On the problem of model reduction in the gap metric

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***Abstract*— This paper deals with the model reduction problem where, for a given linear time-invariant dynamical system of complexity n, a simpler system of complexity $r < n$ is desired such that the gap between their respective behaviors is minimized. We describe dynamical systems as closed, shift invariant subspaces of $\mathcal{H}_2^+$, represented as kernels of rational multiplicative operators that are anti-stable rational elements of $\mathcal{RH}_\infty^-$. Contrary to other approaches this enables to reduce autonomous behaviors. In this paper we will give upper- and lower bounds for the minimal gap between a rational behavior and its optimal approximation in this system class. Bounds are given in terms of its Hankel Singular Values. These bounds only depend on the given system and can be computed in advance due to the use of rational operators describing the dynamical systems. This will be illustrated by a simple example.**

I. Preliminaries

In this paper, dynamical systems are described using the behavioral approach. In the general behavioral framework, introduced by Willems et. al., behaviors are represented by polynomial differential operators that impose restrictions on infinitely smooth trajectories. In this paper, trajectories do not have to be infinitely smooth, but we assume them to be square integrable and to belong to

$$L_2^+ := \{f : \mathbb{R}^+ \to \mathbb{R}^n \mid \|f\|_{L_2^+} := \int_0^\infty f(t)^\top f(t)\mathrm{d}t < \infty\}.$$

Instead of viewing trajectories in the time domain, we will use a frequency domain approach. Therefore, trajectories are equivalently viewed as elements of the Hardy space $\mathcal{H}_2^+$, that is defined as

$$\mathcal{H}_2^+ := \{f : \mathbb{C}^+ \to \mathbb{C}^n \mid f \text{ analytic in } \mathbb{C}^+ \text{ and } \|f\|_{\mathcal{H}_2^+} < \infty\},$$

with $\|f\|_{\mathcal{H}_2^+} := \sup_{\sigma>0} \sqrt{\int_{-\infty}^{\infty} |f(\sigma + j\omega)|^2 \mathrm{d}\omega}$ and where $|\cdot|$ denotes the Euclidean norm. This is a subspace of the Lebesgue space $\mathcal{L}_2$, which is the Laplace transform of the Hilbert space L_2 of square integrable functions on $\mathbb{R}$.

To be able to apply restrictions on $\mathcal{H}_2^+$ trajectories, we consider rational operators in $\mathcal{RH}_\infty^-$, where $\mathcal{H}_\infty^-$ is the set of functions $f : \mathbb{C}^- \to \mathbb{C}^{n\times m}$ analytic in $\mathbb{C}^-$ with finite norm

$$\|f\|_{\mathcal{H}_\infty^-} := \sup_{s\in\mathbb{C}^-} \sigma_{\max}(f(s)).$$

$\mathcal{RH}_\infty^+$ is defined similarly for functions $f : \mathbb{C}^+ \to \mathbb{C}^{n\times m}$, analytic in $\mathbb{C}^+$. A rational element $P \in \mathcal{RH}_\infty^-$ defines a multiplicative operator $P : \mathcal{H}_2^+ \to \mathcal{L}_2$ on the Hilbert space $\mathcal{H}_2^+$ by setting $(Pw)(s) = P(s)w(s)$. Any such P defines a closed linear subspace of $\mathcal{H}_2^+$ that is defined by

$$\mathcal{B} = \{w \in \mathcal{H}_2^+ \mid Pw \in \mathcal{H}_2^-\}. \tag{1}$$

Note that $\mathcal{B} = \ker \Pi_+ P$, where Π_+ is the canonical projection from $\mathcal{L}_2$ to $\mathcal{H}_2^+$.

We refer to (1) as the behavior defined by P, and to P as a kernel representation of $\mathcal{B}$. The class of all systems that can be represented as (1) will be denoted by $\mathbb{B}$. In the frequency domain, the τ-shift operator for trajectories $w \in \mathcal{H}_2^+$ is defined as

$$(\sigma_\tau w)(s) := \begin{cases} \mathrm{e}^{-s\tau}\left(w(s) - \int_0^{-\tau} \hat{w}(t)\mathrm{e}^{-st}\mathrm{d}t\right), & \tau < 0, \\ \mathrm{e}^{-s\tau} w(s), & \tau \geq 0, \end{cases} \tag{2}$$

where $\hat{w} = \mathcal{L}^{-1}(w)$ is the inverse Laplace transform of w. (2) is referred to as a left shift if $\tau < 0$ and to a right shift if $\tau > 0$. For $\mathcal{B}$ as in (1) we then have $\sigma_\tau \mathcal{B} \subseteq \mathcal{B}$ for $\tau \leq 0$ and therefore systems in $\mathbb{B}$ are left shift invariant. We focus on this specific class of systems since it includes autonomous behaviors, contrary to systems represented using stable rational operators in $\mathcal{RH}_\infty^+$.

One of the advantages of using rational operators over polynomial differential matrices is that rational operators allow for scaling in the sense that any $\mathcal{B} \in \mathbb{B}$ admits a co-inner kernel representation, such that $PP^* = I$. For more details on $\mathcal{L}_2$ behaviors represented using rational operators, we refer the reader to [7], [8].

