# Deterministic identification of lossless and dissipative systems

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*Abstract*— **We illustrate procedures to identify a state-space representation of a passive or bounded-real system from noise-free measurements. The basic idea underlying our algorithms is to obtain a state sequence from a rank-revealing factorization of a Gramian-like matrix constructed from the data. The computation of state-space equations is then performed solving a system of linear equations, similarly to what happens in classical deterministic subspace identification methods.**

## I. PROBLEM STATEMENT

We consider the following problem. We are given a discrete-time $\mathtt{w}$-dimensional trajectory $w$ consisting of inputs $u$ and outputs $y$, which we know is produced by a linear finite-dimensional time-invariant system which is passive- or bounded-real. Moreover, we assume that certain identifiability conditions (described further in the paper) are satisfied. The problem is to find a state-space description of the system.

In order to solve this problem, we could use subspace identification methods (see [7]) to compute a state-space sequence $x$ from the data $w$, and then proceed to solve for $(A, B, C, D)$ in the equations

$$\begin{bmatrix} \sigma x \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \ , \tag{1}$$

where $\sigma$ is the backwards shift defined by

$$(\sigma x)(k) := x(k+1) \ .$$

In this paper, we show that a state sequence $x$ can be alternatively obtained from any rank-revealing factorization of a Gramian-like matrix computed from the data $w$; the matrices corresponding to a state-space representation of the system can then be determined solving (1). Our approach can consequently be considered as an alternative to the classical subspace-identification methods that uses energy considerations in order to compute a state sequence, rather than exploiting the idea of intersecting the row-spaces of the 'past' and 'future' matrices of the data.

Different rank-revealing factorizations of the Gramian-like matrix computed from the data produce different state sequences, which in their turn correspond to different state representations $(A, B, C, D)$. In this paper we show how to exploit this fact in order to obtain positive- and bounded-real balanced state-space representations from data. This aspect of our approach makes it particularly interesting for application to the problem of model order reduction from data, i.e. the problem of obtaining a reduced-order model directly from measurements of a system.

In this paper, the concepts and tools of the behavioral approach, and of quadratic difference forms will be put to strenuous use. The reader not familiar with them is referred to [8], [2] for a thorough exposition. In section II we only review the background material strictly necessary for making this paper self-contained. In section III-A we state the main result; we consider the lossless case first for simplicity of exposition. We illustrate the extension of the problem to the case of dissipative systems in section III-B. The paper ends with some concluding remarks, contained in section IV.

*Notation.* We denote the ring of integers with $\mathbb{Z}$, and the set of nonnegative integers $\{z \in \mathbb{Z} \mid z \geq 0\}$ with $\mathbb{Z}_+$. We denote the field of real numbers with $\mathbb{R}$, and the field of complex numbers with $\mathbb{C}$. The space of $\mathtt{n}$ dimensional real vectors is denoted by $\mathbb{R}^{\mathtt{n}}$, and the space of $\mathtt{m} \times \mathtt{n}$ real matrices, by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$. The symbol $\mathbb{R}^{\mathtt{m}\times\bullet}$ denotes the space of real matrices with $\mathtt{m}$ rows, and $\mathbb{R}^{\bullet\times\bullet}$ the space of real matrices with a finite but unspecified number of rows and columns. The symbol $\mathbb{R}^{\mathtt{m}\times\infty}$ denotes the set of real matrices with $\mathtt{m}$ rows and an infinite number of columns. If $A \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$, then $A^\top \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$ denotes its transpose. If $A_1, \ldots, A_n$ are matrices with the same number of columns, we denote with $\mathrm{col}(A_1, \ldots, A_n)$ the matrix obtained by stacking the $A_i$, $i = 1 \ldots, n$ on top of each other.

The space consisting of all sequences from $\mathbb{Z}$ to $\mathbb{R}^{\mathtt{w}}$ is denoted with $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$. On this space we define the *backward shift* $(\sigma w)(t) := w(t+1)$ for all $t \in \mathbb{N}$. The symbol $\ell_2^{\mathtt{w}}(\mathbb{Z})$ denotes the linear space of all square-summable sequences on $\mathbb{Z}$.

The ring of polynomials with real coefficients in the indeterminate $\xi$ is denoted by $\mathbb{R}[\xi]$; the ring of two-variable polynomials with real coefficients in the indeterminates $\zeta$ and $\eta$ is denoted by $\mathbb{R}[\zeta, \eta]$. We denote with $\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi]$ (respectively, $\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi, \xi^{-1}]$) the space of all $\mathtt{r} \times \mathtt{w}$ matrices with entries in the ring $\mathbb{R}[\xi]$ of polynomials in the indeterminate $\xi$ with real coefficients (respectively in the ring $\mathbb{R}[\xi, \xi^{-1}]$ of Laurent polynomials in the indeterminate $\xi$ with real coefficients). We denote with $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta, \eta]$ the space of $\mathtt{n} \times \mathtt{m}$ polynomial matrices in the indeterminates $\zeta$ and $\eta$.

