

State maps from bilinear differential forms

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Abstract—State equations need often to be constructed from a higher-order model of a system, resulting for example from the interconnection of subsystems, or from system identification procedures. In order to compute state equations it is crucial to choose a state variable. One way of doing this is through the computation of a *state map*, introduced in [4]. In this paper we develop an alternative approach to the algebraic characterization of state maps, based on the calculus of *bilinear differential forms (BDFs)*, see [8]. From this approach stem a new algorithm for the computation of state maps, and some new results regarding symmetries of linear dynamical systems.

I. INTRODUCTION

Modeling a complex system is often performed by decomposing it in simpler subsystems, which are then modeled separately; finally, the interconnections of the subsystems are modeled. In the case of linear, time-invariant systems, the end result of such a procedure is a set of higher-order linear constant-coefficient differential equations, possibly with algebraic equations among the variables. Higher-order differential equation representations may also result from identification techniques. State equations, which are of utmost importance for filtering, control, etc., consequently need often to be constructed from such a description of a system, and are often not given a priori.

In order to compute state equations it is crucial to choose a state variable. In [4] the concept of *state map* was introduced; this is a polynomial differential operator $X \left(\frac{d}{dt} \right)$ induced by a polynomial matrix $X(s)$ which, acting on the variables w of a system, produces a state variable x (an n -dimensional vector) for which equations of first order in x and zeroth-order in w can be derived. An algebraic characterization of the polynomial matrices inducing state maps for a given system, and algorithms to compute state maps and state representations from a system description, were also given in [4].

In this paper we develop an alternative approach to the algebraic characterization of state maps, based on the calculus of *bilinear differential forms (BDFs)*, see [8]. We develop a novel algorithm for the computation of state maps, and derive some new results regarding internal (i.e. at the state variable x level) and external (i.e. at the external variables w level) symmetries.

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In order to state the main result of the paper, we need to introduce the setting; this is done in section II. Our alternative approach to the computation of state maps is illustrated in section III. In section IV we discuss the application of our BDF-based approach for the computation of state maps to symmetries of linear dynamical systems. In section V we discuss applications to autonomous Hamiltonian systems.

II. BACKGROUND MATERIAL

We consider *linear differential systems*, described by equations of the form

$$R \left(\frac{d}{dt} \right) w = 0 \quad (1)$$

where $R \in \mathbb{R}^{\bullet \times \mathfrak{w}}[\xi]$, the set of polynomial matrices in the indeterminate ξ with \mathfrak{w} columns and an unspecified (finite) number of rows. Equation (1) describes the *behavior*

$$\mathfrak{B} := \{w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathfrak{w}}) \mid (1) \text{ is satisfied weakly} \},$$

where weak equality, or equivalently equality in the sense of distributions, means that

$$\int_{-\infty}^{+\infty} w(t)^\top R \left(-\frac{d}{dt} \right)^\top f(t) dt = 0$$

for all \mathcal{C}^∞ -testing functions f with compact support.

An important class of behaviors is that of *autonomous behaviors*, which admit kernel representations (1) in which the matrix R is $\mathfrak{w} \times \mathfrak{w}$ and nonsingular. Given an autonomous behavior $\mathfrak{B} \in \mathfrak{L}^{\mathfrak{w}}$, the $\mathfrak{w} \times \mathfrak{w}$ matrices associated with any two kernel representations of \mathfrak{B} have the same Smith form. The invariant polynomials of the Smith form are also called the *invariant polynomials of \mathfrak{B}* ; the product of such polynomials of \mathfrak{B} is denoted by $\chi_{\mathfrak{B}}$ and is called the *characteristic polynomial of \mathfrak{B}* .

