S_2^{\mathrm{T}} & S_1^{\mathrm{T}} \end{pmatrix} > 0.$$

The former condition was supposed before, the latter one is equal to $S_1 S_1^{\mathrm{T}} - S_2 S_2^{\mathrm{T}} > 0$. Use of Lemma 1 concludes the proof.

The left-hand side matrix in (14) is a symmetric trigonometric polynomial matrix $H(w) = H_0 + H_1(w + w^{-1}) + \cdots + H_{\hat{m}}(w^{\hat{m}} + w^{-\hat{m}})$ of size $2\,\hat{n}$. It is positive semidefinite for $|w| = 1$ if and only if there exists a symmetric matrix M of size $2\,\hat{n}\,\hat{m}$ such that

$$\begin{aligned} L(M, H_0, H_1, \ldots, H_{\hat{m}}) = &\left(\begin{array}{c|ccc} H_0 & H_1 & \cdots & H_{\hat{m}} \\ \hline H_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{\hat{m}} & 0 & \cdots & 0 \end{array}\right) + \\ &+ \left(\begin{array}{ccc} I & & \\ \hline & \ddots & \\ & & I \\ 0 & \cdots & 0 \end{array}\right) M \left(\begin{array}{ccc|c} I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{array}\right) - \\ &- \left(\begin{array}{ccc} 0 & \cdots & 0 \\ \hline I & & \\ & \ddots & \\ & & I \end{array}\right) M \left(\begin{array}{c|ccc} 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \end{array}\right) \succ 0, \end{aligned} \tag{15}$$

where in contrast with (6) L depends also on H.

Theorem 3 allows us now to complete the following lemma.

Lemma 3: Consider a plant described by (1) with transfer function (2). A controller with transfer function (10) stabilises the plant if

$$a\,x + b\,y = c$$

is a such polynomial that (15) holds with

$$H(w) = \begin{pmatrix} S_1 & S_2 \\ S_2^{\mathrm{T}} & S_1^{\mathrm{T}} \end{pmatrix},$$

where $S_1 S_1^{\mathrm{T}} - S_2 S_2^{\mathrm{T}}$ is the Schur-Cohn matrix corresponding to c.

Proof: Follows immediately from Theorem 3 and the fact that c is the characteristic polynomial of closed-loop system.

B. Continuous-time case

Due to the assumptions made at the beginning of this section, the polynomial c has the form $c = c_1(w)\,s + c_0(w)$. The sufficient and necessary condition for its stability is

$$\begin{pmatrix} c_0(w) & 0 \\ 0 & c_1(w) \end{pmatrix} \succ 0 \qquad \text{for all } |w| = 1.$$

Positiveness LMI conditions can therefore be applied to the two polynomials separately.

IV. EXAMPLE: CONTROLLED HEAT CONDUCTION IN AN ALUMINIUM ROD

In this section, the above described concept will be demonstrated by means of an example. A controller for a heat conduction in a rod will be designed.

A. Model of the system

A heat conduction in a rod with an array of temperature sensors and actuators is sketched in Fig. 2.

1) Heat equation: The system is described by the well-known *heat equation*, which has the form

$$\frac{\partial u(x,t)}{\partial t} = \kappa \frac{\partial^2 u(x,t)}{\partial x^2} + \hat{q}(t,x), \tag{16}$$

where u denotes temperature (°C), $\hat{q}$ the input heat ($°\mathrm{C\,s^{-1}}$), t and x denote time (s) and a spatial coordinate (m), respectively, and $\kappa = \frac{\varkappa}{\rho\, c_p}$ is a constant ($\mathrm{m^2\,s^{-1}}$), where $\varkappa$ is the thermal conductivity ($\mathrm{W\,m^{-1}\,K^{-1}}$), ρ is the density ($\mathrm{kg\,m^{-3}}$) and c_p is the heat capacity per unit mass ($\mathrm{J\,K^{-1}\,kg^{-1}}$). Reasonable values are $\varkappa = 230$, $\rho = 2700$ and $c_p = 900$.

