On the Geometry and Deformation of Switching Manifolds for Autonomous Hybrid Systems

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***Abstract*— This paper provides a geometrical analysis of autonomous hybrid optimal control systems (HOCS) by studying the properties of switching manifolds in Euclidean space and their associated optimal hybrid trajectories. Motivational examples are to be found in the speed dependent operation of automatic gear shift systems of heavy trucks [1]. In this paper the mathematical formulation of a hybrid system is presented and then the Hybrid Maximum Principle (HMP) necessary conditions for the optimality of a hybrid system trajectory are given, (see [6],[7]). Second order optimality conditions are given in terms of the Hessian matrix of the value function and geometrical data involving the curvature of the switching manifold at its intersection with an optimal trajectory. At the end, the energy of a switching manifold deformation mapping is defined and the hybrid cost optimization is performed with respect to such deformation mappings.**

I. INTRODUCTION

The problem of hybrid systems optimal control (HSOC) has been studied and analyzed in many papers, see e.g [2], [4], [5], [9], [10], [11]. In particular, [5], [12] present an extension of the Maximum Principle for hybrid systems (HMP) and [5] presents an iterative algorithm which is based upon the HMP necessary conditions for optimality. That algorithm is based on a gradient search method for finding the optimal switching state and time on the switching manifold. The HMP algorithm presented in [5] is a general method, implementable for both autonomous and controlled hybrid systems. One important aspect of the optimal control for autonomous hybrid systems is the cost minimization problem with respect to the switching state on the switching manifold, see [5]. This problem is addressed in [6], in which the hybrid cost minimization is converted to a geometrical problem on the switching manifold and second order necessary conditions for optimality are derived based on geometrical properties of the switching manifold. In this paper employing the results presented in [5] and [6], first we define the energy of the switching manifold deformations and second we extend the hybrid cost function in order to include the deformation energy. Necessary conditions for the optimality of the switching state, switching time and the deformation mapping are obtained at the end of the paper.

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II. HYBRID SYSTEMS

The standard hybrid systems framework [2], [5], [9] is as follows:

Definition 2.1: A hybrid systems is a six tuple

$$\mathbf{H} := \{H = Q \times \mathbf{R^{n+1}}, \mathbf{\Gamma}, A, I = \mathbf{\Sigma} \times U, F, \mathcal{M}\} \qquad (1)$$

satisfying:

A0 : $Q = \{1, 2, 3, ..., |Q|\}$ is the finite set of *discrete states*.
H is the *hybrid state space* of $\mathbf{H}$.
$\Gamma : H \times \Sigma \to Q$ is the time independent (partially defined) *discrete transition map*.
$A : Q \to 2^Q$ is the *set valued function* which for a state $q \in Q$ makes transition into $A(q) \subset Q$ under Γ.
$\Sigma = \Sigma_u \dot{\cup} \Sigma_c \dot{\cup} \{id\}$ is a *finite set of distinct autonomous (i.e. uncontrolled) and controlled* discrete event transition labels extended with the identity element $\{id\}$ such that for $i \in Q$, $\sigma_{i,j} \in \Sigma$ only if $j \in A(i)$.
$U \subset R^u$ is the *set of admissible input control values*, where U is an open bounded set in R^u. The set of admissible input control functions is $\mathcal{U} := \mathcal{U}(U, L_\infty[0, T_*))$, the set of all bounded measurable functions on some interval $[0, T_*), T_* \leq \infty$, taking values in U.
$I := \Sigma \times U$ is the *set of system input values*.
F is the *indexed collection of vector fields* $\{f_j\}_{j \in Q}$ such that $f_j : R^{n+1} \times U \to R^{n+1}$ is a uniform Lipschitz vector field assigned to each location.
We assume there exists $K_f < \infty$ such that $max_{j \in Q} sup_{u \in U} ||f_j(0, u)|| \leq K_f, u \in U, j \in Q$.
$\mathcal{M} := \{\tilde{m}^k_\gamma : \gamma \in Q \times Q, k \in \mathbf{Z}_+\}$ is a collection of time dependent manifold subcomponents such that for the ordered pair $\gamma = (p, q), \tilde{m}^k_\gamma$ is a smooth codimension 1 submanifold of R^{n+1}, possibly with boundary $\partial \tilde{m}^k_\gamma$, described locally by $\tilde{m}^k_\gamma = \{x : \tilde{m}^k_\gamma(x) = 0\}$. (See [5] for detailed information concerning the family of manifold subcomponents.)
A *switching manifold* or *guard* $m_{p,q}$ is the union (over k) of a set of switching manifold components $m^k_{p,q} = \bigcup_{k_i; 1 \leq i \leq n(k)} \tilde{m}^{k_i}_{p,q}$. Henceforth, in this paper, the analysis will be assumed to be restricted to a single manifold subcomponent. ■