## II. BACKGROUND MATERIAL

In this paper we consider linear, shift-invariant and 'complete' (see Definition II.4 p. 262 of [9]) subspaces $\mathfrak{B}$ of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$. We call such subspaces *behaviors* and denote the set consisting of all behaviors with $\mathtt{w}$ variables with $\mathfrak{L}^{\mathtt{w}}$. If

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$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, then there exists a polynomial matrix $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that

$$\mathfrak{B} = \{w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}} \mid R(\sigma)w = 0\}. \tag{2}$$

Equation (2) is called a *kernel representation* of the behavior. Another representation of importance is the *hybrid* one, in which besides the *external variable* $w$ also the *latent variable* $\ell$ is present:

$$R(\sigma)w = M(\sigma)\ell \tag{3}$$

Equation (3) has associated a *full behavior* $\mathfrak{B}_f := \{(w,\ell) \mid (w,\ell) \text{ satisfy } (3)\}$ and an *external behavior* $\mathfrak{B} := \{w \mid \exists\ \ell \text{ s.t. } (w,\ell) \text{ satisfy } (3)\}$.

A state representation is a special hybrid representation with a latent variable $x$ satisfying the state property (see Definition VII.1 p. 268 of [9]). It can be shown that any behavior allows a *state representation* $E\sigma x + Fx + Gw = 0$, or a *input-state-output representation* (1) with the *state variable* $x$.

The notion of *controllability* is illustrated thoroughly in section V of [9]; we do not give a formal definition here, since it is not essential for the rest of the paper. However, it is important to remark that if a behavior is controllable, then it contains nonzero $\ell_2^{\mathtt{w}}(\mathbb{Z})$-trajectories, and in particular finite support ones. In the following we denote the subset of $\mathfrak{L}^{\mathtt{w}}$ consisting of all controllable behaviors with $\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$.

We now review some basic concepts regarding *bilinear-* and *quadratic difference forms* relevant for the material presented in this paper; see [2] for a thorough introduction. Let $\Phi \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta,\eta]$; then $\Phi(\zeta,\eta) = \sum_{h,k=0}^{N} \Phi_{h,k}\zeta^h\eta^k$, where $\Phi_{h,k} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$ and $N$ is a nonnegative integer. $\Phi(\zeta,\eta)$ induces the *bilinear difference form* (*BdF*)

$$\begin{aligned} L_\Phi \quad &: \quad (\mathbb{R}^{\mathtt{w}_1})^{\mathbb{Z}} \times (\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}} \longrightarrow \mathbb{R}^{\mathbb{Z}} \\ L_\Phi(w_1,w_2)(t) \quad &:= \quad \sum_{h,k=0}^{N} w_1(t+h)^\top \Phi_{h,k} w_2(t+k)\ . \end{aligned}$$

If $\mathtt{w}_1 = \mathtt{w}_2$, then $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ also induces a *quadratic difference form* (in the following abbreviated with *QdF*)

$$\begin{aligned} Q_\Phi \quad &: \quad (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}} \longrightarrow \mathbb{R}^{\mathbb{Z}} \\ Q_\Phi(w)(t) \quad &:= \quad \sum_{h,k=0}^{N} w(t+h)^\top \Phi_{h,k} w(t+k)\ . \end{aligned}$$

When considering QdFs, without loss of generality we assume the two-variable polynomial matrix $\Phi(\zeta,\eta)$ to be *symmetric*, i.e. $\Phi(\zeta,\eta) = \Phi(\eta,\zeta)^\top$.

The *rate of change* of a QdF $Q_\Phi$, denoted $\nabla Q_\Phi$, is

$$\nabla Q_\Phi(w)(k) := Q_\Phi(w)(k+1) - Q_\Phi(w)(k)\ .$$

In the following we often use the concept of positivity of a QdF. We call $Q_\Phi$ *nonnegative* (denoted $Q_\Phi \geq 0$) if $Q_\Phi(w) \geq 0$ for all $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$. We call $Q_\Phi$ *positive* (denoted $Q_\Phi > 0$) if $Q_\Phi(w) > 0$ for all $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$, $w \neq 0$. If $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and $Q_\Phi$ is nonnegative for all trajectories $w \in \mathfrak{B}$, then we say that $Q_\Phi$ is *nonnegative along* $\mathfrak{B}$, denoted $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$. The notion of positivity along $\mathfrak{B}$ is defined and denoted analogously.

