The Behavioral Approach to Simultaneous Stabilization

Osamu Kaneko

***Abstract*— Simultaneous stabilization is the problem of finding a condition under which there exists a single controller that stabilizes multiple (which is denoted with N in this paper) plants. In this paper, we address the problem of simultaneous stabilization in the behavioral framework. First, we provide a new equivalent condition for a pair of linear systems to be simultaneously stabilizable. We then also present a representation of simultaneous stabilizers under the assumption that the interconnection of these two behavior is stable. By using this result, we address to derive a condition under which a set of three linear behaviors is simultaneously stabilizable. In this case, we show that: if one of the three behaviors stabilizes the other two behaviors, then a set of these three behaviors are simultaneously stabilizable. Moreover, a representation for simultaneous stabilizer in this case is also presented under this assumption.**

I. INTRODUCTION

This paper considers the simultaneous stabilization problem in a behavioral framework. Simultaneous stabilization is the problem of finding a condition under which there exists a single controller that stabilizes multiple (which is denoted by N in this paper) plants. This problem was firstly investigated in [18], and then there have been many studies on this issue in the input/output setting, e.g., [1], [2], [3], [4], [5], [6], [7], [13], [14], [17], [19]. The case of $N = 2$, it is well known that a pair of linear plants is simultaneously stabilizable if and only if there exists a strong stabilizer, – the denominator of the stabilizer is also stable –, for the augmented system constructed using these two systems. The case of $N \geq 3$, it was shown that we can not find a rational[1] necessary and sufficient condition, and thus simultaneous stabilization is known as one of the open problems in systems and control theory (cf. [3], [4], [15]). Indeed, though it is impossible to find any equivalent condition for multiple plants to be simultaneous stabilizable, our goal is to derive a sufficient condition which is closer to necessary one. And then, we also aim to extracting intrinsic ingredients related to system structures from the theoretical perspective.

By the way, J.C. Willems has proposed the behavioral approach, which provides a new viewpoint for dynamical system theory(cf. [20], [21]). In this approach, a control is regarded as an "interconnection"(cf.[21], [12]) which is a generalization of the concept of "control" from a broader perspective. Thus, it is to be expected that the behavioral approach provides new and meaningful insights for an important theoretical issue like the simultaneous stabilization problem. From these backgrounds, the author has been studied this issues from the behavioral perspective. In [9], the author provided a necessary and sufficient condition for a pair of linear plants to be simultaneously stabilizable, and then the author provided a sufficient condition for a set of three linear plants to be simultaneously stabilizable in [10]. However, a sufficient condition provided in [10] requires conservative in the sense of that we require various assumptions which are not intuitively understandable.

In this paper, we address the problem of simultaneous stabilization in the behavioral framework. First, we provide a new equivalent condition for a pair of linear systems to be simultaneously stabilizable. We then also present a representation of simultaneous stabilizers under the assumption that the interconnection of these two behavior is stable. By using this result, we address to derive a condition under which a set of three linear behaviors is simultaneously stabilizable. In this case, we show that: if one of the three behaviors stabilizes the other two behaviors, then a set of these three behaviors are simultaneously stabilizable. Moreover, a representation for simultaneous stabilizer in this case is also presented under this assumption.

[Notations] Let $\mathbb{R}[\xi]$ denote the set of polynomials with real coefficients and $\mathbb{R}^{\bullet\times\bullet}[\xi]$ denote the set of polynomial matrices with real coefficients of suitable sizes, respectively. For a nonsingular polynomial matrix R, all of the roots of $\det(R)$ are located in the open left (right) half plane, R is said to be Hurwitz (anti-Hurwitz, respectively).

II. PRELIMINARIES

In this section, we give brief reviews for behavioral system theory based on the references: [16], [20], [21].