A1: The *initial state* $h_0 := (x(t_0), q_0) \in \mathbf{H}$ is such that $m_{q_0 q_j}(x_0) \neq 0$ for all $q_j \in Q$. It is assumed that for all p, q, whenever a trajectory governed by the controlled vector field f_p meets any given guard manifold $m_{p,q}$ transversally, there is an admissble autonomous switching to the controlled vector field f_q, also transversal to $m_{p,q}$ and conversely any admissble autonomous switching corresponds to a transversal

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intersection. ∎

Definition 2.1: A *hybrid system input* is a triple $I := (\tau, \sigma, u)$ defined on a half open interval $[t_0, T), T \leq \infty$, where $u \in \mathcal{U}$ and (τ, σ) is a hybrid switching sequence $(\tau, \sigma) = ((t_0, \sigma_0), (t_1, \sigma_1), (t_2, \sigma_2), \ldots), t_0 < t_1 <, \cdots,$ of pairs of switching times and discrete input events, $\sigma_0 = id, \sigma_i \in \Sigma, i \geq 1$. $\mathcal{I}_L$ denotes the set of inputs I with L switchings. The corresponding *hybrid state trajectory* is a triple (τ, q, x) consisting of τ, an associated sequence of discrete states $q = (q_0, q_1, q_2, \ldots)$, and a sequence $x(\cdot) = (x_{q_0}(\cdot), x_{q_1}(\cdot), x_{q_2}(\cdot), \ldots)$ of absolutely continuous functions $x_{q_j} : [t_j, t_{j+1}) \to R^{n+1}$. ∎

Let $\{l_j\}_{j \in Q}, l_j \in C^k(R^{n+1} \times U; R_+), k \geq 1$ be a family of loss functions and $h \in C^k(R^{n+1}; R_+), k \geq 1$, a terminal cost satisfying the following.

A2: There exist $K_l < \infty$ and $1 \leq \gamma < \infty$ such that $|l_j(x, u)| \leq K_l(1 + ||x||^\gamma), x \in R^{n+1}, u \in U, j \in Q$, and similarly for $h(.)$. ∎

III. Hybrid Systems Optimal Control Problems

Consider the initial time t_0, final time $t_f < \infty$, initial hybrid state $h_0 = (q_0, x_0)$, and $\bar{L} < \infty$. Let $S_L = ((t_0, \sigma_0), (t_1, \sigma_1), \ldots, (t_L, \sigma_L))$ be a hybrid switching sequence and let $I_L := (S_L, u), u \in \mathcal{U}$, be a hybrid input trajectory subject to *A0, A1*, where $L \leq \bar{L} < \infty$, is the number of switchings. Subject to *A2*, define the *hybrid cost function* as

$$J(t_0, t_f, h_0; I_L, \bar{L}, \mathcal{U}) := \sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s), u(s))ds + h(x_{q_L}(t_f)). \quad (2)$$

The continuous dynamics of the hybrid system are specified as follows:

$$\dot{x}_{q_i}(t) = f_{q_i}(x_{q_i}(t), u(t)), \quad a.e.\, t \in [t_i, t_{i+1}),$$

$$u(t) \in U \subset R^u, u(.) \in L_\infty(U), \quad h_0 = (q_0, x_0), \quad i = 0, 1, \ldots, L,$$

$$x_{q_{i+1}}(t_{i+1}) = lim_{t \to t_{i+1}} x_{q_i}(t), \quad t_{L+1} = t_f < \infty. \quad (3)$$

Then the hybrid system optimal control (HSOC) problem is to find the infimum of $J(t_0, t_f, h_0, I_L, \bar{L}, \mathcal{U})$ i.e

$$J^o(t_0, t_f, h_0, \mathcal{U}) = inf_{I_L \in \mathcal{I}_L, L \leq \bar{L}} J(t_0, t_f, h_0, I_L, \bar{L}, \mathcal{U})$$

Assume that I_L contains only autonomous switchings and let $J^o(t_0, t_f, h_0)$ be the infimized cost function with infimizing control I_L^o and trajectory (x^o, q^o) which are assumed to exist.