We now introduce the concepts of dissipativity, losslessness, and storage functions. $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is *dissipative* with respect to the *supply rate* $Q_\Phi$ if there exists a QdF $Q_\Psi$, called a *storage function*, such that

$$\nabla Q_\Psi(w) \leq Q_\Phi(w) \text{ for all } w \in \mathfrak{B}\ . \tag{4}$$

Equation (4) can be shown to be equivalent (see Proposition 3.3 of [2]) to the existence of a *dissipation function*, i.e. a QdF $Q_\Delta \geq 0$ such that

$$\sum_{k=-\infty}^{+\infty} Q_\Delta(w)(k) = \sum_{k=-\infty}^{+\infty} Q_\Phi(w)(k) \text{ for all } w \in \mathfrak{B} \cap \ell_2^{\mathtt{w}}(\mathbb{Z})\ . \tag{5}$$

Moreover, there is a one-one correspondence between storage functions and dissipation rates, in the sense that for every dissipation function $Q_\Delta$ there exists a unique storage function $Q_\Psi$, and for every storage function $Q_\Psi$ there exists a unique dissipation function $Q_\Delta$, such that for all $w \in \mathfrak{B}$

$$\nabla Q_\Psi(w) + Q_\Delta(w) = Q_\Phi(w)\ . \tag{6}$$

If (4) is an equality, equivalently if $Q_\Delta = 0$ in (6), then $\mathfrak{B}$ is called *lossless with respect to* $Q_\Phi$. If $\sum_{k=-\infty}^{0} Q_\Phi(w)(k) \geq 0$ for all $w \in \mathfrak{B}_{|(-\infty,0]} \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_-)$, then $\mathfrak{B}$ is called *half-line dissipative*; and if a system is lossless and every storage function is positive-definite, then $\mathfrak{B}$ is called *half-line lossless*. Evidently, half-line dissipativity (respectively losslessness) implies dissipativity (respectively losslessness).

In this paper we give special attention to the case of *passive systems*, with $\mathtt{u} = \mathtt{y}$ and the supply rate induced by

$$\Phi = \begin{bmatrix} 0 & I_{\mathtt{u}} \\ I_{\mathtt{u}} & 0 \end{bmatrix}\ ; \tag{7}$$

and to the case of *bounded-real systems*, with

$$\Phi = \begin{bmatrix} I_{\mathtt{u}} & 0 \\ 0 & -I_{\mathtt{y}} \end{bmatrix}\ . \tag{8}$$

In this paper an essential role is played by the fact that under suitable conditions *storage functions for discrete-time systems are quadratic functions of the state*; we now discuss this issue in detail. We say that a storage function $Q_\Psi$ is a quadratic function of the state if, given a state representation for $\mathfrak{B}$ with state variable $x$, there exists $K = K^\top \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that for every trajectory $(x,w) \in \mathfrak{B}_f$ it holds $Q_\Psi(w) = x^\top K x$. While every storage function is a quadratic function of the state for continuous-time systems (see Theorem 5.5 of [11]), it is not in discrete-time, see [3]: additional assumptions are needed. It can be shown that this is the case when the system is lossless (see Theorem 5.3 of [3]) or when every storage function is nonnegative (see Theorem 5.1 of [3]). Another sufficient condition is given in Theorem 5.2 of [3], and a necessary and sufficient condition is given in Proposition 2 of [4]. In particular, storage functions for passive and for bounded-real systems are always a quadratic function of the state.

## III. DETERMINISTIC IDENTIFICATION

In this section and in the rest of the paper we deal with trajectories $w$ defined on $\mathbb{Z}_+$. Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, we denote with $\mathfrak{B}_+ := \{w_{|\mathbb{Z}_+} \mid w \in \mathfrak{B}\}$, and with $\ell_2^{\mathtt{w}}(\mathbb{Z}_+)$ the set of $\mathtt{w}$-dimensional square summable trajectories on $\mathbb{Z}_+$.

The notion of *persistency of excitation* is the only identifiability condition we need for our identification algorithms; we now illustrate it. A sequence $u : \mathbb{Z}_+ \to \mathbb{R}^{\mathtt{m}}$ is said to be *persistently exciting of order* $L$ (abbreviated *p.e. of order* $L$ in the following) if

$$\operatorname{rank} \begin{bmatrix} u(0) & u(1) & \cdots \\ u(1) & u(2) & \cdots \\ \vdots & \vdots & \cdots \\ u(L-1) & u(L) & \cdots \end{bmatrix} = L\mathtt{m} \, .$$

In Corollary 2 of [10] it has been shown that for every trajectory $(u, x)$ of a state-space system $\sigma x = Ax + Bu$ with $\mathtt{n}$-dimensional state vector $x$ and $\mathtt{m}$-dimensional input $u$, if $T$ is "sufficiently large", i.e. $T \geq \mathtt{nm}$, it holds that

$$u \text{ p.e. of order } \mathtt{n} \Longrightarrow \operatorname{rank} \begin{bmatrix} u(0) & \cdots & u(T) \\ x(0) & \cdots & x(T) \end{bmatrix} = \mathtt{n} + \mathtt{m} \quad (9)$$

It follows that if $u$ is p.e. of order $\mathtt{n}$, then $\operatorname{rank} \begin{bmatrix} x(0) & \cdots & x(T) \end{bmatrix} = \mathtt{n}$.