II. Problem Formulation

To formalize a model reduction problem, we need to have a distance measure in $\mathbb{B}$ between any two elements, together with a measure for complexity of dynamical systems in $\mathbb{B}$. In this section, we formalize both and then formulate the problem of model reduction that is discussed in the remainder of this paper.

A. Distance Measure: the Gap

As shown in earlier work [10], the gap metric can be used as distance measure between systems described in the behavioral framework. For two behaviors $\mathcal{B}_1, \mathcal{B}_2 \in \mathbb{B}$, the gap $\delta : \mathbb{B} \times \mathbb{B} \to [0, 1]$ is defined as

$$\delta(\mathcal{B}_1, \mathcal{B}_2) := \max\{\vec{\delta}(\mathcal{B}_1, \mathcal{B}_2), \vec{\delta}(\mathcal{B}_2, \mathcal{B}_1)\},$$

where $\vec{\delta}$ denotes the *directed gap*

$$\vec{\delta}(\mathcal{B}_1, \mathcal{B}_2) = \sup_{w_1\in\mathcal{B}_1, \|w_1\|=1} \inf_{w_2\in\mathcal{B}_2} \|w_1 - w_2\|.$$

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Since we make use of $\mathcal{L}_2$ behaviors in this paper, we would like to use the gap in terms of the rational operators representing these behaviors. Earlier results on the gap are summarized in the following theorem.

Theorem 2.1: Let $\mathcal{B}_1 = \ker \Pi_+ P_1$ and $\mathcal{B}_2 = \ker \Pi_+ P_2$, where both $P_1 \in \mathcal{RH}_\infty^-$ and $P_2 \in \mathcal{RH}_\infty^-$ are co-inner. The directed gap from $\mathcal{B}_1$ onto $\mathcal{B}_2$ is given by

$$\vec{\delta}(\mathcal{B}_1, \mathcal{B}_2) = \sup \left\{ \frac{\langle w_1, w_2 \rangle}{\|w_1\| \|w_2\|},\ w_1 \in \mathcal{B}_1,\ w_2 \in \mathcal{B}_2^\perp \right\} = \|P_2^* \Pi_+ P_2|_{\mathcal{B}_1}\| = \|\Pi_+ P_2|_{\mathcal{B}_1}\|.,$$

where $\perp$ denotes the orthogonal complement. The gap between $\mathcal{B}_1$ and $\mathcal{B}_2$ is a number between 0 and 1 and satisfies

$$\begin{aligned} \delta(\mathcal{B}_1, \mathcal{B}_2) &= \max\left\{ \|\Pi_+ P_1|_{\mathcal{B}_2}\|, \|\Pi_+ P_2|_{\mathcal{B}_1}\| \right\} = \delta(\mathcal{B}_1^\perp, \mathcal{B}_2^\perp) \\ &= \max\left\{ \inf_{Q \in \mathcal{RH}_\infty^+} \|P_1^* - P_2^* Q\|, \inf_{Q \in \mathcal{RH}_\infty^+} \|P_2^* - P_1^* Q\| \right\} \\ &= \|\Pi_{\mathcal{B}_1} - \Pi_{\mathcal{B}_2}\| = \|P_1^* \Pi_+ P_1 - P_2^* \Pi_+ P_2\|, \end{aligned}$$

where the superscript $*$ denotes the adjoint, which is the conjugate transpose of the operator, and where $\Pi_{\mathcal{B}_i}$ denotes the projection of $\mathcal{H}_2^+$ onto the closed subspace $\mathcal{B}_i \subset \mathcal{H}_2^+$. □

For the proof of this Theorem, we refer to [2], [4], [9], [10].

B. Complexity of Dynamical Systems

Now the distance measure between two behaviors is introduced, a measure for complexity has to be given. In general for linear dynamical systems, complexity is defined by the McMillan degree of the transfer function, or equivalently, the state space dimension of one, and hence any, minimal state space representation. Here we want to describe complexity irrespective of the rational operators as used in (1). For this, we need to define the equilibrium response of a system:

Definition 2.1: The *equilibrium response* of a system $\mathcal{B} \in \mathbb{B}$ is the largest right-shift invariant subspace contained in the (left shift invariant) behavior $\mathcal{B}$, denoted by

$$\mathcal{B}^* := \{ w \in \mathcal{B} \mid \sigma_\tau w \in \mathcal{B},\ \tau > 0 \},$$

where σ_τ denotes the shift operator as in (2). □

One can derive that $\mathcal{B}^* = \ker P$ whenever $\mathcal{B} = \ker \Pi_+ P$. With the introduction of the equilibrium response $\mathcal{B}^*$, a decomposition of the behavior $\mathcal{B}$ can be made that will be used to define complexity.

