Existence of optimal solutions of two-directionally continuous repetitive process under convexity assumption

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***Abstract*— In the paper a sufficient condition for the existence of optimal solution to the optimal control problem governed by two-directionally continuous repetitive process and the integral cost functional is given. In the main theorem the crucial assumption is the convexity of the so-called generalized velocities set. The proof of the main theorem is based on the lower closure theorem.**

I. INTRODUCTION

The paper is the second part of our work devoted to the nonlinear continuous version of repetitive process. The first part was presented in the paper *On a nonlinear two-directionally continuous repetitive process* submitted to the first part of the session *Recent Developments in Multi-dimensional Systems, Control and Signals – Theory and Applications*.

In the paper we consider optimal control problem (1) - (3). System (1) is called a two-directionally continuous repetitive process (see [1]). It is a continuous, nonlinear variant of the well-known repetitive process. Repetitive processes have many applications, among others, in modeling of long-wall coal cutting, metal rolling operations [2], [3] and in the theory of ILC schemes [4].

In [5] the authors give a sufficient condition for existence of optimal solution to linear variant of system (1). The result obtained there is proved via continuous dependence of solutions on parameters k, l. The aim of the present paper is to give a sufficient condition for the existence of optimal solutions to nonlinear problem (1)-(3).

II. FORMULATION OF THE PROBLEM

Consider a Lagrange optimal control problem governed by the system

$$\begin{cases} \frac{\partial^2 z}{\partial t \partial x}(t,x) = f^1(t,x,z(t,x), \frac{\partial z}{\partial t}(t,x), \frac{\partial z}{\partial x}(t,x), \\ \qquad\qquad w(t,x), k(t,x)) \\ \frac{\partial w}{\partial t}(t,x) = f^2(t,x,z(t,x), \frac{\partial z}{\partial t}(t,x), \\ \qquad\qquad w(t,x), l(t,x)) \\ \qquad\qquad \text{for a.e. } (t,x) \in P := [0,1] \times [0,1] \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} z(t,0) = 0 & \text{for } t \in [0,1] \\ w(0,x) = 0 & \text{for a.e. } x \in [0,1] \end{cases} \tag{2}$$

and with the cost functional

$$J((z,w),(k,l)) = \int_P f(t,x,z(t,x),z(t,1),w(t,x), k(t,x),l(t,x))dtdx \to \min \tag{3}$$

We define a space of trajectories (z,w) as a set $AC_0^{t,x}(P,\mathbb{R}^{n_1}) \times AC_0^t(P,\mathbb{R}^{n_2})$ where

$$AC_0^{t,x}(P,\mathbb{R}^{n_1}) := \{z : P \to \mathbb{R}^{n_1} : \text{there exists } \zeta \in L^1(P,\mathbb{R}^{n_1}) \text{ such that } z(t,x) = \int_0^t \int_0^x \zeta(t_1,x_1)\,dt_1 dx_1 \text{ for } (t,x) \in P\}$$

and

$$AC_0^t(P,\mathbb{R}^{n_1}) := \{w : P \to \mathbb{R}^{n_2} : \text{there exists } \omega \in L^1(P,\mathbb{R}^{n_2}) \text{ such that } w(t,x) = \int_0^t \omega(t_1,x)\,dt_1 \text{ for } t \in [0,1] \text{ and a.e. } x \in [0,1]\}.$$

As a set of admissible controls (k,l) we consider the space $\mathcal{M}_1 \times \mathcal{M}_2$, where

$$\mathcal{M}_1 := \{k \in L^1(P,\mathbb{R}^{m_1}) : k(t,x) \in M_1 \text{ for a.e. } (t,x) \in P\},$$

$$\mathcal{M}_2 := \{l \in L^1(P,\mathbb{R}^{m_2}) : l(t,x) \in M_2 \text{ for a.e. } (t,x) \in P\}$$

where $M_1 \subset \mathbb{R}^{m_1}, M_2 \subset \mathbb{R}^{m_2}$ are compact.