Let $\mathcal{P}$ denote the behavior of a plant and $\mathtt{q}$ denote the number of the (manifest) variables. Throughout the paper, we assume that a system is linear and time-invariant. Then, the behavior $\mathcal{P}$ is representable by

$$R_N \frac{d^N w}{dt^N} + \cdots + R_1 \frac{dw}{dt} + R_0 w = 0 \quad (1)$$

where $R_i \in \mathbb{R}^{\bullet\times\mathtt{q}}, i = 0, \cdots, N$. This is called a *kernel representation* of $\mathcal{P}$ and the variable w is called a manifest variable. A kernel representation is written as $R(\frac{d}{dt})w = 0$ by using a polynomial matrix $R := R_0 + R_1\xi + \cdots + R_N\xi^N \in \mathbb{R}^{\bullet\times\mathtt{q}}[\xi]$. There are many kernel representations for $\mathcal{P}$. Particularly, we call a kernel representation $R(\frac{d}{dt})w = 0$ *minimal* if R has normal full row rank. Let $\rho(\mathcal{P})$ denote the size of rows of a minimal kernel representation of $\mathcal{P}$ and note that $\rho(\mathcal{P})$ is independent from representations of $\mathcal{P}$. $\rho(\mathcal{P})$ is called the output cardinality of $\mathcal{P}$.

The author is with Institute of Science and Engineering, Kanazawa University, Kakuma-machi, Kanazawa, Ishikwawa, 920-1192, Japan o-kaneko@ec.t.kanazawa-u.ac.jp

[1]The word 'rational condition' implies that arithmetic computation or logical condition.

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$\mathcal{P}$ is said to be *controllable* if for all $w_1, w_2 \in \mathcal{P}$ there exist $w \in \mathcal{P}$ and $T_1, T_2 (\in \mathbb{R})$ such that $w(t) = w_1(t)$ for $t \leq T_1$ and $w(t) = w_2(t)$ for $t > T_2$. In the case of linear time invariant behavior, $\mathcal{P}$ is controllable if and only if a minimal kernel representation is induced by an element of $\mathbb{R}^{\rho(\mathcal{P})\times \mathtt{q}}[\xi]$. The controllability of $\mathcal{P}$ is also equivalent to saying that $\mathcal{P}$ can be described by

$$w = M_L \frac{d^L \ell}{dt^L} + \cdots + M_1 \frac{d\ell}{dt} + M_0 \ell \tag{2}$$

where $M_i \in \mathbb{R}^{\mathtt{q}\times \bullet}, i = 0, \cdots, L$. This is called an *image representation* of $\mathcal{P}$ and ℓ is called a latent variable. Similarly to kernel representations, we use the notation $w = M(\frac{d}{dt})\ell$ by using a polynomial matrix $M := M_0 + M_1\xi + \cdots + M_L\xi^L \in \mathbb{R}^{\mathtt{q}\times\bullet}[\xi]$. Moreover, there are many image representations for $\mathcal{P}$. In addition, ℓ is said to be *observable* from w if $w = 0$ implies $\ell = 0$. A latent variable ℓ in $w = M(\frac{d}{dt})\ell$ is observable from w if and only if $M \in \mathbb{R}^{\mathtt{q}\times(\mathtt{q}-\rho(\mathcal{P}))}[\xi]$. Note that $RM = 0$ and also that there exists $Q \in \mathbb{R}^{\mathtt{q}-\rho(\mathcal{P})\times\mathtt{q}}[\xi]$ such that $[R^{\mathrm{T}}\ Q^{\mathrm{T}}]^{\mathrm{T}}$ is unimodular and $QM = I$. Similarly, there exists a $N \in \mathbb{R}^{\mathtt{q}\times\rho(\mathcal{P})}[\xi]$ such that $[N\ M]$ is unimodular and $RN = I$. Thus,

$$\begin{bmatrix} R \\ Q \end{bmatrix} \begin{bmatrix} N & M \end{bmatrix} = I_{\mathtt{q}}. \tag{3}$$

Throughout of this paper, we assume also controllable behaviors and treat their observable image representations.

$\mathcal{P}$ is said to be stable if $w \in \mathcal{P}$ implies $w(t) \to 0$ as $t \to \infty$. $\mathcal{P}$ is stable if and only if a minimal kernel representation of $\mathcal{P}$ is induced by a Hurwitz polynomial matrix $R \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}}[\xi]$.

In order to stabilize the behavior $\mathcal{P}$, the controller described by a kernel representation $Cw = 0$ must be designed so as to satisfy that the behavior described by

$$\begin{bmatrix} R\left(\frac{d}{dt}\right) \\ C\left(\frac{d}{dt}\right) \end{bmatrix} w = 0. \tag{4}$$

is stable, or equivalently, $\begin{bmatrix} R^{\mathrm{T}} & C^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ must be Hurwitz matrix.

The following lemma is a parameterization of all of the controllers stabilizing $\mathcal{P}$, which is called to stabilizer of $\mathcal{P}$ in this paper, in a behavioral sense derived in [12].