Theorem 2.2: Any $\mathcal{B} \in \mathbb{B}$ admits a decomposition

$$\mathcal{B} = \mathcal{B}^* \oplus (\mathcal{B} \cap \mathcal{B}^{*\perp}),$$

where $\mathcal{B} \cap \mathcal{B}^{*\perp}$ is finite dimensional. □

For the proof of this theorem, we refer the reader to the Appendix. With this result, complexity of systems in the class $\mathbb{B}$ can be defined as follows:

Definition 2.2: The *complexity* of $\mathcal{B} \in \mathbb{B}$ is the function $c : \mathbb{B} \to \mathbb{Z}^+$ defined as

$$c(\mathcal{B}) := \dim(\mathcal{B} \cap \mathcal{B}^{*\perp}).$$

□

Theorem 2.3: Let $\mathcal{B} \in \mathbb{B}$. Then the complexity $c(\mathcal{B})$ of $\mathcal{B} = \ker \Pi_+ P$ is equal to the McMillan degree of $\mathrm{degree}(P)$. More precisely,

$$\begin{aligned} c(\mathcal{B}) &= \dim(\Pi_+ P^* \mathcal{H}_2^-) \\ &= \text{rank } (\Gamma_P) = \mathrm{degree}(P), \end{aligned}$$

where Γ_P is the Hankel operator with symbol P defined as $\Gamma_P := \Pi_- P \Pi_+$. □

Also the proof of this theorem can be found in the Appendix.

C. Problem Formulation

With the introduction of complexity of systems in $\mathbb{B}$ and the distance measure between behaviors in $\mathbb{B}$, we can formulate the problem of model order reduction of systems in the class $\mathbb{B}$ as follows.

Problem Formulation: Given $\mathcal{B} \in \mathbb{B}$ of complexity $c(\mathcal{B}) = n$ and a positive integer $r < n$. Find, if it exists, $\mathcal{B}_r \in \mathbb{B}$ of complexity $c(\mathcal{B}_r) = r$ such that the gap between the complex and approximated behavior $\delta(\mathcal{B}, \mathcal{B}_r)$ is minimal. In particular, determine

$$\gamma_r^* := \inf\{ \delta(\mathcal{B}, \mathcal{B}_r) \mid \mathcal{B}_r \in \mathbb{B} \text{ and } c(\mathcal{B}_r) = r \}$$

as the smallest distance. □

Model order reduction using the gap metric is not completely new, since in [1], [2], [3], [5], [10] this problem is also discussed. This has resulted in bounds for the minimal gap γ_r^* between the original and approximated systems, however this is done for systems represented by kernels of stable rational operators. In this paper, we make use of a different system class namely left shift invariant systems, where we will also show bounds for this gap, in terms of Hankel singular values. Furthermore, we will discuss computational aspects of using state space representations.

III. Upper and Lower bounds

Given $\mathcal{B} \in \mathbb{B}$ with $c(\mathcal{B}) = n$, we are interested in finding for any $1 \le r < n$ the bounds γ_r^- and γ_r^+ such that

$$\gamma_r^- \le \gamma_r^* \le \gamma_r^+ \tag{3}$$

for a system in the class $\mathbb{B}$ that has to be reduced to complexity r. The bounds we have obtained for systems in $\mathbb{B}$ are given in the following theorem.

For any $G \in \mathcal{RL}_\infty$ we define the Hankel operator with symbol G by $\Gamma_G : \Pi_- G \Pi_+$. The k^{th} singular value of Γ_G is denoted by $\sigma_k(G)$ and is referred to as the k^{th} Hankel singular value. By $\sigma(G)$ we denote the collection of all Hankel singular values of G.

Theorem 3.1: Given a system $\mathcal{B} = \ker \Pi_+ P \in \mathbb{B}$, where $P \in \mathcal{RH}_\infty^-$ is co-inner, with $c(\mathcal{B}) = n$ and a positive integer $r < n$. Then, the lower and upper bounds for the minimal gap γ_r^* are given by

$$\gamma_r^- = \sigma_{r+1}(P^*P) \quad \text{and} \quad \gamma_r^+ = \sum_{k>r} \sigma_k(P),$$

where $\sigma_i(P)$ is the i^{th} Hankel singular value of P. □

The proof of this theorem is given in the Appendix of this paper. Note that the upper and lower bounds are explicit in terms of the given dynamical system $\mathcal{B} \in \mathbb{B}$. There are similarities between the derived results in this theorem and the ones obtained in earlier research, where stable rational operators are used (e.g. [5]). This approach has the advantage that left shift invariant systems can be taken into account, which can be useful when reducing autonomous ("closed-loop") systems.

Another result that could be notified is that we can formulate an equivalent problem with the one sketched in the previous section. For the problem of model order reduction, we can also try to approximate the orthogonal complement of the behavior $\mathcal{B}^\perp$.