### A. The lossless case

State sequences are computed from the data by performing a rank-revealing factorization of the $S$-matrix, which we now introduce.

*Definition 1:* Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ , $w \in \mathfrak{B}_+ \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$ and $\Phi = \Phi^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. The *S-matrix* associated with $w$ and $\Phi$ is the infinite matrix

$$[S(w)]_{i,j=0,\ldots} := \sum_{k=0}^{\infty} L_\Phi(\sigma^i w, \sigma^j w)(k) \quad (10)$$

Since $w \in \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$ the $S$-matrix is well-defined, and each of its entries is a real number.

The following result is the foundation for the deterministic identification procedures we illustrate in this paper.

*Proposition 2:* Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, and let $\mathfrak{B}_f$ be a state representation of $\mathfrak{B}$ with state variable $x$. Assume that $\mathfrak{B}$ is half-line lossless with respect to the supply rate induced by $\Phi = \Phi^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Then there exists a nonsingular $K = K^\top \in \mathbb{R}^{\bullet\times\bullet}$ such that for every $w = \operatorname{col}(u, y) \in \mathfrak{B}_+ \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$ with associated state trajectory $x$, i.e. $(x, u, y) \in \mathfrak{B}_f$ the following equality holds:

$$S(w) = \begin{bmatrix} x(0)^\top \\ x(1)^\top \\ \vdots \end{bmatrix} K \begin{bmatrix} x(0) & x(1) & \cdots \end{bmatrix} . \quad (11)$$

*Proof:* Since $w \in \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, it follows that $\lim_{k\to\infty} w(k) = 0$. From the losslessness of $\mathfrak{B}$ and Theorem 5.3 of [3] it follows that every storage function is a quadratic function of the state. Consequently, there exists a real symmetric matrix $K'$ such that for every pair of trajectories $(x_i, w_i)$, $i = 1, 2$ in the full behavior $\mathfrak{B}_f$ it holds

$$\begin{aligned} \sum_{k=0}^{+\infty} w_1(k)^\top \Phi w_2(k) &= -x_1(0)^\top K' x_2(0) \\ &+ x_1(1)^\top K' x_2(1) - x_1(1)^\top K' x_2(1) + \ldots \\ &= -x_1(0)^\top K' x_2(0) \, , \end{aligned}$$

where we have used the fact that since $\lim_{k\to\infty} w(k) = 0$, also $\lim_{k\to\infty} x(k) = 0$. It is easy to see that this implies $\sum_{k=0}^{+\infty} (\sigma^i w)(k)^\top \Phi (\sigma^j w)(k) = -x(i)^\top K' x(j)$ for $i, j = 0, \ldots$. This proves equation (11), with $K := -K'$. In order to prove nonsingularity of $K$, recall that $\mathfrak{B}$ is half-line lossless, and consequently every storage function is positive-definite. ■

The following result is a direct consequence of Proposition 2.

*Corollary 3:* Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ be half-line lossless with respect to the supply rate induced by $\Phi = \Phi^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, and let $\mathfrak{B}_f$ be a minimal state representation of $\mathfrak{B}$ with state variable $x$. Let $w = \operatorname{col}(u, y) \in \mathfrak{B}_+ \cap l_2^{\mathtt{w}}(\mathbb{Z}_+)$, and assume that $u$ is p.e. of order $\mathtt{n}(\mathfrak{B})$. Then the $S$-matrix has rank $\mathtt{n}(\mathfrak{B})$.

*Proof:* Recall that the storage function matrix $K' \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$ is positive definite. Now use the persistency of excitation of $u$ in order to conclude that $\operatorname{rank} \begin{bmatrix} x(0) & x(1) & \cdots \end{bmatrix} = \mathtt{n}(\mathfrak{B})$. Finally, use (11). ■