Lemma 1 *Let $\mathcal{P}$ be a controllable behavior and let $R(\xi) \in \mathbb{R}^{\rho(\mathcal{P})\times\mathtt{q}}[\xi]$ induce a minimal kernel representation of $\mathcal{P}$. Then, all of the stabilizers of $\mathcal{P}$ can be parameterized by*

$$C := \begin{bmatrix} F & B \end{bmatrix} \begin{bmatrix} R \\ Q \end{bmatrix} \tag{5}$$

where $B \in \mathbb{R}^{\mathtt{q}-\rho(\mathcal{P})\times\mathtt{q}-\rho(\mathcal{P})}[\xi]$ is an arbitrary Hurwitz matrix and $F \in \mathbb{R}^{\mathtt{q}-\rho(\mathcal{P})\times\rho(\mathcal{P})}[\xi]$ is an arbitrary matrix.□

The next lemma is one of the equivalent conditions for the system described by $C(\frac{d}{dt})w = 0$ to be a stabilizer for $\mathcal{P}$ derived in [12].

Lemma 2 *Assume that $\mathcal{P}$ is controllable. Let M induce an observable image representation for $\mathcal{P}$. Consider $C \in \mathbb{R}^{\mathtt{q}-\rho(\mathcal{P})\times\mathtt{q}}[\xi]$. Then C is stabilizers for $\mathcal{P}$ if and only if CM is a Hurwitz matrix.*□

The next lemma is a straightforward application of Lemma 2, which is used for the statements in our main theorems.

Lemma 3 *Consider two controllable behaviors $\mathcal{P}_1$ and $\mathcal{P}_2$. Let R_1 and M_2 induce a minimal kernel representation of $\mathcal{P}_1$ and an observable image representation of $\mathcal{P}_2$, respectively. Assume that the output cardinalities of both behaviors are the same, i.e., $\rho(\mathcal{P}_1) = \rho(\mathcal{P}_2) =: \mathtt{p}$. Then $\mathcal{P}_1 \cap \mathcal{P}_2$ is stable if and only if R_1M_2 is Hurwitz.* □

III. Problem Formulation

As stated in the above sentences, we assume linear, time-invariant, and controllable behaviors throughout this paper. In addidion, we focus on the case of $\mathtt{q} = 2$. These settings imply that $\mathcal{P}$ is also described by a single input and singe output transfer function. However, it should be also noticed that we have no input output partitions in the variables differently conventional control theory.

Under these settings, the problem we consider in this paper can be formalized as follows:

Problem 1 *Let $\mathcal{P}_i$ denote a set of N linear plants. Find a condition under which there exists a single controller $\mathcal{C}$ such that $\mathcal{P}_i \bigcap \mathcal{C}$ for all $i \in \{1, 2, \cdots, N\}$ are stable.*

Problem 1 can be formalized in terms of polynomial matrices as follows.

Problem 2 *Let $R_i \in \mathbb{R}^{1\times 2}[\xi]$ induce kernel representations of $\mathcal{P}_i$, $i \in \{1, 2, \cdots, N\}$. Then, for all $i \in \{1, 2, \cdots, N\}$, find a condition under which there exists a polynomial matrix $C \in \mathbb{R}^{1\times 2}[\xi]$ such that*

$$\begin{pmatrix} R_i(\xi) \\ C(\xi) \end{pmatrix} \tag{6}$$

are Hurwitz.

In the following, M_i denotes a matrix inducing an observable image representation of $\mathcal{P}_i$ for each $i = 1, 2, \cdots, N$. Similarly, Q_i and N_i denote matrices satisfying Eq.(3)

IV. Simultaneous stabilization for a pair of linear systems

A. Review

Before going to the main result of this paper, we give a brief review of simultaneous stabilization in the case of $N = 2$ ([9]) in the behavioral framework. It should be noticed that the following theorem holds for the case of $\mathtt{q} > 2$.