Theorem 3.2: Let $\mathcal{B} \in \mathbb{B}$ and let $r < c(\mathcal{B})$ be a positive integer. Then $\mathcal{B}^\perp \in \mathbb{B}$ and $c(\mathcal{B}^\perp) = n$ and $\mathcal{B}_r$ is an optimal approximation of $\mathcal{B}$ of complexity $c(\mathcal{B}_r) = r$ if and only if $\mathcal{B}_r^\perp$ is an optimal approximation of $\mathcal{B}^\perp$ of complexity $c(\mathcal{B}_r^\perp) = r$. □

IV. Computational aspects

In this section we will show how the lower and upper bounds of the gap in Theorem 3.1 can be computed. Since any rational operator can be written as $P(s) = C(sI - A)^{-1}B + D$ for suitable real matrices (A, B, C, D), we will compute the bounds by algebraic operations only. In this section, we assume that $\mathcal{B} \in \mathbb{B}$ is of complexity $c(\mathcal{B}) = n$ and is represented by $P \in \mathcal{RH}_\infty^-$, in the sense that $\mathcal{B} = \ker \Pi_+ P$. In addition, let $\mathcal{B}_r \in \mathbb{B}$ have complexity $c(\mathcal{B}_r) = r < n$ and $\mathcal{B}_r = \ker \Pi_+ P_r$ with $P_r \in \mathcal{RH}_\infty^-$. Moreover, assume (without loss of generality) that P and P_r are co-inner.

A. Computation of lower bound γ_r^-

The lower bound $\gamma_r^- = \sigma_{r+1}(P^*P)$ of the minimal gap can be computed using the Hankel singular values of the operator P^*P. As discussed in the introduction, we consider a minimal state space realization of $(P^*P)(s) = C(sI - A)^{-1}B + D$. Consider the following system

$$\begin{cases} \dot{x} = Ax + Bw, \\ v = Cx + Dw, \end{cases}$$

where w and v are inputs and outputs, resp., x is the state variable of dimension $2n$. Since P^*P is self-adjoint, the eigenvalues of A are symmetric with respect to the imaginary axis and it not straightforward to calculate Hankel singular values of this system. Consider a decomposition

$$P^*P = P_{\text{stab}} + P_{\text{anti-stab}},$$

where $P_{\text{stab}} \in \mathcal{RH}_\infty^+$ and $P_{\text{anti-stab}} \in \mathcal{RH}_\infty^-$. This decomposition can also be obtained using algebraic state space operations. Consider an eigenvalue decomposition of A in case the eigenvalues of A are real, or a real Jordan decomposition for $A = U^{-1}VU$ if A has complex eigenvalues. Let $\hat{x} = Ux = \left[\begin{smallmatrix} x_+ \\ x_- \end{smallmatrix}\right]$ with x_+ and x_- of dimension n and consider the equivalent system

$$\begin{aligned} \begin{bmatrix} \dot{x}_+ \\ \dot{x}_- \end{bmatrix} &= \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \end{bmatrix} + \begin{bmatrix} B_+ \\ B_- \end{bmatrix} w, \\ v &= \begin{bmatrix} C_+ & C_- \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \end{bmatrix}, \end{aligned} \tag{4}$$

where $\lambda(A_-) \subset \mathbb{C}^-$ and $\lambda(A_+) \subset \mathbb{C}^+$. Then P_{stab} and $P_{\text{anti-stab}}$ are represented by the state space representations with the matrices (A_-, B_-, C_-) and (A_+, B_+, C_+) respectively. With the partition into P_{stab} and $P_{\text{anti-stab}}$, we have that

$$\begin{gathered} \Pi_+ P^*P \Pi_- = \Pi_+ P_{\text{stab}} \Pi_- \\ \text{and} \\ \Pi_- P^*P \Pi_+ = \Pi_- P_{\text{anti-stab}} \Pi_+. \end{gathered} \tag{5}$$

It is possible to calculate the singular values of the Hankel operators (5) using the given state space representations in (4). For this, let Q_- and P_- be the unique positive definite solutions of

$$\begin{aligned} A_-^* Q_- + Q_- A_- + C_-^* C_- &= 0, \\ A_- P_- + P_- A_-^* + B_- B_-^* &= 0, \end{aligned} \tag{6a}$$

and, similarly, Q_+ and P_+ solutions of

$$\begin{aligned} A_+^* Q_+ + Q_+ A_+ - C_+^* C_+ &= 0, \\ A_+ P_+ + P_+ A_+^* - B_+ B_+^* &= 0. \end{aligned} \tag{6b}$$

The singular values of the Hankel operators (5) are then given by

$$\sigma(P_{\text{stab}}) = \sqrt{\lambda(P_- Q_-)} \text{ and } \sigma(P_{\text{anti-stab}}) = \sqrt{\lambda(P_+ Q_+)}.$$

With this result, we have two "different" sets of Hankel singular values for P_{stab} and $P_{\text{anti-stab}}$. With this, we state the following theorem:

Theorem 4.1: Let $P \in \mathcal{RH}_\infty^-$ be co-inner and let P^*P be decomposed as $P_{\text{stab}} + P_{\text{anti-stab}}$. Then $\Pi_+ P^*P \Pi_- = \Pi_+ P_{\text{stab}} \Pi_-$ and $\Pi_- P^*P \Pi_+ = \Pi_- P_{\text{anti-stab}} \Pi_+$. Moreover, the observability and reachability Gramians P_-,Q_-,P_+ and Q_+ corresponding to the stable and anti-stable elements P_{stab} and $P_{\text{anti-stab}}$ satisfy

$$\begin{aligned} \sqrt{\lambda(P_- Q_-)} &= \sigma(P_{\text{stab}}) = \sigma(P^*P) \\ &= \sigma(P_{\text{anti-stab}}) = \sqrt{\lambda(P_+ Q_+)}. \end{aligned}$$