The following theorem gives the Hybrid Maximum Principle in a restricted class of the cases treated in [5], specifically the autonomous switchings case extended to the time varying guards case.

Theorem 1: ([5]) Consider a hybrid system satisfying the assumptions *A0-A2* above and define

$$H_q(x, \sigma, u, \lambda) = \lambda^T f_{\sigma(q)}(x, u) + l_{\sigma(q)}(x, u), \quad \lambda \in R^{n+1}, u \in U, q \in Q.$$

Let I_L^o have L autonomous switchings $L \leq \bar{L}$ at the switching times along the optimal trajectory.
Finally, assume that almost everywhere along an optimal trajectory the continuous state x satisfies the controllability condition given in [5]. Then:
(i) There exists a piecewise absolutely continuous adjoint process satisfying

$$\dot{\lambda}_j^o = -\frac{\partial H_j}{\partial x}(x^o, \sigma^o, \lambda, u^o), \quad u_t^o \in U \quad a.e., \quad t \in (t_j, t_{j+1}).$$

(ii) At the switching times the adjoint process and Hamiltonain function satisfy

$$\lambda_j(t_j^-) = \lambda_{j+1}(t_j^+) + p_j \nabla_x m_{j,j+1}(x(t_j), t_j), \quad 1 \leq j \leq L, \quad (4)$$

$$H_j(t_j^-) = H_{j+1}(t_j^+) - p_j \nabla_t m_{j,j+1}(x(t_j), t_j), \quad 1 \leq j \leq L. \quad (5)$$

(iii) Along the optimal trajectory the Hamiltonian minimization condition is satisfied

$$H_j(x^o(t), \sigma^o(t), \lambda_j(t), u^o(t)) \leq H_j(x^o(t), \sigma^o(t), \lambda_j(t), v), \quad \forall v \in U, \quad t \in [t_j, t_{j+1}), j \in [0, 1, , ..., L]. \quad (6)$$

i.e.

$$\lambda^{oT}(t) f_{\sigma^o(q^o(t-))}(x_t^o, u_t^o) + l_{\sigma^o(q^o(t-))}(x_t^o, u_t^o) \leq \lambda^{oT}(t) f_{\sigma^o(q^o(t-))}(x_t^o, v) + l_{\sigma^o(q^o(t-))}(x_t^o, v), \quad \forall v \in U.$$

∎

For simplicity of analysis we only consider the case of one autonomous switching from $q_0 \in Q$ to $q_1 \in Q$, and so $I_L^o \equiv u_t^o \equiv (u_1^o, u_2^o, t)$; the extension to the general case is straightforward but engenders significant complexity, see [5].

IV. Necessary Conditions for Optimality

Here in this subsection we present the results of [8], and, based upon these, formulate necessary conditions for optimal trajectories for autonomous hybrid systems consisting of two modes and one time invariant switching manifold. Recalling (2), we have

$$v(x, t) = inf_{u \in \mathcal{U}} J(t_0, t_f, h_0; x_{t_s}, t_s, u)|_{t_s = t, x_{t_s} = x}. \quad (7)$$

This formulation displays the hybrid value function dependence on the switching state and switching time on the switching manifold $\mathcal{M}$.

Lemma 1: ([7]) Let $(x(t_s), t_s) = (x(t_s^o), t_s^o)$ be the optimal switching state and time subject to the hypotheses of the HSOC problem and the hypotheses of Theorem 1, then

$$(i) \quad \frac{\partial v(x, t)}{\partial t}|_{(x(t_s), t_s)} = 0, \quad (8)$$

$$(ii) \quad \nabla_x v(x,t)|_{(x(t_s),t_s)} \perp T_{x(t_s)}\mathcal{M}, \tag{9}$$

and hence

$$\nabla_x v(x,t)|_{(x(t_s),t_s)} = \mu^{-1}N, \tag{10}$$

where $T_{x(t_s)}\mathcal{M}$ is the tangent space at the switching state $x(t_s)$ and μ is a scalar.