It follows from Proposition 2 and Corollary 3 that in order to compute a minimal state sequence corresponding to the data $w$, one can proceed as follows. Define a *rank-revealing factorization* of $S(w)$ to be any factorization $S(w) = U\Delta U^\top$ with $U \in \mathbb{R}^{\infty\times\mathtt{n}(\mathfrak{B})}$ and $\Delta \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$ both having full rank, equal to $\mathtt{n}(\mathfrak{B}) = \operatorname{rank} S(w)$. It follows from (11) that a state sequence $x(0)$, $x(1)$, $\ldots$ can be obtained from such a rank-revealing factorization as

$$\begin{bmatrix} x(0) & x(1) & \ldots \end{bmatrix} := U^\top$$

Moreover, the matrices $A$,$B$, $C$, $D$ corresponding to a minimal input-state-output representation of $\mathfrak{B}$ can be obtained solving the set of linear equations

$$\begin{aligned} &\begin{bmatrix} x(1) & x(2) & x(3) & \cdots \\ y(0) & y(1) & y(2) & \cdots \end{bmatrix} \\ &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(0) & x(1) & x(2) & \cdots \\ u(0) & u(1) & u(2) & \cdots \end{bmatrix} . \quad (12) \end{aligned}$$

In order to obtain a state representation $E\sigma x + Fx + Gw = 0$, the system of linear equations in the matrices $E$, $F$, and $G$ needs to be solved:

$$\begin{bmatrix} E & F & G \end{bmatrix} \begin{bmatrix} x(1) & x(2) & x(3) & \cdots \\ x(0) & x(1) & x(2) & \cdots \\ w(0) & w(1) & w(2) & \cdots \end{bmatrix} = 0 \, . \quad (13)$$

Note that solutions $(A, B, C, D)$ and $(E, F, G)$ to equations (12) and (13) always exists since $x$ is a state variable, see for example Prop. VII.3 of [9].

*Remark 4:* When considering the application of the results of Proposition 2 and Corollary 3 to real data in order to compute a state representation of $\mathfrak{B}$, we need to consider that only a finite number of measurements of $w$ is available, and that consequently only an *approximation* of the entries of the $S$-matrix can be computed. It follows that a rank-revealing factorization of this approximate $S$-matrix only corresponds to an *approximation* of an actual state sequence of the data-producing system. An expedient solution is to assume that a "sufficiently large" time window of the data is given; since $w \in \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, $w(T) \simeq 0$ for "large enough" $T$ and the approximation error becomes negligible if the data is known from 0 to $T$. The assumption of "sufficiently large" $T$T is made also in [5]. This stratagem however cannot be considered entirely satisfactory, and a thorough investigation on the finite-measurements issues is required; we will pursue this elsewhere.

Different rank-revealing factorizations of the Gramian-like matrix produce different state sequences, which in their turn correspond to different state representations. We next show how one can exploit this in order to obtain balanced state space representations from data. We first explain in what sense "balanced state space representation" must be understood. Assume that every storage function is positive definite; we say that a minimal state space representation of $\mathfrak{B}$ is *balanced* if the matrices $K_-$ and $K_+$ corresponding to the minimal and the maximal storage functions $x^\top K_- x$ and $x^\top K_+ x$ are diagonal and inverse of each other, i.e. $K_- = K_+^{-1} = \Lambda$ for some diagonal matrix $\Lambda$. In the case of $\Phi$ given by (7), respectively (8), this definition of balanced state representation coincides with the classical passive, respectively bounded-real, balanced realization. Note that in the lossless case, the maximal- and minimal storage functions coincide, and consequently a realization is balanced if the matrix $K$ corresponding to this unique storage function is the identity.

We now show that by choosing appropriately the rank-revealing factorization (11) and solving the corresponding equations (12), a balanced realization of the data-producing system $\mathfrak{B}$ can be obtained. In order to do this, the following result is instrumental.

*Proposition 5:* Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$, and let $\mathfrak{B}_f$ be a minimal input/state/output representation of $\mathfrak{B}$ with state variable $x$, associated with the matrices $A$, $B$, $C$, $D$. Assume that $\mathfrak{B}$ is half-line lossless with respect to the supply rate induced by $\Phi = \Phi^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Then the matrix $K = K^\top \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$ satisfying equation (11) is equal to the unique real symmetric solution of the equations

$$\begin{aligned} B(-K)B^\top - \Phi_{uu} - D^\top\Phi_{uy}^\top - \Phi_{uy}D - D^\top\Phi_{yy}D &= 0 \\ A^\top(-K)B - \Phi_{uy}C - D^\top\Phi_{yy}C &= 0 \\ A^\top(-K)A - (-K) - C^\top\Phi_{yy}C &= 0 \end{aligned}$$

*Proof:* Follows in a straightforward manner from the Kalman-Yakubovich-Popov lemma. ■

It follows from Proposition 5 that under the assumption that $u$ is p.e. of order $\mathtt{n}(\mathfrak{B})$, if the matrix $K$ satisfying (11) equals $-I_{\mathtt{n}(\mathfrak{B})}$, i.e. if the factorization of the $S$-matrix is of the form $S(w) = -UU^\top$, then the realization obtained by solving (12) is balanced. This is interesting in view of the application of our ideas to the problem of *model reduction from data*: given a sequence $w$ produced by an unknown, high-order system, compute from it a reduced-order model for the system. These refinements will be pursued elsewhere.