□

We can now summarize the computation of the lower bound in the following algorithm:

Algorithm 4.1: [Computation of the lower bound γ_r^-]
Given: $P \in \mathcal{RH}_\infty^-$ is co-inner, corresponding to $\mathcal{B} \in \mathbb{B}$ with $c(\mathcal{B}) = n$ and $r < n$.
Find: The lower bound γ_r^- for an approximation $\mathcal{B}_r \in \mathbb{B}$ with complexity $c(\mathcal{B}_r) = r < n$ as in (3).
Step 1: Convert the operator P^*P into a minimal state space representation.
Step 2: Decompose $P^*P = P_{\mathrm{stab}} + P_{\mathrm{anti-stab}}$ using an eigenvalue decomposition or a real Jordan decomposition as suggested in (4).
Step 3: Calculate the observability and reachability Gramians of P_{stab} or $P_{\mathrm{anti-stab}}$ according to (6a) and (6b), resp.
Step 4: Compute the Hankel singular values of P^*P using

$$\sigma(P^*P) = \sqrt{\lambda(P_-Q_-)} \quad \text{or} \quad \sigma(P^*P) = \sqrt{\lambda(P_+Q_+)}.$$

Result: Define $\gamma_r^- = \sigma_{r+1}(P^*P)$ as the lower bound of the gap between $\mathcal{B}$ and $\mathcal{B}_r$ in the class $\mathbb{B}$. □

B. Computation of upper bound γ_r^+

As shown in Theorem 3.1, the upper bound is given by a sum of Hankel singular values of P. Since $P \in \mathcal{RH}_\infty^-$, we know that any (minimal) state space realization of this operator will have eigenvalues in the right half plane. Therefore, the computation of the Gramians Q and P can be fulfilled using the Lyapunov equations in (6b), where one has to discard the subscript $-$. With the found Gramians, the Hankel singular values can then again be calculated as the square root of the eigenvalues of the product PQ. For completeness, the calculation of the upper bound is summarized in the following algorithm:

Algorithm 4.2: [Computation of the upper bound γ_r^+]
Given: $P \in \mathcal{RH}_\infty^-$ corresponding to $\mathcal{B} \in \mathbb{B}$ with $c(\mathcal{B}) = n$.
Find: The upper bound γ_r^+ for an approximation $\mathcal{B}_r \in \mathbb{B}$ with complexity $c(\mathcal{B}_r) = r < n$ as in (3).
Step 1: Convert the operator P into a minimal state space representation.
Step 2: Compute the Gramians of this (anti-stable) operator as in (6b).
Step 3: Calculate the Hankel singular values using the obtained Gramians.
Result: The upper bound γ_r^+ is then defined as the sum of the smallest $n - r$ Hankel singular values. □

One is able to verify whether the calculated bounds are correct by applying an optimal Hankel norm approximation on P_{stab} as well as $P_{\mathrm{anti-stab}}$. Then, it is possible to calculate $P_r^* P_r$ and hence one can compare the norm

$$\|P^*P - P_r^*P_r\|_\infty$$

with the found bounds γ_r^- and γ_r^+. This will be done in the example that is discussed in the next section of this paper.

V. Illustrative example

The computation of the bounds introduced in the previous section will be done for a random generated system in MATLAB. For simplicity, the order of the system $\mathcal{B} = \ker \Pi_+ P$, with $P \in \mathcal{RH}_\infty^-$, is chosen to be $n = 5$ and $\lim_{|s|\to\infty} P(s) = 0$. In this example, we are interested in finding bounds for an approximation $\mathcal{B}_r \in \mathbb{B}$ of complexity $c(\mathcal{B}_r) = r = 2$.

The first step is to convert the rational operator P, and also P^*P, into a state space representation. For the chosen system P, the corresponding state space realization is given by $(A, B, C, 0)$ with

$$A = \begin{bmatrix} 0.3738 & 0.0631 & 0.4564 & 0.0633 & 0.2604 \\ 0.2585 & 0.8165 & 0.9102 & 0.7162 & 0.0099 \\ 0.2606 & 0.0684 & 0.5507 & 0.7199 & 0.3501 \\ 0.4725 & 0.2976 & 0.4292 & 0.7412 & 0.8962 \\ 0.1948 & 0.7876 & 0.3991 & 0.6894 & 0.7432 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.3010 & 0.2269 & 0.9027 \\ 0.3799 & 0.0218 & 0.1966 \\ 0.4727 & 0.0155 & 0.6090 \\ 0.2226 & 0.7838 & 0.5229 \\ 0.3207 & 0.0186 & 0.7067 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.3532 & 0.5056 & 0.3625 & 0.9011 & 0.5210 \\ 0.7589 & 0.7658 & 0.9250 & 0.5517 & 0.1790 \end{bmatrix}.$$

