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A minicourse on noncommutative rational functions and noncommutative convexity

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***Abstract*— One of the biggest recent revolutions in optimization, called semidefinite programming, is a methodology for solving Linear Matrix Inequalities (LMIs) proposed in 1994 by Nesterov and Nemirovski. As it turns out, most optimization problems appearing in systems and control are dimension-independent. Namely, the natural variables are matrices (rather than just collections of scalars) and the problem involves rational expressions in these matrix variables which have therefore the same form independent of the matrix sizes. Hence the study of LMIs in systems and control leads not so much to classical convex analysis and positivity, but rather to the newly emerging areas of (free) noncommutative convexity and noncommutative positivity, with the polynomials and rational functions in commuting variables replaced by noncommutative polynomials and noncommutative rational functions. The purpose of this minicourse is to provide an introduction to noncommutative rational functions and their realization theory on one hand, and to noncommutative positivity and noncommutative convexity, including noncommutative LMIs, on the other.**

I. BACKGROUND AND PURPOSE

One of the biggest recent revolutions in optimization, called semidefinite programming, is a methodology for solving Linear Matrix Inequalities (LMIs) proposed in the 1994 book by Nesterov and Nemirovski [19]; see, e.g., Skelton–Iwasaki–Grigoriadis [22] and the recent survey Nemirovski [18]. As it turns out, most optimization problems appearing in systems and control are *dimension-independent*. Namely, the natural variables are matrices (rather than just collections of scalars) and the problem involves rational expressions in these matrix variables (rather than arbitrary expressions in the matrix entries) which *have therefore the same form independent of the matrix sizes*; see Helton [11] and Helton–McCullough–Putinar–Vinnikov [13] for a detailed discussion. Hence the study of LMIs in systems and control leads not so much to classical convex analysis and positivity, but rather to the newly emerging areas of (free) noncommutative convexity and noncommutative positivity, with the polynomials and rational functions in commuting variables replaced by noncommutative polynomials and noncommutative rational functions. This provides motivation and guidance for the development of a noncommutative semialgebraic geometry; see Helton [10], Helton–McCullough [12], Helton–Putinar [15].

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Historically, noncommutative rational functions first appeared in system theory in the context of recognizable formal power series in the theory of formal languages and finite automata; see Kleene [16], Schützenberger [20], [21], and Fliess [7], [8], [9] (where the motivation comes also from applications to certain classes of nonlinear systems), and Berstel–Reutenauer [6] for a survey. In particular, noncommutative rational functions admit a good state space realization theory. These realizations have important applications to LMIs in the context of dimension-independent problems; see Helton–McCullough–Vinnikov [14]. In addition, state space realizations of noncommutative rational expressions in Hilbert space operators (modelling structured possibly time varying uncertainty) have figured prominently in work on robust control of linear systems; see Beck [4], Beck–Doyle–Glover [5], Lu–Zhou–Doyle [17]. Ball–Groenewald–Malakorn [1], [2], [3] introduced a unified framework of *structured noncommutative multidimensional linear systems* for different kinds of realization formulae.

The purpose of this minicourse is to provide an introduction, on the one hand, to noncommutative rational functions and their realization theory, and on the other hand, to noncommutative semialgebraic geometry: inequalities such as noncommutative positivity, noncommutative convexity, and LMIs in matrix unknowns.

II. PLAN OF THE MINICOURSE

The minicourse will consist of four one-hour lectures. There will be a modest attempt to keep each one self contained.

Lecture 1: Noncommutative rational functions (V. Vinnikov)

Noncommutative rational functions as equivalence classes of noncommutative rational expressions under the evaluation equivalence on tuples of square matrices of all sizes. Difference-differential calculus for noncommutative rational functions. Special cases and applications: directional derivatives, backward shifts, finite difference formulae, higher order difference-differential operators, and connections with formal power series.

Lecture 2: Realization theory of noncommutative rational functions (D. Kaliuzhnyi-Verbovetskyi)

Motivations for realization theorems for noncommutative rational functions: realization formulae for recognizable (= noncommutative rational) formal power series in the theory of finite automata and formal languages, robust control of systems with structured time-variant uncertainties. Structured

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noncommutative multidimensional systems and basic results of realization theory. Bounded real lemma and lossless bounded real lemma. The description of the singularity sets for noncommutative rational functions in terms of their minimal realizations.

Lecture 3: Noncommutative inequalities (I. Klep and J. W. Helton)

Physical problems in matrix and operator unknowns which call for a noncommutative semialgebraic geometry: design of optimal linear systems, Bessis–Moussa–Villani (BMV) conjecture from statistical mechanics. Noncommutative polynomials: positivity and sums of (hermitian) squares, noncommutative Positivstellensatz, convexity of noncommutative rational functions and applications of and to semidefinite programming.

Lecture 4: Basic techniques and noncommutative sets (I. Klep and J. W. Helton)

Positivstellensätze: separation and Gelfand–Naimark–Segal (GNS) construction. Positive noncommutative second derivatives and convexity. Noncommutative convex domains, noncommutative LMI representations and noncommutative separation.

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