A linear differential system can also be represented in *hybrid form*, i.e. as

$$R \left(\frac{d}{dt} \right) w = M \left(\frac{d}{dt} \right) \ell \quad (2)$$

where $M \in \mathbb{R}^{\bullet \times 1}[\xi]$ and ℓ is a *latent variable*. Equation (2) defines the *full behavior*

$$\mathfrak{B}_f := \{(w, \ell) \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathfrak{w}+1}) \mid (2) \text{ is satisfied weakly}\}$$

and the *external behavior*

$$\mathfrak{B} := \{w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathfrak{w}}) \mid \exists \ell \text{ s.t. } (2) \text{ is satisfied weakly}\}.$$

A differential system with latent variable is called a *state system* if the latent variable satisfies the *property of state*: for

all $(w_1, \ell_1), (w_2, \ell_2) \in \mathfrak{B}_f$ such that ℓ_1, ℓ_2 are continuous at $t = 0$, it holds that

$$[\ell_1(0) = \ell_2(0)] \implies [(w_1, \ell_1) \wedge (w_2, \ell_2) \in \mathfrak{B}_f] ,$$

where \wedge denotes concatenation at zero:

$$(w_1, \ell_1) \wedge (w_2, \ell_2) := \begin{cases} (w_1, \ell_1)(t) & \text{for } t < 0 \\ (w_2, \ell_2)(t) & \text{for } t \geq 0. \end{cases}$$

We call a polynomial differential operator $X \left(\frac{d}{dt} \right)$ induced by a polynomial matrix $X \in \mathbb{R}^{\bullet \times w}[\xi]$ a *state map* if the hybrid representation

$$\begin{aligned} R \left(\frac{d}{dt} \right) w &= 0 \\ X \left(\frac{d}{dt} \right) w &= x \end{aligned}$$

defines a state system. The state map is called *minimal* if it induces a minimal state variable. In [4] (see Algorithm 1 p. 1066) it has been shown how to compute a state map from a given kernel representation; in section III we illustrate an alternative approach based on bilinear differential forms, which we now introduce.

Let $\Phi \in \mathbb{R}^{w_1 \times w_2}[\zeta, \eta]$, the set of $w_1 \times w_2$ polynomial matrices in the two indeterminates ζ and η ; then $\Phi(\zeta, \eta) = \sum_{h,k=0}^N \Phi_{h,k} \zeta^h \eta^k$, where $\Phi_{h,k} \in \mathbb{R}^{w_1 \times w_2}$ and N is a nonnegative integer. The two-variable polynomial matrix Φ induces the bilinear functional acting on w_1 -, respectively, w_2 -dimensional infinitely differentiable trajectories, defined as $L_\Phi(w_1, w_2) = \sum_{h,k=0}^N \left(\frac{d^h w_1}{dt^h} \right)^T \Phi_{h,k} \frac{d^k w_2}{dt^k}$. Such a functional is called a *bilinear differential form* (BDF). L_Φ is *skew-symmetric*, meaning $L_\Phi(w_1, w_2) = -L_\Phi(w_2, w_1)$ for all w_1, w_2 , if and only if Φ is a *skew-symmetric* two-variable polynomial matrix, i.e., if $w_1 = w_2$ and $\Phi(\zeta, \eta) = -\Phi^T(\eta, \zeta)$. A two-variable polynomial matrix $\Phi(\zeta, \eta)$ is called *symmetric* if $w_1 = w_2 = w$ and $\Phi(\zeta, \eta) = \Phi^T(\eta, \zeta)$. If Φ is symmetric then it induces also a quadratic functional acting on w -dimensional infinitely smooth trajectories as $Q_\Phi(w) := L_\Phi(w, w)$. We will call Q_Φ the *quadratic differential form* (QDF) associated with Φ .

Given a BDF L_Ψ we define its *derivative* as the BDF L_Φ defined by $L_\Phi(w_1, w_2) := \frac{d}{dt}(L_\Psi(w_1, w_2))$ for all w_1, w_2 . In terms of the two-variable polynomial matrices associated with the BDFs, the relationship between a BDF and its derivative is expressed as $\Phi(\zeta, \eta) = (\zeta + \eta)\Psi(\zeta, \eta)$. The notion of a derivative of a QDF is analogous and algebraically characterized in the same way; we will not repeat its definition here.