For the calculation of the lower bound γ_r^-, we need to make the decomposition as in (4). In this case, we need to make a real Jordan decomposition since the eigenvalues of the A matrix corresponding to the realization of P^*P contains complex numbers. The way how this is done in MATLAB is shown in the algorithm at the end of the Appendix. With this decomposition, the Gramians in (6a) or (6b) can be computed, which results in the Hankel singular values

$$\sigma(P^*P) = \{6.3218,\ 4.5748,\ 0.3002,\ 0.2780,\ 0.0809\}.$$

This gives us the lower bound of minimal the gap γ_r^*, namely

$$\gamma_r^- = \sigma_{r+1}(P^*P) = 0.3002.$$

When computing the upper bound, we also need to calculate the Gramians (using (6b)) and Hankel singular values, but now for the operator P. This resulted in the values

$$\sigma(P) = \{3.1289,\ 1.9767,\ 0.5121,\ 0.3395,\ 0.0994\},$$

where the upper bound of the gap is then equal to

$$\gamma_r^+ = \sum_{k>r} \sigma_k(P) = 0.95099.$$

This implies that for a reduction of $\mathcal{B}$ to an approximation $\mathcal{B}_r$ with complexity $r = 2$, we have that the optimal gap value is bounded between

$$0.3002 \leq \gamma_r^* \leq 0.95099.$$

To verify whether this is the case, we apply an optimal Hankel norm approximation on P such that the approximated system, with rational operator P_r, has the desired complexity. This has been done using the MATLAB command `hankmr()`. With this approximation $\mathcal{B}_r$, we can compute

$$\gamma = \|P^*P - P_r^*P_r\|_\infty,$$

which should be between the found upper and lower bounds of the optimal gap metric γ_r^*. In this example, the calculated value equals $\gamma = 0.63703$, which is indeed within the computed bounds.

VI. CONCLUSIONS

In this paper we have discussed the problem of model order reduction for systems represented using the behavioral framework. Representation-free notions of complexity and distance between systems have been introduced. In our work, we focused on behaviors in $\mathcal{H}_2^+$, which corresponds to square integrable trajectories over the positive time domain, that are represented as kernels of anti-stable rational operators in $\mathcal{RH}_\infty^-$. This class of systems is denoted by $\mathbb{B}$. We have shown that the gap and the notion of complexity can be expressed in terms of anti-stable rational kernel representations of systems. One of the advantages of using anti-stable rational functions is that this model class allows to represent autonomous systems. This contrasts the setting in [10] where right shift invariant systems are considered.

With the notions of complexity and the gap metric, we have stated the problem of model order reduction of systems in the class $\mathbb{B}$. For a complex system $\mathcal{B} \in \mathbb{B}$ that should be reduced to a system with lower complexity $\mathcal{B}_r$, we have given upper and lower bounds of the gap, describing the "error" between the full and approximated systems, in terms of the rational operator describing $\mathcal{B}$ and the desired order r of the approximation. In this paper, we discussed the computational aspects of these bounds. Since we are making use of rational operators, it is possible to use state space representations that make computations easier. This is illustrated when computing the bounds in the example, where we also have verified that the gap using optimal Hankel norm approximation is within the bounds.

APPENDIX:

Proof: [Theorem 2.2]
First two observations:

i. $\mathcal{B}^{*\perp} = \mathrm{cl}\,(\Pi_+ P^* \mathcal{L}_2)$. This claim follows from

$$\begin{aligned}
\mathcal{B}^* = \ker P &= \{w \in \mathcal{H}_2^+ \mid Pw = 0 \in \mathcal{L}_2\} \\
&= \{w \in \mathcal{H}_2^+ \mid \langle Pw,\ v\rangle_{\mathcal{L}_2} = 0,\ \forall v \in \mathcal{L}_2\} \\
&= \{w \in \mathcal{H}_2^+ \mid \langle w,\ P^*v\rangle_{\mathcal{L}_2} = 0, \forall v \in \mathcal{L}_2\} \\
&= \{w \in \mathcal{H}_2^+ \mid \langle w,\ \Pi_+ P^*v\rangle_{\mathcal{H}_2^+} = 0, \forall v \in \mathcal{L}_2\} \\
&= \{w \in \mathcal{H}_2^+ \mid w \perp \Pi_+ P^* v, \forall v \in \mathcal{L}_2\} \\
&= \mathrm{cl}\left([\Pi_+ P^*\mathcal{L}_2]^\perp\right) = [\Pi_+ P^* \mathcal{L}_2]^\perp .
\end{aligned}$$

ii. $\mathcal{B}^\perp = \mathrm{cl}\left(P^*\mathcal{H}_2^+\right)$. This follows from

$$\begin{aligned}
\mathcal{B} &= \{w \in \mathcal{H}_2^+ \mid Pw \in \mathcal{H}_2^-\} \\
&= \{w \in \mathcal{H}_2^+ \mid \langle Pw,\ v\rangle_{\mathcal{L}_2} = 0, \forall v \in \mathcal{H}_2^+\} \\
&= \{w \in \mathcal{H}_2^+ \mid \langle w,\ P^*v\rangle_{\mathcal{H}_2^+} = 0, \forall v \in \mathcal{H}_2^+\} \\
&= \mathrm{cl}\left(\left[P^*\mathcal{H}_2^+\right]^\perp\right) = \left[P^*\mathcal{H}_2^+\right]^\perp .
\end{aligned}$$