2) Discretisation: The corresponding transfer function can be derived as follows [1]. Discretisation of (16) using finite difference methods [38] approximates partial derivatives by differences, i.e.

$$\begin{aligned} \left(\frac{\partial u}{\partial t}\right)_{k,i} &= \frac{u_{k+1,i} - u_{k,i}}{T}, \\ \left(\frac{\partial^2 u}{\partial x^2}\right)_{k,i} &= \frac{u_{k,i-1} - 2\,u_{k,i} + u_{k,i+1}}{h^2}, \end{aligned}$$

where $T > 0$ is the sampling (time) period and $h > 0$ denotes the distance between the nodes along the rod. Substitution of the above formulae into (16) gives partial recurrence equation

$$u_{k+1,i} = \frac{T\kappa}{h^2} u_{k,i-1} + \left(1 - 2\frac{T\kappa}{h^2}\right) u_{k,i} + \frac{T\kappa}{h^2} u_{k,i+1} + q_{k,i}, \tag{17}$$

where k corresponds to discrete time and i to the coordinate of the node and, for simplicity, $q_{k,i} = T\,\hat{q}_{k,i}$.

3) Selection of sampling rage using von Neumann's stability analysis: Clearly, for any subsequent analysis which should produce acceptable results, the discretized model must give a sufficiently accurate approximation to the original model described by (16). A standard tool for this is the von Neumann's analysis of the (numerical) stability of an iterative scheme (17). This will give a relation between T and h to guarantee the convergence.

To proceed, consider the case when zero external heat is applied. Then (17) has the form

$$u_{k+1,i} = \frac{T\kappa}{h^2} u_{k,i-1} + \left(1 - 2\,\frac{T\kappa}{h^2}\right) u_{k,i} + \frac{T\kappa}{h^2} u_{k,i+1}.$$

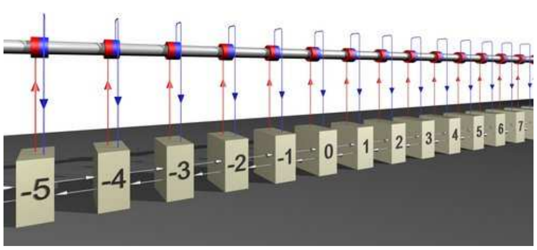

Fig. 2. Distributed control of a distributed parameter system: a rod with an array of heaters and temperature sensors and a distributed controller (an array of controllers)

Also, replace $u_{k,i}$ by $g^k\, e^{j\,i\,\theta}$ to obtain

$$g^{k+1}\, e^{j\,i\,\theta} = \frac{T\kappa}{h^2} g^k \left(e^{j\,(i-1)\,\theta} + e^{j\,(i+1)\,\theta}\right) + \\ + \left(1 - 2\,\frac{T\kappa}{h^2}\right) g^k\, e^{j\,i\,\theta},$$

where θ is the spatial frequency and $j = \sqrt{-1}$. The parameter g is termed *the amplification factor* and the recurrence equation is stable if and only if $|g| \le 1$, see [38] for details. Using Euler's formula and routine simplification now gives

$$\begin{aligned} g &= \frac{T\kappa}{h^2}\left(e^{-j\,\theta} + e^{j\,\theta}\right) + \left(1 - 2\,\frac{T\kappa}{h^2}\right) \\ &= \frac{T\kappa}{h^2}\, 2\cos\theta + \left(1 - 2\,\frac{T\kappa}{h^2}\right). \end{aligned}$$

Hence, $|g| \le 1$ when $\frac{T}{h^2} < \frac{1}{2\,\kappa}$.