Theorem 1 *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ denote controllable behaviors and $\rho(\mathcal{P}_1) = \rho(\mathcal{P}_2) =: \mathtt{p}$. Define the new behavior $\mathcal{P}_{12}$ described by following new kernel representation*

$$R_2(\ \tfrac{d}{dt}\ )\left(\ N_1(\tfrac{d}{dt}) \quad M_1(\tfrac{d}{dt})\ \right) w = 0. \tag{7}$$

Then, $\mathcal{P}_1$ and $\mathcal{P}_2$ are simultaneously stabilizable if and only if there exist $F_{12} \in \mathbb{R}^{(\mathtt{q}-\mathtt{p})\times\mathtt{q}}[\xi]$ and Hurwitz $H_{12} \in \mathbb{R}^{(\mathtt{q}-\mathtt{p})\times(\mathtt{q}-\mathtt{p})}[\xi]$ such that

$$\left(\begin{array}{cc} F_{12} & H_{12} \end{array} \right) \tag{8}$$

induces a stabilizer for $\mathcal{P}_{12}$. $\square$

If we regard that the H_{12} corresponds to the "denominator" of the stabilizer for $\mathcal{P}_{12}$, Eq.(8) in the above condition can be regarded as a "strong stabilizer" for $\mathcal{P}_{12}$.

B. New results of simultaneous stabilization for a pair of linear systems

In this subsection, we provide some new results on simultaneous stabilization for the case of $N = 2$, which are also crucial roles in the case of $N \geq 3$ afterwards.

Let $\mathcal{C}_{12}^{\mathrm{aug}}$ denote the behavior of the stabilizer for $\mathcal{P}_{12}$ stated in Theorem 1. It follows from Lemma 2 that

$$\left(\begin{array}{cc} F_{12} & H_{12} \end{array} \right) \left(\begin{array}{c} R_1 \\ Q_1 \end{array} \right) M_2$$

is also Hurwitz, which is denoted with H_{21}. In other words, $\mathcal{P}_1$ and $\mathcal{P}_2$ are simultaneously stabilizable if and only if there exist Hurwitz H_{12} and H_{21} such that

$$\left(\begin{array}{cc} F_{12} & H_{12} \end{array} \right) \left(\begin{array}{c} R_1 \\ Q_1 \end{array} \right) M_2 = H_{21}. \tag{9}$$

Thus, we obtain the following equivalent condition for a pair of linear systems to be simultaneously stabilizable with respect to the solvability of a polynomial equation.

Theorem 2 *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ denote controllable behaviors. Then, $\mathcal{P}_1$ and $\mathcal{P}_2$ are simultaneously stabilizable if and only if the following polynomial equation*

$$H_{12}Q_1M_2 - H_{21} = -F_{12}R_1M_2 \tag{10}$$

is solvable with respect to Hurwitz polynomials H_{12} and H_{21}, and a polynomial F_{12}.

In Eq.(9), we focus on

$$\left(\begin{array}{cc} F_{12} & H_{12} \end{array} \right) \left(\begin{array}{c} R_1 \\ Q_1 \end{array} \right). \tag{11}$$

Since H_{12} is Hurwitz, it follows from Lemma 1 that Eq.(11) induces a kernel representation of the stabilizer for $\mathcal{P}_1$. In addition, from Lemma 2, Eq.(9) implies that Eq.(11) also induces a kernel representation of the stabilizer for $\mathcal{P}_2$. Namely, if F_{12} and H_{12} are solutions of Eq.(9) or Eq.(10), then Eq.(11) induces a kernel representation of the simultaneous stabilizer for $\mathcal{P}_1$ and $\mathcal{P}_2$. Thus, we obtain the following theorem on the representation of simultaneous stabilizer for a pair of linear systems

Theorem 3 *Assume that Eq.(10) is solvable with respect to Hurwitz polynomials H_{12} and H_{21}, and a polynomial F_{12}. Then Eq.(11) induces a kernel representation of simultaneous stabilizer of $\mathcal{P}_1$ and $\mathcal{P}_2$.*

Theorem 3 plays crucial roles in this paper.

Next, we consider the condition under which Eq.(10) is solvable. For this purpose, we assume that $\mathcal{P}_1$ stabilizes $\mathcal{P}_2$, or equivalently, $\mathcal{P}_2$ stabilizes $\mathcal{P}_1$, which is also equivalent to

$$\mathcal{P}_1 \cap \mathcal{P}_2 \text{ is stable.} \tag{12}$$

From Lemma 2, the assumption of Eq.(12) is also written as

$$R_1M_2 = R_2M_1 =: B_{12} \tag{13}$$

where B_{12} is Hurwitz. Under this assumption, Eq.(10) can be described as

$$H_{12}Q_1M_2 - H_{21} = -F_{12}B_{12}. \tag{14}$$

Note that H_{12} and H_{21} should be Hurwitz. Thus, they can include B_{12} as

$$B_{12}H'_{12}Q_1M_2 - B_{12}H'_{21} = -F_{12}B_{12} \tag{15}$$

where H'_{12} and H'_{21} are Hurwitz. The above Eq.(15) is always solvable with respect to Hurwitz polynomials H'_{12} and H'_{21}, and a polynomial F_{12}. Clearly, for arbitrary Hurwitz H'_{12} and H'_{21},