Finally, we introduce the notion of coefficient matrix of a B/QDF, and that of canonical factorization. With every $\Phi \in \mathbb{R}^{w_1 \times w_2}[\zeta, \eta]$ we associate its *coefficient matrix* $\tilde{\Phi}$, which is defined as the infinite matrix $\tilde{\Phi} := (\Phi_{i,j})_{i,j=0,\dots}$. Observe that although $\tilde{\Phi}$ is infinite, only a finite number of its entries are nonzero. Note also that Φ is skew-symmetric if and only if $\tilde{\Phi}^T = -\tilde{\Phi}$; also, Φ is symmetric if and only if $\tilde{\Phi}^T = \tilde{\Phi}$.

Given $\Phi \in \mathbb{R}^{w_1 \times w_2}[\zeta, \eta]$, we can factor its coefficient matrix as $\tilde{\Phi} = \tilde{N}^T \tilde{M}$, with \tilde{N} and \tilde{M} infinite matrices

having a finite number of rows and all but a finite number of nonzero entries. This factorization of $\tilde{\Phi}$ induces the following factorization of $\Phi(\zeta, \eta)$:

$$\Phi(\zeta, \eta) = [I_{w_1} \quad \zeta I_{w_1} \quad \dots] \tilde{N}^T \tilde{M} \begin{bmatrix} I_{w_2} \\ \eta I_{w_2} \\ \vdots \end{bmatrix} =: N(\zeta)^T M(\eta) \tag{3}$$

If we take \tilde{N} and \tilde{M} to be full row rank, then their number of rows equals the rank of $\tilde{\Phi}$; in this case we call (3) a *canonical factorization* of Φ . If Φ is symmetric, then it admits a *canonical symmetric factorization* $\Phi(\zeta, \eta) = M(\zeta)^T \Sigma M(\eta)$, with Σ a signature matrix.

III. BILINEAR DIFFERENTIAL FORMS AND STATE MAPS

In [4] an algebraic characterization of state maps has been given for systems in represented in kernel (1), hybrid (2) and in image form. Crucial in that characterization was the use of the property of state in order to determine necessary and sufficient conditions under which a trajectory is concatenable with zero. We follow a similar strategy in this section, but use an approach based on BDFs.

The main result of this paper is the following Theorem.

Theorem 1: Let $\mathfrak{B} \subset \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^w)$ be the set of weak solutions of (1). Define $\Phi(\zeta, \eta) := R(-\zeta) - R(\eta) \in \mathbb{R}^{\bullet \times w}[\zeta, \eta]$. Then there exists a polynomial matrix $\Psi(\zeta, \eta) \in \mathbb{R}^{\bullet \times w}[\zeta, \eta]$ such that

$$(\zeta + \eta)\Psi(\zeta, \eta) = \Phi(\zeta, \eta) .$$

Let $\Psi(\zeta, \eta) = L(\zeta)^T M(\eta)$ be a canonical factorization of $\Psi(\zeta, \eta)$. Then $M(\xi)$ defines a state map for \mathfrak{B} . Moreover, a polynomial matrix $X \in \mathbb{R}^{n \times w}[\xi]$ induces a state map for \mathfrak{B} if and only if there exist $A \in \mathbb{R}^{\bullet \times n}$ and $F \in \mathbb{R}^{\bullet \times w}[\xi]$ such that

$$M(\xi) = AX(\xi) + F(\xi)R(\xi) .$$

Proof: The fact that a two-variable polynomial matrix Ψ exists such that $(\zeta + \eta)\Psi(\zeta, \eta) = \Phi(\zeta, \eta)$ follows from the fact that $\Phi(-\xi, \xi) = 0$, and from Theorem 3.1 of [8].