Definition 1: Fix a function $(k,l) \in \mathcal{M}_1 \times \mathcal{M}_2$. We say that a function $(z,w) \in AC_0^{t,x}(P,\mathbb{R}^{n_1}) \times AC_0^t(P,\mathbb{R}^{n_2})$ is a solution to problem (1)-(2) corresponding to the control (k,l) if (z,w) and (k,l) satisfy (1)-(2). We denote such a solution by $(z,w)_{(k,l)}$.

III. THE SET OF SOLUTION TO REPETITIVE PROCESS.

We impose the following assumptions:

(A1) There exists a constant $L > 0$ such that

$$\left|f^1(t,x,z,z_1,z_2,w,k) - f^1(t,x,\bar{z},\bar{z}_1,\bar{z}_2,\bar{w},k)\right| \le L(|z-\bar{z}| + |z_1-\bar{z}_1| + |z_2-\bar{z}_2| + |w-\bar{w}|)$$

and

$$\left|f^2(t,x,z,z_1,w,l) - f^2(t,x,\bar{z},\bar{z}_1,\bar{w},l)\right| \le L(|z-\bar{z}| + |z_1-\bar{z}_1| + |w-\bar{w}|)$$

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for a.e. $(t,x) \in P$, and any $z, z_1, z_2, \bar{z}, \bar{z}_1, \bar{z}_2 \in \mathbb{R}^{n_1}, w, \bar{w} \in \mathbb{R}^{n_2}, k \in \mathbb{R}^{m_1}, l \in \mathbb{R}^{m_2}$.

(A2) The functions $f^1(\cdot,\cdot,z,z_1,z_2,w,k)$ and $f^2(\cdot,\cdot,z,z_1,w,l)$ are measurable on P for any $z, z_1, z_2 \in \mathbb{R}^{n_1}, w \in \mathbb{R}^{n_2}, k \in \mathbb{R}^{m_1}, l \in \mathbb{R}^{m_2}$.

(A3) The functions $f^1(t,x,z,z_1,z_2,w,\cdot)$ and $f^2(t,x,z,z_1,w,\cdot)$ are continuous on $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$ resp. for a.e $(t,x) \in P, z, z_1, z_2 \in \mathbb{R}^{n_1}, w \in \mathbb{R}^{n_2}$.

(A4) There exists a constant $b > 0$ such that

$$\left|f^1(t,x,0,0,0,0,k)\right| + \left|f^2(t,x,0,0,0,l)\right| \leq b$$

for a.e $(t,x) \in P, k \in M_1, l \in M_2$

Assume (A1)–(A4), then one can prove the following properties

Theorem 1: For any control (k,l) there exists a unique solution $(z,w)_{k,l} \in AC_0^{t,x}(P,\mathbb{R}^{n_1}) \times AC_0^t(P,\mathbb{R}^{n_2})$ to (1)-(2) corresponding to the controls.

Theorem 2: Let $\{(k_s,l_s)\}_{s\in\mathbb{N}} \subset L^1(P,\mathbb{R}^{m_1}) \times L^1(P,\mathbb{R}^{m_2})$ be a sequence of controls such that $(k_s,l_s) \to (k_0,l_0)$ in the space $L^1(P,\mathbb{R}^{m_1}) \times L^1(P,\mathbb{R}^{m_2})$ as $s \to \infty$. Then $(z_s,w_s) \to (z_0,w_0)$ in the space $AC_0^{t,x}(P,\mathbb{R}^{n_1}) \times AC_0^t(P,\mathbb{R}^{n_2})$ as $s \to \infty$, where (z_s,w_s) denotes the solution to (1)-(2) corresponding to the control (k_s,l_s) for $s = 0,1,2,\ldots$.