4) Transfer function: Performing the two z-transforms, the transfer function is

$$P = \frac{z}{1 + \left(2\,\frac{T\kappa}{h^2} - 1\right) z - \frac{T\kappa}{h^2}(w + w^{-1})z}, \tag{18}$$

where the system output is the temperature at the particular place and time and the system input is the heat (power brought to the system at the given place and time). Choosing $T = 1$ s and $h = \frac{1}{59}$ m in agreement with the above analysis, the transfer function is

$$P = \frac{b(z,w)}{a(z,w)} = \frac{z}{1 - 0.34\,z - 0.32(w + w^{-1})z}. \tag{19}$$

B. Controller design

Consider a controller with the transfer function

$$R(z,w) = r_0 + r_1(w + w^{-1}), \tag{20}$$

where r_0 and r_1 are real constants. This controller behaves as a proportional controller in time and a first-order controller in space, that is, it takes the measurement of the temperature from the neighbouring node into consideration. The characteristic polynomial of closed-loop system has the form

$$c(z,w) = 1 + (r_0 - 0.34)z + (r_1 - 0.32)\,(w + w^{-1})z. \tag{21}$$

Next, we find some constants r_0 and r_1 stabilising the closed-loop system. The method described in Sec. III gives

$$\begin{aligned} S_1 &= 1, \\ S_2 &= r_0 + \left(r_1 - \tfrac{8}{25}\right)(w + w^{-1}) - \tfrac{17}{50}, \\ H_0 &= \begin{pmatrix} 1 & \mathrm{r}_0 - \frac{17}{50} \\ \mathrm{r}_0 - \frac{17}{50} & 1 \end{pmatrix}, \\ H_1 &= \begin{pmatrix} 0 & \mathrm{r}_1 - \frac{8}{25} \\ \mathrm{r}_1 - \frac{8}{25} & 0 \end{pmatrix}. \end{aligned}$$

Using SeDuMi and Yalmip [40], [26] we can check that *some* matrix M in (15) exists and r_0, r_1 are, for example, $r_0 = 0.2$, $r_1 = 0.28$. The characteristic polynomial c is then $c = 1 - 0.14\,z - 0.04(w + w^{-1})z$. Solving Diophantine equation $ax + by = c$ for x and y with c gives $x = 1$ and $y = 0.2 + 0.28(w + w^{-1})$.

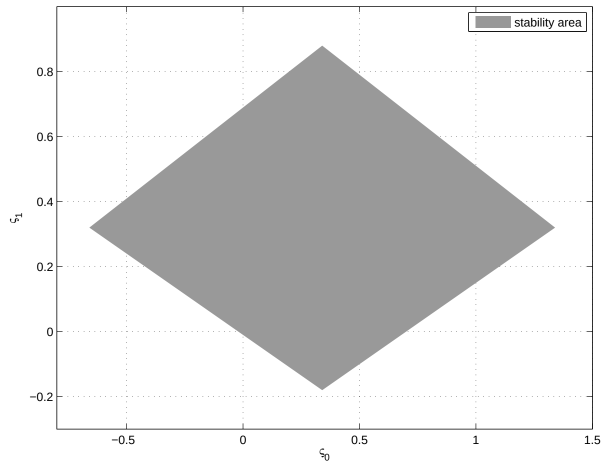

Fig. 3. The values of r_0 and r_1 for which the polynomial (21) is stable

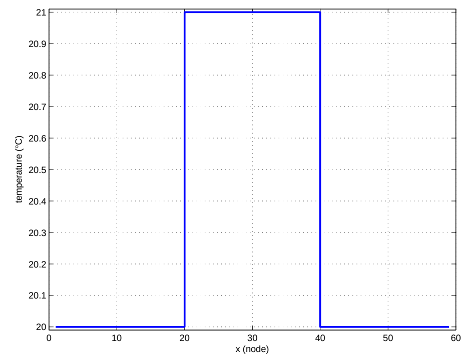

Fig. 5. Initial temperature profile for the numerical example.