■

We note the similarities between (9) and (4); the next lemma presents the actual relationship between them.

Lemma 2: ([7]) For the HSOC problem defined in Theorem 2 the following relations hold:

$$\frac{\partial v(x,t)}{\partial t}|_{(x(t_s),t_s)} = H_1(t_s^-) - H_2(t_s^+), \tag{11}$$

$$\nabla_x v(x(t_s),t_s) = \lambda_2(t_s^+) - \lambda_1(t_s^-). \tag{12}$$

■

The results of Lemma 1 and the chain rule imply

$$\frac{\partial v(x^o,t^o)}{\partial \omega^i} = \nabla_x v(x^o,t^o)\frac{\partial x}{\partial \omega^i}, \quad i=1,...,n, \tag{13}$$

where ω^i is the ith local coordinate of x on $\mathcal{M}$. Using the results above, the next theorem is proved:

Theorem 2: ([7]) At the optimal switching state x^o on the switching manifold, $\mathcal{M}$, we have

$$\begin{aligned} -H_{ik} &= \mu\frac{\partial x^T}{\partial \omega^i}\frac{\partial^2 v(x^o,t^o)}{\partial x^2}\frac{\partial x}{\partial \omega^k} + T_i^T\frac{\partial x}{\partial \omega^k} \\ &= \mu\frac{\partial x^T}{\partial \omega^k}\frac{\partial^2 v(x^o,t^o)}{\partial x^2}\frac{\partial x}{\partial \omega^i} + T_k^T\frac{\partial x}{\partial \omega^i}, \end{aligned} \tag{14}$$

where $T_i, T_k \in T_{x^o}\mathcal{M}, \quad i,k = 1,...,n$ and μ is the discontinuity parameter appearing in the adjoint process boundary condition at the switching time, and H_{ik} is the (i,k) entry of the second fundamental form matrix of $\mathcal{M}$ at x^o, see (10) and (12).

■

This yields the following result:

Theorem 3: ([7]) In the local coordinates of $T_{x^o}\mathcal{M}$, we have

$$\frac{\partial x^T}{\partial \omega^i}\frac{\partial^2 v(x^o,t^o)}{\partial x^2}\frac{\partial x}{\partial \omega^k} = \frac{\partial^2 v(x^o,t^o)}{\partial \omega^i \partial \omega^k} - \mu^{-1}H_{ik}, \quad i,k=1,...,n. \tag{15}$$

■

The vectors T_i, T_k appearing in the statement of Theorem 2 are as yet unspecified. The following lemma and theorem give expressions for those vectors in a specific coordinate system around the optimal switching state.

Lemma 3: ([7]) The local components, $t_{ij}, \quad i,j = 1,...,n,$ of the vector

$$T_i = \sum_{j=1}^{n} t_{ij}\frac{\partial x}{\partial \omega^j}, \quad i=1,..,n, \tag{16}$$

in Theorem 2 are given by:

$$t_{ij} = -\mu\frac{\partial \eta^i}{\partial \omega^j}, \quad i,j=1,...,n, \tag{17}$$

where $\bar{\eta} = [\eta^1,...,\eta^n]$ is the projection onto $T_{x^o}\mathcal{M}$ of the gradient of the value function with respect to the switching state at the optimal switching state.

■

It should be mentioned here that all the results above are valid only at the switching state which optimizes i.e. minimizes the hybrid cost function on the switching manifold.

A. Example 1

Consider a hybrid system consisting of two modes with the following dynamics:

$$S_1 \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u \tag{18}$$

$$S_2 \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} u, \tag{19}$$

for which the cost function and boundary conditions are defined as:

$$J = \frac{1}{2}\int_0^{10} u^2(t)dt, \quad x_0 = (0,0,0), \quad x_f = (4,1,3).$$