We conclude this subsection with the statement of an algorithm for the identification of a state-space representation of a lossless system from noise-free data; note that we gloss over the issue of finite measurements (see Remark 4). We use Matlab© notation: if $U$ is a matrix, then $U(1:j,:)$ is its submatrix consisting of the first $j$ rows of $U$.

**Algorithm 1**

Input: $w = \mathrm{col}(u, y) \in \mathfrak{B}_+ \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, with $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ half-line lossless with respect to the supply rate induced by $\Phi = \Phi^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$.

Output: A minimal input-state-output representation of $\mathfrak{B}$.

Step 1: Compute the $S$-matrix (10).
Step 2: Compute $\mathtt{n} := \mathrm{rank}\ S(w)$.
Step 3: Factorize $S(w) = U\Delta U^\top$ with $U \in \mathbb{R}^{\infty\times\mathtt{n}}$, $\Delta \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. (For a balanced realization do Step 3 with $\Delta = -I_{\mathtt{n}}$.)
Step 4: Define $X := U(1:\infty,:)^\top$, $\sigma X := U(2:\infty,:)^\top$.
Step 5: Solve (12) in the unknowns $A, B, C, D$.
Step 6: Return $A, B, C, D$.

*B. The dissipative case*

In some areas of application, e.g. when dealing with mechanical systems, a dissipation function is often either known or can be computed on the basis of physical considerations. In other situations the given data $w$ is of a special nature; for example, $w$ is known to be a trajectory of zero dissipation, i.e. $Q_\Delta(w)$ is identically zero. In situations like these it makes sense to consider the extension of the approach illustrated in section III-A to the case of dissipative systems. This extension is rather straightforward, and it is based on the following intuition: if a system is dissipative with respect to a supply rate $Q_\Phi$, then it is lossless with respect to the supply rate $Q_\Phi - Q_\Delta$, with $Q_\Delta$ a dissipation function. Consequently, if $Q_\Delta$ is known (or if the data consists of a zero-dissipation function), and if some additional assumptions guaranteeing that storage functions are a quadratic function of the state are satisfied, then the procedure illustrated in the previous section can be easily adapted to the dissipative case. We now formalize this intuition.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ be $\Phi$-half-line dissipative with $\Phi = \Phi^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, and assume that a dissipation function induced by a two-variable polynomial $\Delta \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ is known. Moreover, assume that the storage function $Q_\Psi$ associated with the dissipation function $Q_\Delta$ is a quadratic function of the state. As explained already in section II, note that while this is always true in continuous-time, it is not in discrete-time; see section II for the statement of several sufficient conditions for this purpose.

From the dissipation equality (6) it follows in a straightforward manner that for every $w_1, w_2 \in \mathfrak{B}$

$$L_\Phi(w_1, w_2) = L_\Delta(w_1, w_2) + \nabla L_\Psi(w_1, w_2) ,$$

where $L_\Phi$, $L_\Delta$, and $L_\Psi$ are the *bilinear* difference forms associated with $\Phi$, and with the two-variable polynomial matrices $\Psi(\zeta, \eta)$ and $\Delta(\zeta, \eta)$ corresponding to the storage function and the dissipation function respectively. Now let $\mathfrak{B}_f$ be a state representation of $\mathfrak{B}$ with state variable $x$; if $w_1, w_2 \in \mathfrak{B}_+ \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, with associated full trajectories $(w_i, x_i) \in \mathfrak{B}_f$, $i = 1, 2$, then

$$\begin{aligned} &\sum_{k=0}^{\infty} w_1(k)^\top \Phi w_2(k) \\ &= \sum_{k=0}^{\infty} L_\Delta(w_1, w_2)(k) - x_1(0)^\top K' x_2(0) \end{aligned}$$

where $K' = K'^\top \in \mathbb{R}^{\bullet \times \bullet}$ is the matrix corresponding to the storage function $Q_\Psi$ and the state variable $x$. Now define the *generalized $S$-matrix* as

$$\begin{aligned} [S(w)]_{i,j=0,\ldots} \quad := \quad & \sum_{k=0}^{\infty} L_\Phi(\sigma^i w, \sigma^j w)(k) \\ & - \sum_{k=0}^{\infty} L_\Delta(\sigma^i w, \sigma^j w)(k) \end{aligned} \quad (14)$$

and note that

$$S(w) \quad = \quad \begin{bmatrix} x(0)^\top \\ x(1)^\top \\ \vdots \end{bmatrix} K \begin{bmatrix} x(0) & x(1) & \cdots \end{bmatrix} , \quad (15)$$

where $K := -K'$.