$$F_{12} = H'_{12}Q_1M_2 - H'_{21} \tag{16}$$

is a solution of Eq.(15). Moreover, together with Eq.(16), Theorem 3 can be modified as follows under the assumption that $\mathcal{P}1 \cap \mathcal{P}_2$is stable.

Theorem 4 *Assume that $\mathcal{P}_1 \cap \mathcal{P}_2$ is stable and define Hurwitz polynomial $B_{12} := R_1M_2 = R_2M_1$. Then, a kernel representation of a simultaneous stabilizer for $\mathcal{P}_1$ and $\mathcal{P}_2$ is induced by*

$$\left(\begin{array}{cc} H'_{12}Q_1M_2 - H'_{21} & B_{12}H'_{12} \end{array} \right) \left(\begin{array}{c} R_1 \\ Q_1 \end{array} \right) \tag{17}$$

for arbitrary Hurwitz polynomials H'_{12} and H'_{21}.

V. Simultaneous stabilization for a set of three linear systems

Let $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ be the behaviors of three linear systems. We assume Eq.(12). Moreover, we also assume that

$$\mathcal{P}_1 \cap \mathcal{P}_3 \text{ is stable.} \tag{18}$$

In other words, $\mathcal{P}_1$ is a simultaneous stabilizer for $\mathcal{P}_2$ and $\mathcal{P}_3$. Similarly to B_{12} for $\mathcal{P}_1$ and $\mathcal{P}_2$, we also define Hurwitz polynomials

$$B_{13} := R_1M_3 = R_3M_1. \tag{19}$$

Under this assumption, applying Theorem 5 to $\mathcal{P}_1$ and $\mathcal{P}_3$ yields that a kernel representation of a simultaneous stabilizer for $\mathcal{P}_1$ and $\mathcal{P}_3$ is induced by

$$\left(\begin{array}{cc} H'_{13}Q_1M_3 - H'_{31} & B_{13}H'_{13} \end{array} \right) \left(\begin{array}{c} R_1 \\ Q_1 \end{array} \right) \tag{20}$$

for arbitrary Hurwitz polynomials H'_{13} and H'_{31}. By comparing the simultaneous stabilizer for $\mathcal{P}_1$ and $\mathcal{P}_2$ described by Eq.(17) with the simultaneous stabilizer for $\mathcal{P}_1$ and $\mathcal{P}_3$ described by Eq.(20), if the following two identical equations

$$H'_{12}Q_1M_2 - H'_{21} = H'_{13}Q_1M_3 - H'_{31} \tag{21}$$

and

$$B_{12}H'_{12} = B_{13}H'_{13} \tag{22}$$

hold, then these two stabilizers are the same. In other words, Eq.(17) or Eq.(20) induces also simultaneous stabilizer for $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$. From this reason, we consider the problem of whether Eq.(21) and Eq.(22) are solvable for Hurwitz polynomials H'_{12}, H'_{21}, H'_{13}, and H'_{31}.

Firstly, we focus on the solvability of Eq.(22). At this point, since we just have to guarantee that H'_{12} and H'_{13} are Hurwitz, we can select them as

$$\begin{aligned} H'_{12} &= B'_{13} && (23)\\ H'_{13} &= B'_{12}. && (24) \end{aligned}$$

Next, we focus on the solvability of Eq.(21). Since we have selected H'_{12} and H'_{21} so as to satisfy Eq.(23) and Eq.(24), Eq.(21) is written as

$$B'_{13}Q_1M_2 - H'_{21} = B'_{12}Q_1M_3 - H'_{31} \tag{25}$$

or equivalently

$$B'_{13}Q_1M_2 - B'_{12}Q_1M_3 = H'_{21} - H'_{31}. \tag{26}$$

The left-hand side of Eq.(26) has already been determined. This implies that the problem is whether the polynomial $B'_{13}Q_1M_2 - B'_{12}Q_1M_3$ can be described by the subtraction of Hurwitz polynomials. In the case in which $B'_{13}Q_1M_2 - B'_{12}Q_1M_3$ is Hurwitz, for instance, we can choose