The simulation is performed with respect to the following switching manifold:

$$\mathcal{M} = \{(x_1,x_2,x_3); \quad x_1^2 + 2x_2^2 - x_3 = 4\}, \tag{20}$$

By direct calculation we have

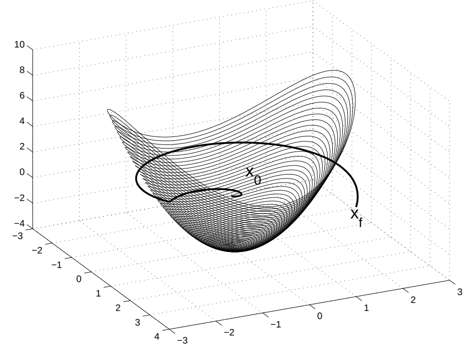

Fig. 1. State Trajectory of Example 1

$$H = \frac{1}{\sqrt{(1+4x^2+16y^2)}}\begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}, \tag{21}$$

where at the optimal state we have

$$H^o = \begin{bmatrix} -0.4770 & 0 \\ 0 & -0.9539 \end{bmatrix}. \tag{22}$$

By applying the *GGAP-HMP* algorithm introduced in [8] to the example above, the optimal switching time and state are as follows:

$$t^o = 6.1900, \quad x^o = [-1.1771, -0.8307, -1.23420], \mu = -27.8. \tag{23}$$

(GGAP-HMP algorithm denotes the *Geodesic-Gradient Along Parameterization-Hybrid Maximum Principle* algorithm whose convergence properties are proven in [8]). The evolution of the state trajectory in the example is shown in Figure 1 and the Hessian of the value function at x^o is obtained by direct calculation as

$$\frac{\partial^2 v(x^o,t^o)}{\partial x^2} = \begin{bmatrix} 2.8235 & -1.5010 & 0.7973 \\ -1.4993 & 4.5418 & -1.4824 \\ 0.7987 & -1.4827 & 2.8875 \end{bmatrix}. \quad (24)$$

Employing Lemma 3 and then by solving a system of differential equations introduced in [6] we get

$$\begin{aligned} T_1 &= [16.0188, 40.2344, -171.4023], \quad (25)\\ T_2 &= [-155.4266, 207.9389, -325.0340]. \end{aligned}$$

Inserting T_1, T_2 and $\frac{\partial^2 v(x^o,t^o)}{\partial x^2}$ in (14) we obtain

$$[H_{ij}] = \begin{bmatrix} -0.4458 & -0.0521 \\ 0.0500 & -1.0430 \end{bmatrix} \cong \begin{bmatrix} -0.4770 & 0 \\ 0 & -0.9539 \end{bmatrix}, \quad (26)$$

which (to two decimal places) is consistent with (22). The lack of symmetry in (26) results from the lack of symmetry in (24) which is due to the level of precision chosen in the numerical calculation.

V. Energy of Switching Manifold Deformations

In this section we introduce hybrid systems for which the total cost is the summation of the cost defined by (2) and the energy of the switching manifold deformation mapping. As shown in [6], for a given switching manifold, it is in general possible to decrease the value function by a local perturbation around the nominal switching state x^o. In practice, changing the switching manifold may imposes an extra cost. Consider an automatic gear changing system which changes at a certain speed. Changing the speed level at which the switching happens requires a change in the mechanical structure of the gear box which may not be feasible or may make manufacture more expensive.

This motivates us to include an extra term in hybrid cost which corresponds to the cost deformations of switching manifolds. This extra cost depends on the nature of the hybrid system and may differ from a system to another. We shall assign a positive cost for changing the nominal switching manifold M to a new switching manifold $F(M)$. Figure 2 provides a picture of such deformations in three dimensional space.

We now define an energy function for the mapping $F : M \to N, \quad F \in C^\infty(M,N)$. In general we consider (M,g) to be the m dimensional domain Riemannian manifold with the corresponding Riemannian metric g and (N,h) to be the n dimensional image Riemannian manifold with the corresponding Riemannian metric h. The differential of the mapping F at point $x \in M$ is defined to be the linear map from T_xM to $T_{u(x)}N$ given by:

$$dF_x : T_xM \to T_{u(x)}N. \quad (27)$$

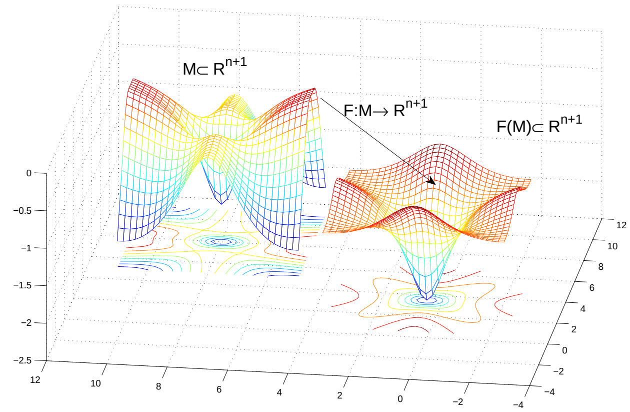

Fig. 2. Switching Manifold Deformation

The local coordinates of $x \in M$ and $F(x) \in N$ are $\{(\frac{\partial}{\partial x^i})\}_{i=1,...,m}$ and $\{(\frac{\partial}{\partial y^\alpha})\}_{\alpha=1,...,n}$, and so the push forward of F is given by

$$dF_x((\frac{\partial}{\partial x^i})) = \sum_{\alpha=1}^{n}(\frac{\partial F^\alpha}{\partial x^i})(x)(\frac{\partial}{\partial y^\alpha}), \quad i = 1,...,m. \quad (28)$$

The space $Hom(T_xM, T_{u(x)}N)$, which is the space of all linear maps from T_xM to $T_{u(x)}N$, is linearly isomorphic to $T_xM^* \otimes T_{u(x)}N$, see [3], and therefore dF_x is locally described by:

$$dF_x \in Hom(T_xM, T_{u(x)}N) \cong T_xM^* \otimes T_{u(x)}N, \quad (29)$$

where

$$dF_x = \sum_{i=1}^{m}\sum_{\alpha=1}^{n}(\frac{\partial F^\alpha}{\partial x^i})(x)(dx^i) \otimes (\frac{\partial}{\partial y^\alpha}). \quad (30)$$

The norm g^* on T_xM^* and h on N induces an inner product on the tensor product space given by

$$< dx^i \otimes (\frac{\partial}{\partial y^\alpha}), dx^j \otimes (\frac{\partial}{\partial y^\beta}) >= g^{ij}h_{\alpha\beta}(F(x)), \quad (31)$$

where

$$g^* = [g_{ij}]^{-1}, \quad i,j = 1,...,n. \quad (32)$$

Employing (30) and (31), the norm of dF_x is then defined as follows:

$$\begin{aligned} |dF_x|^2 &= < dF_x, dF_x > \\ &= \sum_{i,j=1}^{m}\sum_{\alpha,\beta=1}^{n} g^{ij}(x)h_{\alpha\beta}(F(x))(\frac{\partial F^\alpha}{\partial x^i})(x)(\frac{\partial F^\beta}{\partial x^j})(x), \end{aligned}$$

and the mapping dF is defined as the following section

$$dF \in \Gamma(T^*M \otimes F^{-1}TN), \quad (33)$$

where $F^{-1}TN$ is the induced tangent bundle by F and Γ is the cross section of the vector bundle $T^*M \otimes F^{-1}TN$.

The norm of dF over the manifold M is written as

$$|dF|^2 = \sum_{i,j=1}^{m}\sum_{\alpha,\beta=1}^{n} g^{ij}h_{\alpha\beta}(F)(\frac{\partial F^\alpha}{\partial x^i})(\frac{\partial F^\beta}{\partial x^j}), \quad (34)$$

where $h_{\alpha\beta}$ is the metric on N. The energy density of F is finally defined as

$$e(F)(x) = \frac{1}{2}|dF|^2(x), \quad x \in M, \tag{35}$$

with the corresponding energy functional

$$E(F) = \int_M e(F) d\mu_g, \tag{36}$$

where μ_g is Lebesgue measure defined by the Riemannian metric g_{ij} on the manifold M, see [3].

The minimization of the energy function E with respect to a parametrized mapping may be analyzed using variational methods from the calculus of variations, [3].