This argument proves the following result, analogous to Proposition 2 of section III-A.

*Proposition 6:* Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, and let $\mathfrak{B}_f$ be a state representation of $\mathfrak{B}$ with state variable $x$. Assume that $\mathfrak{B}$ is half-line dissipative with respect to $\Phi = \Phi^\top \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$, and let $\Delta \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}_s[\zeta, \eta]$ induce a dissipation function for $\mathfrak{B}$. Assume that the storage function associated with $Q_\Delta$ is a quadratic function of the state. Then there exists $K = K^\top \in \mathbb{R}^{\bullet \times \bullet}$ such that for all $w = \text{col}(u, y) \in \mathfrak{B}_+ \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, with associated state trajectory $x$, i.e. $(x, u, y) \in \mathfrak{B}_f$, the generalized $S$-matrix (14) satisfies (15).

From the factorization (15) of the generalized $S$-matrix (14) and from the persistency of excitation of $u$ it follows that if $\mathfrak{B}_f$ is a minimal state representation of $\mathfrak{B}$ and if $K$ is sign-definite, then $S(w)$ has rank $\mathtt{n}(\mathfrak{B})$. In order to ensure that any storage function is sign-definite, we assume that the system is half-line dissipative, and that the number $\mathtt{m}(\mathfrak{B})$ of input variables equals the number $\sigma_+(\Phi)$ of positive eigenvalues of the supply rate matrix $\Phi$. Indeed in this case, using Theorem 5.3 of [3] and the same argument of the proof of Theorem 6.4 of [11] for the continuous-time case, it can be shown that all storage functions are positive, in the sense that if a storage function is given by $x^\top K x$, then $K > 0$. Note that the assumption $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Phi)$ holds for passive and for bounded-real systems, see equations (7) and (8) respectively.

These considerations lead to the following result.

*Proposition 7:* Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, and let $\mathfrak{B}_f$ be a minimal state representation of $\mathfrak{B}$ with state variable $x$. Assume that $\mathfrak{B}$ is $\Phi$-half-line dissipative with $\Phi = \Phi^\top \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$, and let $\Delta \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}_s[\zeta, \eta]$ induce a dissipation function for $\mathfrak{B}$. Assume that the storage function associated with $Q_\Delta$ is a quadratic function of the state, and moreover assume that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Phi)$. Then the matrix $K = K^\top \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}) \times \mathtt{n}(\mathfrak{B})}$ corresponding to the storage function is positive-definite. Moreover, let $w = \text{col}(u, y) \in \mathfrak{B}_+ \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, and assume that $u$ is p.e. of order $\mathtt{n}(\mathfrak{B})$. Then $\text{rank } S(w) = \mathtt{n}(\mathfrak{B})$.

*Proof:* Proven analogously to Proposition 3. ■

*Remark 8:* Propositions 6 and 7, allow to formulate a procedure for deterministic identification of a state-space representation of a dissipative systems completely analogous to Algorithm 1; indeed, the only modification to be performed is in Step 1, where the generalized $S$-matrix (14) is used in place of the $S$-matrix (10).

*Remark 9:* The problem of identifying a dissipation function arises in many areas of application, for example in vibration theory for mechanical systems, where the identification of the damping coefficient is a standard problem usually solved using physical insight, see [1]. We are currently investigating whether the framework illustrated in this paper offers an alternative approach to this problem. A possible way to restrict the search for suitable candidate dissipation functions could be to use the dissipation equality (6) in order to conclude that a dissipation function, besides being nonnegative, should also make the generalized $S$-matrix corresponding to it to have rank equal to $\mathtt{n}(\mathfrak{B})$ if this is known, or "as low as possible" if this additional information on the McMillan degree of the system is unavailable. An open research question is to find and if possible to characterize such functionals, given as few as possible arbitrary assumptions on the form of the functional. This will be pursued elsewhere.

Finally, we conclude this section discussing the computation of balanced realizations from data. We begin with the following result.

*Proposition 10:* Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, and let $\mathfrak{B}_f$ be a minimal input/state/output representation of $\mathfrak{B}$ with state variable $x$ associated with the matrices $(A, B, C, D)$. Assume that $\mathfrak{B}$ is $\Phi$-half-line dissipative, and let $\Delta \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}_s[\zeta, \eta]$ induce a dissipation function for $\mathfrak{B}$. Assume that the storage function associated with $Q_\Delta$ is a quadratic function of the state, and moreover that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Phi)$. Let $w = \text{col}(u, y) \in \mathfrak{B} \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, and let $S(w)$ be defined as in (14). Define

$$\begin{aligned} R \quad &:= \quad \Phi_{uu} + D^\top \Phi_{uy}^\top + \Phi_{uy} D + D^\top \Phi_{yy} D \\ S^\top \quad &:= \quad \Phi_{uy} C + D^\top \Phi_{yy} C \\ Q \quad &:= \quad C^\top \Phi_{yy} C . \end{aligned} \quad (16)$$