Lemma 1: There exists a constant $c > 0$ such that for any control (k,l) we have

$$|z_{k,l}(t,x)| \leq c, \text{ for } (t,x) \in P \text{ and } |w_{k,l}(t,x)| \leq c, \quad \text{for a.e. } (t,x) \in P$$

Moreover, for any control (k,l) we have that

$$\left|\frac{\partial^2 z_{k,l}(t,x)}{\partial t \partial x}\right| \leq c \text{ and } \left|\frac{\partial w_{k,l}(t,x)}{\partial t}\right| \leq c \quad \text{for a.e. } (t,x) \in P,$$

where c does not depend on (k,l).

Theorem 3: The family

$$\mathcal{S} := \{(z,w)_{k,l} \in AC_0^{t,x}(P,\mathbb{R}^{n_1}) \times AC_0^t(P,\mathbb{R}^{n_2}) : (k,l) \in L^1(P,\mathbb{R}^{m_1}) \times L^1(P,\mathbb{R}^{m_2})\}$$

of all solutions to (1)-(2) is relatively sequentially compact with respect to the uniform convergence and the weak topology of $L^1(P,\mathbb{R}^{n_2})$. It means that for any sequence $\{(z_s,w_s)\}_{s\in\mathbb{N}} \subset \mathcal{S}$ there exists a subsequence $\{(z_{s_i},w_{s_i})\}_{i\in\mathbb{N}} \subset \mathcal{S}$ and a function $(z_0,w_0) \in AC_0^{t,x}(P,\mathbb{R}^{n_1}) \times AC_0^t(P,\mathbb{R}^{n_2})$ such that $z_{s_i} \to z_0$ uniformly and $w_{s_i} \rightharpoonup w_0$ weakly in $L^1(P,\mathbb{R}^{n_2})$ as $i \to \infty$.

IV. The main result

Definition 2: A control $(k,l) \in \mathcal{M}_1 \times \mathcal{M}_2$ is said to be an admissible control if there exists a trajectory $(z,w)_{(k,l)} \in AC_0^{t,x}(P,\mathbb{R}^{n_1}) \times AC_0^t(P,\mathbb{R}^{n_2})$ corresponding to (k,l) such that there exists an integral

$$\int_P f(t,x,z_{(k,l)}(t,x),z_{(k,l)}(t,1), w_{(k,l)}(t,x),k(t,x),l(t,x))dtdx$$

(we identify $\left(z_{(k,l)},w_{(k,l)}\right) = (z,w)_{(k,l)}$). A trajectory $(z,w)_{(k,l)}$ is said to be an admissible trajectory in such a situation.

Definition 3: A pair $\left((z,w)_{(k,l)},(k,l)\right)$, where (k,l) is an admissible control and $(z,w)_{(k,l)}$ is an admissible trajectory corresponding to (k,l) will be called admissible pair.

The set of all admissible pairs will be denoted by $\mathcal{A}$.

Definition 4: A pair $\left((z,w)_{(k,l)},(k,l)\right)$ that solves problem (1)–(3) is called an optimal pair.

Next, we impose the following assumptions

(A5) There exists a constant $\alpha \in \mathbb{R}$, such that

$$\mathcal{A}_\alpha := \{((z,w),(k,l)) \in \mathcal{A} : J((z,w),(k,l)) \leq \alpha\} \neq \emptyset.$$

(A6) There exists s function $\lambda \in L^1(P,\mathbb{R})$ such that for any $((z,w),(k,l)) \in \mathcal{A}_\alpha$

$$f(t,x,z(t,x),z(t,1),w(t,x),k(t,x),l(t,x)) \geq \lambda(t,x) \text{ for } a.e \ (t,x) \in P$$

(A7) The set

$$\begin{aligned} &Q(t,x,z,z_1) := \\ &\{(\eta,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5) \in \mathbb{R} \times \mathbb{R}^{n_1} \times (\mathbb{R}^{n_2})^2 \times (\mathbb{R}^{n_1})^2 : \\ &\quad \text{there exist } k \in M_1, l \in M_2 : \\ &\quad \eta \geq f(t,x,z,z_1,\xi_3,k,l), \\ &\quad \xi_1 = f^1(t,x,z,\xi_4,\xi_5,\xi_3,k), \\ &\quad \xi_2 = f^2(t,x,z,\xi_4,\xi_3,l)\} \end{aligned}$$

is convex for a.e. $(t,x) \in P$ and every $(z,z_1) \in (\mathbb{R}^{n_1})^2$

The main result of our work is the following

Theorem 4: The optimal control problem (1)-(3) possesses at least one optimal solution.