To prove the claims, we first prove that

$$\mathcal{B} \cap \mathcal{B}^{*\perp} = \Pi_+ P^* \mathcal{H}_2^-, \tag{7}$$

whenever $\mathcal{B} = \ker \Pi_+ P \in \mathbb{B}$ with P co-inner in $\mathcal{RH}_\infty^-$.

To prove (7), let $w \in \Pi_+ P^* \mathcal{H}_2^-$. Then, obviously, $w \in \Pi_+ P^* \mathcal{L}_2 = \mathcal{B}^{*\perp}$ (by *i.*) and we can write $w = \Pi_+ P^* v$ for some $v \in \mathcal{H}_2^-$. Let $u := P^*v$. Then u can be decomposed as $u = u_- + u_+$, where $u_- \in \mathcal{H}_2^-$ and $u_+ \in \mathcal{H}_2^+$. Clearly, $w = u_+ = u - u_- = P^*v - u_-$, and we get that $Pw = P(P^*v - u_-) = PP^*v - Pu_- = v - Pu_-$ which belongs to $\mathcal{H}_2^-$ since $v \in \mathcal{H}_2^-$ and $P \in \mathcal{RH}_\infty^-$ maps $\mathcal{H}_2^-$ to elements in $\mathcal{H}_2^-$. (Note that we used that P is co-inner). Conclude that $Pw \in \mathcal{H}_2^-$ or, phrased differently, $\Pi_+ Pw = 0$, i.e. $w \in \mathcal{B}$. This shows that $w \in \mathcal{B} \cap \mathcal{B}^{*\perp}$ and since $w \in \Pi_+ P^* \mathcal{H}_2^-$ is arbitrary, we establish that

$$\mathcal{B} \cap \mathcal{B}^{*\perp} \supseteq \Pi_+ P^* \mathcal{H}_2^- .$$

Now to prove the converse, let $w \in \mathcal{B} \cap \mathcal{B}^{*\perp}$. Then, using *i.*, $w = \Pi_+ P^* v$ for some $v \in \mathcal{L}_2$. Now decompose v according to $v = v_- + v_+$ with $v_- \in \mathcal{H}_2^-$ and $v_+ \in \mathcal{H}_2^+$. We complete the proof if we can show that $v_+ = 0$. Therefore, suppose that $v_+ \neq 0$. Then,

$$w = \underbrace{\Pi_+ P^* v_-}_{w_1} + \underbrace{\Pi_+ P^* v_+}_{w_2} . \tag{8}$$

Since $w_2 = \Pi_+ P^* v_+ \subseteq \Pi_+ P^* \mathcal{H}_2^+ = \mathcal{B}^\perp$ (by *ii.*), we infer that $w_2 \in \mathcal{B}^\perp$. As $w \in \mathcal{B}$, we conclude that $w_2 = 0$. Conclude that $v_+ = 0$ (or can be chosen to be 0) in (8). Hence, $w = w_1 = \Pi_+ P_+^* v_-$ for some $v_- \in \mathcal{H}_2^-$, i.e.

$$\mathcal{B} \cap \mathcal{B}^{*\perp} \subseteq \Pi_+ P^* \mathcal{H}_2^- .$$

Thus, $\mathcal{B} \cap \mathcal{B}^{*\perp}$ is the image of the Hankel operator $\Gamma_{P^*} : \mathcal{H}_2^- \to \mathcal{H}_2^+$, which is defined by $\Gamma_{P^*} = \Pi_+ P^*$. Since $P^* \in \mathcal{RH}_\infty^+$ is rational, hence the image of the Hankel is finite dimensional.

Lastly, $\mathcal{B} = \mathcal{B} \cap (\mathcal{B}^* \oplus \mathcal{B}^{*\perp}) = (\mathcal{B} \cap \mathcal{B}^{*\perp}) \oplus (\mathcal{B} \cap \mathcal{B}^*) = (\mathcal{B} \cap \mathcal{B}^{*\perp}) \oplus \mathcal{B}^*$, as $\mathcal{B}^* \subset \mathcal{B}$ implies that $\mathcal{B} \cap \mathcal{B}^* = \mathcal{B}^*$. ■

Proof: [Theorem 2.3]
As shown in the proof of Theorem 2.2, we have $\mathcal{B} \cap \mathcal{B}^{*\perp} = \Pi_+ P^* \mathcal{H}_2^-$, which is the Hankel operator Γ_{P^*} mapping past inputs to future outputs. We know that the dimension of $\Pi_+ P^* \Pi_-$ equals the rank of the Hankel operator Γ_{P^*}. Similarly, the Hankel operator $\Gamma_P : \mathcal{H}_2^+ \to \mathcal{H}_2^-$ can be obtained, defined by $\Gamma_P = \Pi_- P$, which has the same rank. It is then known that the rank of Γ_P is equal to the McMillan degree of the operator P. ■

Proof: [Theorem 3.1]
First we will show the condition for the lower bound γ_r^-. Let $\mathcal{B}_r$ be the optimal approximation of order r when $c(\mathcal{B}) = n$ and let $\gamma_r = \delta(\mathcal{B}, \mathcal{B}_r)$. Assume that P and P_r are normalized co-prime kernels of $\mathcal{B}$ and $\mathcal{B}_r$, respectively.