In order to prove the rest of the claim, observe first that for linear systems, the property of state can be equivalently stated in terms of concatenability with zero: for all $(w, \ell) \in \mathfrak{B}_f$ such that ℓ is continuous at $t = 0$, it holds that

$$[\ell(0) = 0] \implies [(0, 0) \wedge (w, \ell) \in \mathfrak{B}_f] .$$

We consequently study the conditions under which a given trajectory $w \in \mathfrak{B}$ is concatenable with zero in the past. Observe that $0 \wedge w \in \mathfrak{B}$ if and only if for all testing functions f it holds that

$$\begin{aligned} & \int_{-\infty}^{+\infty} (0 \wedge w)(t)^T R \left(-\frac{d}{dt} \right)^T f(t) dt \\ &= \int_0^{+\infty} w(t)^T R \left(-\frac{d}{dt} \right)^T f(t) dt = 0 . \end{aligned}$$

Now repeatedly integrate by parts the last expression, obtaining

$$\begin{aligned} & \int_0^{+\infty} w(t)^\top R\left(-\frac{d}{dt}\right)^\top f(t) dt \\ &= \int_0^{+\infty} \left(R\left(\frac{d}{dt}\right) w \right)^\top f(t) dt + \text{remainder at zero}, \end{aligned}$$

where the remainder at zero is readily verified to be of the form

$$\left[w(0)^\top \quad \frac{dw}{dt}(0)^\top \quad \dots \quad \frac{d^i w}{dt^i}(0)^\top \dots \right] \tilde{\Psi} \begin{bmatrix} f(0) \\ \frac{df}{dt}(0) \\ \vdots \\ \frac{d^i f}{dt^i}(0) \dots \end{bmatrix}$$

where the constant matrix $\tilde{\Psi}$ equals

$$\tilde{\Psi} := \begin{bmatrix} -R_1 & -R_2 & \dots \\ R_2 & R_3 & \dots \\ \vdots & \ddots & \ddots \end{bmatrix},$$

with R_i the matrix coefficient of the i -th power of ξ in $R(\xi)$.

Given that $\left(R\left(\frac{d}{dt}\right)w\right)(t) = 0$ for $t \geq 0$, it follows that w is concatenable with zero in the past if and only if the remainder at zero equals zero; this in turn is equivalent, given the arbitrariness of the testing function f , to

$$\begin{bmatrix} w(0) \\ \frac{dw}{dt}(0) \\ \vdots \end{bmatrix} \in \ker \tilde{\Psi}.$$

It is a matter of straightforward verification to check that $\tilde{\Psi}$ is precisely the coefficient matrix of the two-variable polynomial matrix $\Psi(\zeta, \eta)$ such that $(\zeta + \eta)\Psi(\zeta, \eta) = R(-\zeta) - R(\eta)$. Now consider a canonical factorization of $\Psi(\zeta, \eta) = L(\zeta)^\top M(\eta)$; it follows from the previous argument that w is concatenable with zero in the past if and only if $\left(M\left(\frac{d}{dt}\right)w\right)(0) = 0$.

In order to prove the last part of the claim, we use the following results, contained in Lemma B.2 p. 1081 of [4], which we state again here in order to make the paper self-contained.

Lemma 2: Let $\mathfrak{B} = \ker R\left(\frac{d}{dt}\right)$, and $X_i \in \mathbb{R}^{\bullet \times w}[\xi]$, $i = 1, 2$. Assume that for all $w \in \mathfrak{B} \cap \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w)$ it holds that

$$\left[\left(X_1 \left(\frac{d}{dt} \right) w \right) (0) = 0 \right] \implies \left[\left(X_2 \left(\frac{d}{dt} \right) w \right) (0) = 0 \right];$$

then there exist $A \in \mathbb{R}^{\bullet \times \bullet}$ and $B \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ such that

$$X_2(\xi) = AX_1(\xi) + B(\xi)R(\xi).$$

In order to conclude the proof of the Theorem, apply Lemma 2 with $X_1 = X$ and $X_2 = M$. This concludes the proof. ■

The result of Theorem 1 suggests the following algorithm for the computation of a minimal state map.