The proof of the above theorem is based on the lower closure theorem (see [7]).

Note, that the set Q defined in (A7) could be convex despite the fact that neither of f^1 and f^2 are convex.

Example 1: Let $M = [0,1]$ and

$$f^1 : P \times \mathbb{R}^4 \times M^2 \to \mathbb{R},$$
$$f^1(t,x,z,\xi_4,\xi_5,\xi_3,k,l) = z\sqrt{k}$$

$$f^2 : P \times \mathbb{R}^3 \times M^2 \to \mathbb{R},$$
$$f^2(t,x,z,\xi_4,\xi_3,l) = \sqrt{l}$$

$$f : P \times \mathbb{R}^3 \times M^2 \to \mathbb{R},$$
$$f(t,x,z,z_1,\xi_3,k,l) = \sqrt{k} + \sqrt{l}.$$

Then

$$Q(t,x,z,z_1) := \{(\eta,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5) \in \mathbb{R}^6 : \text{there exist } k,l \in M : \eta \geq \sqrt{k}+\sqrt{l}, \xi_1 = z\sqrt{k}, \xi_2 = \sqrt{l}$$

is convex for a.e. $(t,x) \in P$ and evert $(z,z_1) \in \mathbb{R}$.

Indeed, if $\xi = (\eta,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5) \in Q(t,x,z,z_1)$ then there are $k,l \in M$ such that $\eta \geq \sqrt{k}+\sqrt{l}$, $\xi_1 = z\sqrt{k}$ and $\xi_2 = \sqrt{l}$. Similarly, if $\bar{\xi} = (\bar{\eta},\bar{\xi}_1,\bar{\xi}_2,\bar{\xi}_3,\bar{\xi}_4,\bar{\xi}_5) \in Q(t,x,z,z_1)$ then $\bar{\eta} \geq \sqrt{\bar{k}}+\sqrt{\bar{l}}$, $\bar{\xi}_1 = z\sqrt{\bar{k}}$ and $\bar{\xi}_2 = \sqrt{\bar{l}}$ for some $\bar{k},\bar{l} \in M$. Consider $\tilde{\xi} = (\tilde{\eta},\tilde{\xi}_1,\tilde{\xi}_2,\tilde{\xi}_3,\tilde{\xi}_4,\tilde{\xi}_5) = \lambda\xi + (1-\lambda)\bar{\xi}$ with $\lambda \in [0,1]$. Let $\tilde{k},\tilde{l} \in M$ be such that $\sqrt{\tilde{k}} = \lambda\sqrt{k} + (1-\lambda)\sqrt{\bar{k}}$ and $\sqrt{\tilde{l}} = \lambda\sqrt{l} + (1-\lambda)\sqrt{\bar{l}}$. Then $\tilde{\xi}_1 = \lambda\xi_1 + (1-\lambda)\bar{\xi}_1 = z(\lambda\sqrt{k} + (1-\lambda)\sqrt{\bar{k}}) = z\sqrt{\tilde{k}}$, $\tilde{\xi}_2 = \lambda\xi_2 + (1-\lambda)\bar{\xi}_2 = (\lambda\sqrt{l} + (1-\lambda)\sqrt{\bar{l}}) = \sqrt{\tilde{l}}$ and $\tilde{\eta} = \lambda\eta + (1-\lambda)\bar{\eta} \geq \lambda(\sqrt{k}+\sqrt{l}) + (1-\lambda)(\sqrt{\bar{k}}+\sqrt{\bar{l}}) = \sqrt{\tilde{k}}+\sqrt{\tilde{l}}$.

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