Let $K$ be such that (15) holds, and assume that $R - B^\top K B > 0$; then the matrix $K' := -K$ satisfies the algebraic Riccati equation

$$0 = A^\top K' A - K' + Q - \left(A^\top K' B + S\right)\left(B^\top K' B + R\right)^{-1}\left(B^\top K' A + S^\top\right)$$

*Proof:* The result follows from the well-known relationship between storage functions and solutions of the algebraic Riccati equation. ■

Now assume that the dissipation functions $\Delta_+$ and $\Delta_-$ corresponding to the maximal and the minimal storage functions $\Psi_+$ and $\Psi_-$ (see section 4 of [2] and section 3 of [4] for details) are known, and that the corresponding storage functions are quadratic functions of the state. Assume also that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Phi)$, so that if $x^\top K x$ is a storage function, then $K > 0$. Assume also that $u$ is persistently exciting of order $\mathtt{n}(\mathfrak{B})$. Then we can obtain a balanced realization directly from the data $w = (u, y)$, as we now show.

We first compute two generalized $S$-matrices:

$$\begin{aligned} [S_-(w)] &:= \left[L_\Phi(\sigma^i w, \sigma^j w) - L_{\Delta_-}(\sigma^i w, \sigma^j w)\right]_{i,j=0,\dots} \\ [S_+(w)] &:= \left[L_\Phi(\sigma^i w, \sigma^j w) - L_{\Delta_+}(\sigma^i w, \sigma^j w)\right]_{i,j=0,\dots} \end{aligned} \quad (17)$$

We then compute a rank-revealing factorization of $S_-(w)$ as $S_-(w) = -V^\top V$. Recall that the columns of the matrix $V$ form a minimal state sequence $x(0)$, $x(1)$, $\dots$; consequently, there exists $K_+ = K_+^\top \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}) \times \mathtt{n}(\mathfrak{B})}$, $K_+ > 0$, such that $S_+(w) = V^\top(-K_+)V$. It is immediate to verify that $K_+$ can be computed from the data as $K_+ = -\left(VV^\top\right)^{-1} V S_+(w) V^\top \left(VV^\top\right)^{-1}$. Now compute a singular value decomposition of $-\left(VV^\top\right)^{-1} V S_+(w) V^\top \left(VV^\top\right)^{-1}$:

$$-\left(VV^\top\right)^{-1} V S_+(w) V^\top \left(VV^\top\right)^{-1} =: U \Sigma U^\top , \quad (18)$$

and observe that both $U$ and $\Sigma$ are square and nonsingular. Now define $T := U\Sigma^{-\frac{1}{4}}$; then it is a matter of straightforward verification to check that $S_-(w) = V^\top \Sigma^{-\frac{1}{2}} V$ and that $S_+(w) = V^\top \Sigma^{\frac{1}{2}} V$, where $V' := T^{-1} V = \Sigma^{\frac{1}{4}} U^\top V$. Since $T$ is nonsingular and the columns of $V$ form a (minimal) state sequence, also the columns of $V'$ form a (minimal) state sequence; moreover, the matrices $(A, B, C, D)$ corresponding to this state sequence are such that the matrices associated with the minimal and the maximal storage functions are respectively $-\Sigma^{-\frac{1}{2}}$ and $-\Sigma^{\frac{1}{2}}$. It follows that $(A, B, C, D)$ is a balanced input/state/output representation of $\mathfrak{B}$.

## IV. CONCLUSIONS

We have illustrated a novel approach to identify lossless- and dissipative systems; the basic idea is to compute first a state trajectory from a rank-revealing factorization of a "Gramian" matrix associated with the measured data, and then compute the matrices $(A, B, C, D)$ corresponding to this state sequence solving (12). The cases of bounded-real and positive-real systems are important special cases for the application of the algorithms presented in this paper.

If the data-producing system is lossless, this identification technique does not require any additional knowledge about the data-generating system except the supply rate; in the dissipative case, in order for the technique to be applied also the dissipation function must be known. Our identification procedure has been shown to yield in a straightforward manner balanced state representations.

Current research is aimed in several directions. Firstly, we want to consider the application of this procedure to the problem of model reduction from data. Secondly, it is necessary to carry out a detailed analysis of the computational costs involved in the identification, and to investigate efficient and numerically accurate algorithms for the implementation of the procedure. Thirdly, we aim at investigating the case of noisy measurements. Fourthly, the important generalization of our approach to the case of finite measurements must be pursued. Finally, we want to address some of the research questions described in Remark 9.

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