Algorithm

Input: $R \in \mathbb{R}^{\bullet \times w}[\xi]$.

Output: A minimal state map $X \in \mathbb{R}^{\bullet \times w}[\xi]$ for $\mathfrak{B} = \ker R\left(\frac{d}{dt}\right)$.

Step 1: Compute $\Phi(\zeta, \eta) := \frac{R(-\zeta) - R(\eta)}{\zeta + \eta}$.

Step 2: Compute a canonical factorization $\Phi(\zeta, \eta) = L(\zeta)^\top M(\eta)$.

Step 3: Return $X(\xi) := M(\xi)$.

The result of Theorem 1 has several interesting consequences; in the next sections we explore two of them, regarding respectively internal and external symmetries, and Hamiltonian systems.

IV. APPLICATION: STATIC EXTERNAL AND INTERNAL SYMMETRIES

We begin by summarizing the results of [1], [2] which are more relevant to our investigation; see also [6]. Let $J \in \mathbb{R}^{w \times w}$ be an involution, i.e. $J^2 = I_w$, and let \mathfrak{B} be a differential behavior with w external variables. \mathfrak{B} is called *J-symmetric* if

$$[w \in \mathfrak{B}] \iff [Jw \in \mathfrak{B}],$$

equivalently if

$$J\mathfrak{B} := \{w' \mid \exists w \in \mathfrak{B} \text{ s.t. } w' = Jw\} = \mathfrak{B}.$$

It follows from standard arguments of behavioral system theory that if $R \in \mathbb{R}^{\bullet \times w}[\xi]$ is a full row rank matrix such that $\mathfrak{B} = \ker R\left(\frac{d}{dt}\right)$, then there exists a unimodular matrix $U(\xi)$ such that $R(\xi)J = U(\xi)R(\xi)$. The insight of [1], [2] is that there exists a representation (1) for which the matrix U is not only unimodular, but in fact a constant matrix; we review now this result.

It can be proven following the same argument used for Th. 5.3 of [1] that there exists a polynomial matrix R such that $\mathfrak{B} = \ker R\left(\frac{d}{dt}\right)$ and a signature matrix $\Sigma = \begin{bmatrix} I_{n_+} & 0 \\ 0 & -I_{n_-} \end{bmatrix}$ such that

$$R(\xi)J = \Sigma R(\xi), \quad (4)$$

and moreover that the integers n_+ and n_- only depend on the behavior \mathfrak{B} , and not on the particular matrix R . It also follows from the same argument that this particular matrix R can be chosen to be row proper (see section 6.3 of [3] for a definition). We can now state the following result.

Theorem 3: Let \mathfrak{B} be a *J-symmetric* linear differential system, with $J \in \mathbb{R}^{w \times w}$ a static involution. There exist a state representation \mathfrak{B}_f of \mathfrak{B} with state variable x and a signature matrix S such that

$$[(w, x) \in \mathfrak{B}_f] \iff [(Jw, Sx) \in \mathfrak{B}_f].$$

Proof: Choose a row proper matrix $R \in \mathbb{R}^{\bullet \times w}[\xi]$ such that $\mathfrak{B} = \ker R\left(\frac{d}{dt}\right)$ such that (4) holds. Compute Ψ as in Theorem 1, and recall that the coefficient matrix of Ψ equals

$$\begin{bmatrix} -R_1 & -R_2 & \dots \\ R_2 & R_3 & \dots \\ \vdots & \ddots & \ddots \end{bmatrix};$$

since R is row proper, it is straightforward to verify that the nonzero rows of this matrix are linearly independent. Equating the coefficients of the same powers of ξ on the right and the left of the equality $R(\xi)J = \Sigma R(\xi)$ it follows that $R_i J = \Sigma R_i$, $i = 0, \dots, L$, where L is the highest power of ξ in $R(\xi)$. Conclude that

$$\begin{bmatrix} -\Sigma R_1 & -\Sigma R_2 & \dots \\ \Sigma R_2 & \Sigma R_3 & \dots \\ \vdots & \ddots & \ddots \end{bmatrix} = \begin{bmatrix} -R_1 J & -R_2 J & \dots \\ R_2 J & R_3 J & \dots \\ \vdots & \ddots & \ddots \end{bmatrix}.$$