For any system, the Hankel norm is always smaller than the L_∞ norm, e.g.

$$\|\cdot\|_H = \sup_{u_-} \frac{\|y_+\|}{\|u_-\|} \leq \sup_{u=u_- \wedge\ u_+} \frac{\|y_+\|}{\|u\|} \leq \sup_{u} \frac{\|y_+ \wedge\ y_-\|}{\|u\|},$$

which equals the L_∞ norm $\|\cdot\|_\infty$, where $\wedge$ denotes the concatenation of two signals and the subscripts $+,-$ denote L_2^+ and L_2^- signals respectively. We also have, as shown in the proof of Lemma 1 in [4], that

$$\|P^*P - P_r^*P_r\|_\infty \leq \|P^*\Pi_+P - P_r^*\Pi_+P_r\|_\infty = \delta(\mathcal{B}, \mathcal{B}_r).$$

Combining this, we have that the Hankel norm of $E_r := P^*P - P_r^*P_r$ is a lower bound of γ_r^-. There is given that the rank of the Hankel operator with symbol P^*P (as $\Pi_+P^*P\Pi_-$) is n and that the rank of the Hankel $P_r^*P_r$ equals r. E_r is the difference between them, which rank is never smaller then the $(r+1)$st singular value of the Hankel operator associated with symbol P^*P. This is, by definition, equal to

$$\gamma_r^- := \sigma_{r+1}(P^*P),$$

which is the normal singular value of the Hankel operator $\Pi_-P^*P\Pi_+$ (or equivalently $\Pi_+P^*P\Pi_-$).

To show that the given upper bound γ_r^+ holds, we construct the optimal Hankel approximation of P^*, denoted as N_r^*. Glover has shown in [6], that

$$\|P^* - N_r^*\|_\infty \leq \gamma_r^+,$$

with γ_r^+ the sum of the last $n-r$ Hankel singular values of P^*. Let $N_r^* = P_r^*Q_r^*$ be an inner-outer factorization of N_r and define $\mathcal{B}_r := \ker \Pi_+P_r^*$ en $Q_0^* = [Q_r^*]^{-1}$. Then P_r co-inner, $Q_0^* \in \mathcal{H}_\infty^+$ and

$$\begin{aligned}\gamma_r \leq \delta(\mathcal{B}, \mathcal{B}_r) &= \inf_{Q^* \in \mathcal{H}_\infty^+} \|P^* - P_r^*Q^*\|_\infty \\ &\leq \|P^* - P_r^*Q_0^*\|_\infty = \|P^* - N_r^*\|_\infty \leq \gamma_r^+,\end{aligned}$$

where knowing that $\sum \sigma(P) = \sum \sigma(P^*)$ completes the proof for both bounds. ∎

Algorithm A: [Real Jordan Decomposition]
Given: A full rank matrix $A \in \mathbb{R}^{n\times n}$.
Find: A decomposition of A into two blocks containing the stable and anti-stable eigenvalues, as in (4).
Step 1: Apply an eigenvalue decomposition on A such that $AV = VE$, where V contains the eigenvectors corresponding to the eigenvalues in E.
Step 2: Store the eigenvectors corresponding to the negative and positive real valued eigenvalues in $V_{\text{stab,r}}$ and $V_{\text{anti}-\text{stab,r}}$, respectively.
Step 3: For any complex pair eigenvalues, find the corresponding two eigenvectors. Store one of them in $V_{\text{stab,cp}}$ and $V_{\text{anti}-\text{stab,cp}}$, depending on the real part of the eigenvalue. Remark that these eigenvectors can be complex valued.
Step 4: Define the new transformation matrix $\tilde{V}$ as

$$\begin{aligned}\tilde{V} = [&V_{\text{stab,r}}, \Im(V_{\text{stab,cp}}), \Re(V_{\text{stab,cp}}), \ldots \\ &V_{\text{anti}-\text{stab,r}}, \Im(V_{\text{anti}-\text{stab,cp}}), \Re(V_{\text{anti}-\text{stab,cp}})],\end{aligned}$$

where $\Im$ and $\Re$ denote the imaginary and real parts of the eigenvectors.
Step 5: Compute $\tilde{V}^{-1}$ and use it with $\tilde{V}$ to transform the state vector. This will yield that $\tilde{E} = \tilde{V}^{-1}A\tilde{V}$ is the decomposed A matrix as required. □

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