$$\begin{aligned} H_{21} &:= \gamma B'_{13}Q_1M_2 - B'_{12}Q_1M_3\\ H_{31} &:= (\gamma-1)B'_{13}Q_1M_2 - B'_{12}Q_1M_3 \end{aligned}$$

for arbitrary $\gamma \in \mathbb{R}$. In general, since $B'_{13}Q_1M_2 - B'_{12}Q_1M_3$ is not necessarily Hurwitz, we have to regard it as the product of Hurwitz polynomial and anti-Hurwitz one. The Hurwitz part can be included as the factor of both H'_{21} and H'_{31} Thus, the problem is whether the anti-Hurwitz part can be decomposed as the subtraction of Hurwitz polynoimials. As for this problem, we can obtain the following lemma.

Lemma 4 *Let $a \in \mathbb{R}[\xi]$ be an arbitrary anti-Hurwitz polynomial. Then there exist Hurwitz polynomials b_1 and $b_2 \in \mathbb{R}[\xi]$ such that*

$$a = b_1 - b_2. \tag{27}$$

The proof will be contained in the future publications by the author (cf.[11]).

Based on Lemma 4, it is possible to guarantee that there exist Hurwitz H'_{21} and H'_{31} such that

$$B'_{13}Q_1M_2 - B'_{12}Q_1M_3 = H'_{21} - H'_{31}. \tag{28}$$

Therefore, if $\mathcal{P}_1 \cap \mathcal{P}_2$ and $\mathcal{P}_1 \cap \mathcal{P}_3$ are stable, then $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ are simultaneous stabilizable. Moreover, by using Theorem 5, we can obtain the following theorem.

Theorem 5 *Assume that $\mathcal{P}1 \cap \mathcal{P}_2$ and $\mathcal{P}1 \cap \mathcal{P}_3$ are stable. Define Hurwitz polynomial $B_{12} := R_1M_2 = R_2M_1$) and $B_{13} := R_1M_3 = R_3M_1$). Then, a kernel representation of a simultaneous stabilizer for $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ is induced by*

$$\begin{pmatrix} B'_{13}Q_1M_2 - H'_{21} & B_{12}B'_{13} \end{pmatrix} \begin{pmatrix} R_1 \\ Q_1 \end{pmatrix} \tag{29}$$

or

$$\begin{pmatrix} B'_{12}Q_1M_3 - H'_{31} & B_{13}B'_{12} \end{pmatrix} \begin{pmatrix} R_1 \\ Q_1 \end{pmatrix} \tag{30}$$

where H'_{21} and H'_{31} are Hurwitz polynomials obtained by the decomposition as

$$B'_{13}Q_1M_2 - B'_{12}Q_1M_3 = H'_{21} - H'_{31}. \tag{31}$$

Of course, the same condition holds if the role of $\mathcal{P}_1$ and $\mathcal{P}_2$ (or $\mathcal{P}_3$) are replaced. Consequently, we obtain the main theorem of this paper as follows.

Theorem 6 *Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ denote a set of three behaviors. If one of them ($\mathcal{P}_i$) stabilizes the other two behaviors ($\mathcal{P}_j, \mathcal{P}_k$) , then these three behaviors are simultaneously stabilizable. Moreover, the following matrix*

$$\begin{pmatrix} B'_{ik}Q_iM_j - H'_{ji} & B_{ij}B'_{ik} \end{pmatrix} \begin{pmatrix} R_i \\ Q_i \end{pmatrix} \tag{32}$$

or

$$\begin{pmatrix} B'_{ij}Q_iM_k - H'_{ki} & B_{ik}B'_{ij} \end{pmatrix} \begin{pmatrix} R_i \\ Q_i \end{pmatrix} \tag{33}$$

where H'_{ji} and H'_{ki} are Hurwitz polynomials obtained by the decomposition as

$$B'_{ik}Q_iM_j - B'_{ij}Q_iM_k = H'_{ji} - H'_{ki}. \tag{34}$$

induces a kernel representation of simultaneous stabilizers.