Consider the following *variation* of F:

$$\hat{F} : I \times M \to N, \quad I = (-\epsilon, \epsilon) \in R,$$
$$\hat{F}(x,0) = F(x) \in N, \quad \forall x \in M, \hat{F} \in C^\infty(I \times M, N). \tag{37}$$

We shall write $F_s = \hat{F}(., s), s \in (-\epsilon, \epsilon)$. *Harmonic maps* are solutions to the differential equation:

$$\frac{d}{ds} E(F_s)|_{s=0} = 0. \tag{38}$$

The *tension field* of F in the induced bundle $F^{-1}TN$, in the local coordinates of M and N, $\tau(F)$ is written as (see [3]):

$$\begin{aligned} \tau(F)^\alpha &= \sum_{i,j=1}^{m} g^{ij} \Big(\frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \sum_{k=1}^{m} \Gamma_{ij}^k \frac{\partial F^\alpha}{\partial x^k} \\ &+ \sum_{\beta,\gamma=1}^{n} \Gamma_{\beta\gamma}^\alpha(F) \frac{\partial F^\beta}{\partial x^i} \frac{\partial F^\gamma}{\partial x^j} \Big), \quad \alpha = 1, ..., n, \end{aligned} \tag{39}$$

where $\Gamma_{ij}^k, \Gamma_{\beta\gamma}^\alpha$ are Christofel symbols defined on (M, g) and (N, h) respectively. The following results give an expression for the harmonic maps from M to N.

Theorem 4: ([3]) Let $F_s = \hat{F}(., s)$ be the C^∞ variation of $F = F_0 : M \to N$, then

$$\frac{d}{ds} E(F_s)|_{s=0} = -\int_M < Y, \tau(F) > d\mu_g, \tag{40}$$

where $Y = \frac{d}{ds} F_s|_{s=0}$ is a variation vector field of F and $\tau(F)$ is the tension field of F. ■

VI. Extended Hybrid Systems

Employing the notion of the energy of the mapping between manifolds, we modify the cost function defined in (2) to the following formula which includes the energy of the deformation of the nominal manifold M in the hybrid cost,

$$\begin{aligned} J^F(t_0, t_f, I_L, \bar{L}, \mathcal{U}) &:= \int_{t_0}^{t_s} l_{q_1}(x_{q_1}(s), u(s)) ds + \\ \int_{t_s}^{t_f} & l_{q_2}(x_{q_2}(s), u(s)) ds + \\ h(x_{q_L}(t_f)) &+ ||E(F_{M\to N}) - E(I_{M\to N})||^2, \end{aligned} \tag{41}$$

where $I_{M\to N}$ is the identity map from M to N. For a given F_s, the hybrid value function for the cost J^{F_s} is defined as:

$$\begin{aligned} v(x, t, F_s) = inf_{u\in U} J^{F_s}(t_0, t_f, I_L, \bar{L}, \mathcal{U})|_{x_s=x, t_s=t}, \\ x \in M, t \in R, F \in C^\infty(M, N). \end{aligned} \tag{42}$$

Since the first two terms on the right hand side of (41) are computable separately, the value function defined by (42) can be split into two terms as follows:

$$v(x, t, F) = v_F(x, t) + ||E(F_s) - E(I_{M\to N})||^2, \tag{43}$$

where $v_F(x, t)$ is the value function of the hybrid system without considering the deformation cost. In order to be able to compute the sensitivity of $v(x, t, F_s)$ at (x, t, F_s) we use the notion of the variation, F_s, introduced in (37) . The sensitivity of $v(x, t, F_s)$ with respect to (x, t, F_s) is given as follows:

$$\frac{\partial v(x, t, F_s)}{\partial x} = \frac{\partial v_{F_s}(x, t)}{\partial x} + \frac{\partial ||E(F_s) - E(I_{M\to N})||^2}{\partial x}, \tag{44}$$

and similarly for t and s we have

$$\frac{\partial v(x, t, F_s)}{\partial t} = \frac{\partial v_{F_s}(x, t)}{\partial t} + \frac{\partial ||E(F_s) - E(I_{M\to N})||^2}{\partial t}, \tag{45}$$

$$\frac{\partial v(x, t, F_s)}{\partial s} = \frac{\partial v_{F_s}(x, t)}{\partial s} + \frac{\partial ||E(F_s) - E(I_{M\to N})||^2}{\partial s}. \tag{46}$$

$E(F_s)$ does not depend on x and t and so the second terms in (44) and (45) vanish; hence

$$\frac{\partial v(x, t, F_s)}{\partial x} = \frac{\partial v_{F_s}(x, t)}{\partial x} = \frac{\partial v(F_s(x), t)}{\partial F_s(x)} W(x), \tag{47}$$

where $v(F_s(x), t)$ is the hybrid value function based on the switching time t, the switching state is such that $F_s(x) \in F(M)$, and

$$\frac{\partial v(F_s(x), t)}{\partial F_s(x)} = \lambda^+ - \lambda^-, \quad W(x) := \frac{\partial F_s(x)}{\partial x}, \tag{48}$$

where $W(x)$ is the Jacobian matrix of the coordinate changes.