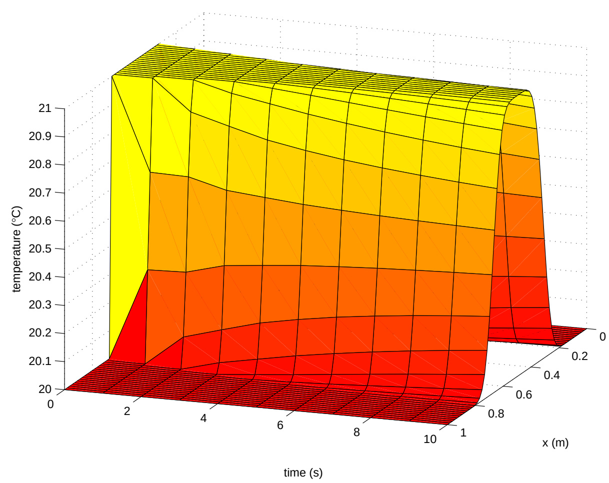

Fig. 4. Response of the uncontrolled system to the initial temperature profile.

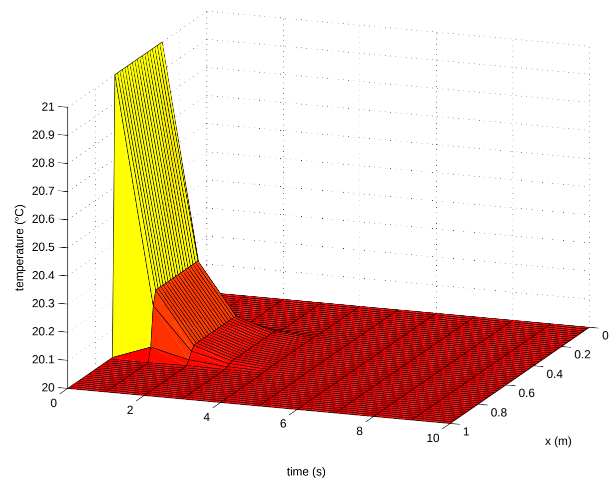

Fig. 6. Response of the controlled system to the initial temperature profile.

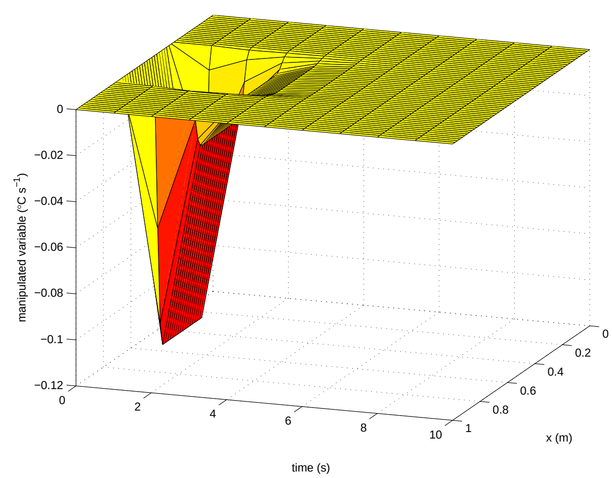

Fig. 7. Controller outputs in response to the initial temperature profile.

The Fig. 3 shows the region in the parameter space r_0 and r_1 for which the polynomial (21) is stable. Apparently, the region is convex, which is the nice fact discovered in this paper.

C. Numerical simulations

Although the procedure presented in this paper (find some stable closed loop polynomial and then solve for the feedback controller that achieves it) is not meant as a truly practical procedure for a controller design but rather as a basic building block for such procedures based on convex optimisation, a few numerical simulations follow.

Fig. 4 gives the response of the uncontrolled system to the initial conditions (temperature profile) in Fig. 5. Next, the controller $R = 0.2 + 0.28(w + w^{-1})$ is considered. The closed-loop system response and manipulated variable are shown in Fig. 6 and 7, respectively.

V. CONCLUSIONS AND FUTURE WORK

A convex characterisation of linear, time and shift invariant, spatially distributed controllers stabilising a spatially invariant system was given. It consists in describing a stabilising region in the coefficient space of the closed-loop characteristic polynomial using an LMI condition. The underlying technique is based on an LMI condition on positiveness of a polynomial matrix on the unit circle. Complementing this with Diophantine equation $ax + by = c$ with polynomials, a stabilising controller is fully characterised. Alternatively, a structure of a distributed feedback controller can be specified first, and then a "stabilising" region in the controller parameter space can be characterised using convex constraints.

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