Remark V.1 *In [10], the author also derived a sufficient condition for a set of linear behaviors to be simultaneously stabilzable. However, addition to the assumption that $\mathcal{P}1 \cap \mathcal{P}_2$ and $\mathcal{P}1 \cap \mathcal{P}_3$ are stable, we required $\mathcal{P}_1 \cap \mathcal{P}_2^{\perp}$ and $\mathcal{P}_1 \cap \mathcal{P}_3^{\perp}$ should be stable. where $\mathcal{P}_i^{\perp}$ is described by a kernel representation $Q_iw = 0$. Hence, the main result of this paper is less conservative than that in [10].*

VI. Example

In this section, we give a simple example in order to illustrate how the result in this paper works in the simultaneous stabilization. Let $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ be described as kernel representations induced by

$$R_1 = \begin{bmatrix} -2 & \xi - 1 \end{bmatrix}, \tag{35}$$

$$R_2 = \begin{bmatrix} \xi & 2\xi + 1 \end{bmatrix}, \tag{36}$$

$$R_3 = \begin{bmatrix} \xi - 3 & 3\xi - 1 \end{bmatrix}, \tag{37}$$

respectively. A matrix Q_1 such that $[R_1^{\mathrm{T}}\ Q_1^{\mathrm{T}}]^{\mathrm{T}}$ is unimodular can be easily computed as

$$Q_1 = \begin{bmatrix} 0 & -\frac{1}{2} \end{bmatrix}. \tag{38}$$

Firstly, we check whether $\mathcal{P}_1 \cap \mathcal{P}_2$ and $\mathcal{P}_1 \cap \mathcal{P}_3$ are stable. Actually, from simple computations, we see that

$$R_1 M_2 = -\xi^2 - 2\xi - 2 \quad (39)$$

$$R_1 M_3 = -\xi^2 - 2\xi - 1. \quad (40)$$

Thus, the sufficient condition for a triple of the behaviors to be simultaneously stabilizable is satisfied. We compute

$$B'_{13} Q_1 M_2 - B'_{12} Q_1 M_3 = -\xi^2 - 4\xi - 3 \quad (41)$$

One of the candidate of the decomposition described by Eq.(41) is

$$\begin{array}{rcl} H'_{21} & = & 2(-\xi^2 - 4\xi - 3) \\ H'_{31} & = & -H'_{21} \end{array}$$

Then, we obtain

$$B'_{13} Q_1 M_2 - H'_{21} = B'_{12} Q_1 M_3 - H'_{31} = \frac{1}{2}\xi^3 - \xi^2 - \frac{15}{2}\xi - 6.$$

Hence, the behavior of a simultaneous stabilizers, say $\mathcal{C}$, is induced by

$$\begin{array}{rcl} C & := & \left(\begin{array}{cc} B'_{13} Q_1 M_2 - H'_{21} & B'_{12} B'_{13} \end{array} \right) \left(\begin{array}{c} R_1 \\ Q_1 \end{array} \right) \\ & = & \left(\begin{array}{cc} -\xi^3 + 2\xi + 15\xi + 12 & -4\xi^3 - 11\xi - 2\xi + 5 \end{array} \right). \end{array}$$

Indeed, we can validate that C induces simultaneous stabilizers for $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$. As for $\mathcal{P}_1 \cap \mathcal{C}$, we see that $CM_1 = B'_{12} B'_{13}$, which is stable. The roots of polynomials of CM_2 and CM_3 are $\{-1, -1, -1, -3\}$ and $\{-1, -1, -2, -3\}$, respectively, which implies that they are also Hurwitz. Thus, we see that C described by Eq.(41) induces a simultaneous stabilizers for $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$.

VII. CONCLUSIONS AND FUTURE WORKS

In this paper, we have treated simultaneous stabilization problem in the behavioral framework. We have provided a new equivalent condition for a pair of linear systems to be simultaneously stabilizable, and then also presented a representation of simultaneous stabilizer under the assumption that $\mathcal{P}_1 \cap \mathcal{P}_2$ is stable. By using this result, we have also considered the case of $N = 3$. In the case of $N = 3$, we have also shown that if one of the behavior stabilizes the other two behaviors then these three behaviors are simultaneously stabilizable. Moreover, a representation for simultaneous stabilizer in the case of $N = 3$ have also presented under this assumption.

The future directions of this studies are as follows. Firstly, we are now trying to derive the "parameterization" of simultaneous stabilizers, i.e., we consider the problem whether arbitrary simultaneous stabilizer can be induced in the polynomial form presented in this paper. Secondly, the problem how the results in this paper can be extended for the case of $N \geq 4$ should be considered. Thirdly, less conservative sufficient condition is to be obtained.

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