By the results of Lemma 2, the first term on the right hand side of Equation (46) is given by the following lemma:

Lemma 4: For the extended hybrid cost function defined in (41), $\frac{\partial v_F(x,t)}{\partial s}$ is given by

$$\frac{\partial v_F(x, t)}{\partial s} = (\lambda^+ - \lambda^-) Y(x), \tag{49}$$

where $(\lambda^+ - \lambda^-)$ is the adjoint process discontinuity at $F(x) \in F(M)$ and $Y(x) = \frac{dF_s(x)}{ds}|_{s=0}$.

Proof: In the case of general Riemannian manifolds M and N, the sensitivity function defined in (44) is a linear map from the tangent space of M to R. By ***A0***, $x \in R^{n+1}$ and so in our analysis $N = R^{n+1}$. Since by the definition (42), t and x are chosen independently of s, we have

$$\frac{\partial v_{F_s}(x, t)}{\partial s}|_{s=0} = lim_{s\to 0} \frac{v(x, t, F_s(x)) - v(x, t, F_0(x))}{s}, \tag{50}$$

where $F_s(x)$ is the switching state on $F_s(M)$. Applying the chain rule we have

$$\frac{\partial v_{F_s}(x,t)}{\partial s} = \frac{\partial v(x,t,F_s(x))}{\partial F_s(x)}|_{s=0}.\frac{dF_s(x)}{ds}. \tag{51}$$

Lemma 2 gives

$$\frac{\partial v(x,t,F_s(x))}{\partial F_s(x)}|_{s=0} = \lambda^+ - \lambda^-, \tag{52}$$

so (52) and the notation $Y(x) = \frac{dF_s(x)}{ds}|_{s=0}$ yield (49). ■

We observe that Theorem 4 implies that the second term of the right hand side of (46) is given by:

$$\frac{\partial ||E(F_s) - E(I_{M\to N})||^2}{\partial s} = -2(E(F_s) - E(I_{M\to N})). \int_M < Y, \tau(F) > d\mu_g. \tag{53}$$

Now if (x, t, F_0) corresponds to the minimum of $v(x, t, F)$ then

$$\frac{\partial v(x,t,F_0)}{\partial x} \perp T_x M, \quad \frac{\partial v(x,t,F_0)}{\partial t} = 0, \tag{54}$$

and

$$\frac{\partial v(x,t,F_s)}{\partial s}|_{s=0} = 0, \tag{55}$$

where (54) holds by Lemma 1.

Hence (55), Lemma 4 and Theorem 4 yield

$$0 = \frac{\partial v(x,t,F_s)}{\partial s}|_{s=0} = (\lambda^+ - \lambda^-).Y(x) -2(E(F_0) - E(I_{M\to N})). \int_M < Y, \tau(F_0) > d\mu_g, \tag{56}$$

and consequently

$$(\lambda^+ - \lambda^-).Y(x) = 2(E(F_0) - E(I_{M\to N})) \times \int_M < Y, \tau(F_0) > d\mu_g. \tag{57}$$

We observe that equation (57) must be satisfied for all C^∞ *variational vector fields* $Y(.)$, the left hand side of (57) only depends on $Y(.)$ at the optimal x and the right hand side of (57) is independent of x.

Clearly $\lambda^+ - \lambda^- = 0, \tau(F_0) = 0$, satisfy both (54) and (55). Hence the necessary condition (57) for optimally with respect to joint hybrid control and switching manifold perturbations is satisfied for the mapping F_0 for which $F_0(M)$ passes through the optimal controlled switching states, i.e $\lambda^+ - \lambda^- = 0$ (see [6]), and F_0 is a harmonic map, i.e. $\tau(F_0) = 0$, (see [3]). Using the variational method it is shown that $\lambda^+ - \lambda^- = 0$ and $\tau(F_0) = 0$ is the unique solution for (57).

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