Now select from the matrix on the left-hand side of this equality the submatrix consisting of its nonzero rows. It is easily seen that we can write this submatrix as the product of a signature matrix S and the submatrix \tilde{M} of $\tilde{\Psi}$ consisting of its nonzero rows. Moreover, from the last equality it follows also that $S\tilde{M} = \tilde{M}J$. From this it follows also that $SM(\xi) = M(\xi)J$, where

$$M(\xi) = \tilde{M} \begin{bmatrix} I_w \\ \xi I_w \\ \vdots \end{bmatrix}$$

is a state map. Conclude from this argument that for the state representation

$$\begin{aligned} R \left(\frac{d}{dt} \right) w &= 0 \\ M \left(\frac{d}{dt} \right) w &= x \end{aligned} \quad (5)$$

it holds that

$$[(w, x) \text{ satisfy (5)}] \iff [(Jw, Sx) \text{ satisfy (5)}].$$

This concludes the proof of the claim. \blacksquare

The result of Theorem 3 illustrates how *external symmetries* at the level of the manifest variable w , and represented by the involution J , are reflected in *internal symmetries* at the level of state, represented by the signature matrix S . Note that this is deduced from first principles, and need not be considered a mere consequence of a set-up essentially based on the use of *a priori* available state-space representations.

A consequence of the correspondence between internal and external symmetries is that external symmetries are also reflected in state-space representations. For example, let \mathfrak{B} be an autonomous behavior and x a state variable for it; it has been shown in [4] that there exist constant matrices A and C such that

$$\begin{aligned} \frac{d}{dt} x &= Ax \\ w &= Cx. \end{aligned} \quad (6)$$

Now assume that \mathfrak{B} is J -symmetric, that x is minimal, and that the conditions of Theorem 3 are satisfied, i.e. that (w, x) is a full trajectory if and only if (Jw, Sx) is a full trajectory. It is a matter of straightforward verification to check that this implies that the following equalities hold for any full trajectory (w, x) :

$$\begin{aligned} \frac{d}{dt} x &= Ax = S^{-1}ASx \\ Jw &= JCx = CSx. \end{aligned}$$

Using the minimality of x , we conclude that $S^{-1}AS = A$ and that $JC = CS$. This argument leads to the following result.

Corollary 4: Let \mathfrak{B} be a J -symmetric linear differential system, with $J \in \mathbb{R}^{w \times w}$ a static involution. There exist a signature matrix S and a state representation (6) such that $S^{-1}AS = A$ and $JC = CS$.

A thorough investigation of whether the approach proposed herein can be useful to the study of how external symmetries reflect in state space equations will be pursued elsewhere.

V. APPLICATION: AUTONOMOUS HAMILTONIAN SYSTEMS

A behavioral approach to linear Hamiltonian systems has been given in [5]; see also [7]. In this section we restrict ourselves to the case of autonomous Hamiltonian systems. In order to define autonomous Hamiltonian systems, we need the notion of nondegeneracy of a BDF: a BDF L_Φ restricted to act on trajectories of an autonomous behavior \mathfrak{B} , denoted $L_\Phi|_{\mathfrak{B}}$, is called *nondegenerate* if for all $w \in \mathfrak{B}$ it holds that

$$[L_\Phi(v, w)(0) = 0 \text{ for all } v \in \mathfrak{B}] \iff [w = 0].$$

An autonomous behavior \mathfrak{B} is called *Hamiltonian* if there exists a skew-symmetric BDF L_Ψ such that such that

- (i) $\frac{d}{dt} L_\Psi(w_1, w_2) = 0$ for all $w_1, w_2 \in \mathfrak{B}$;
- (ii) L_Ψ is skew-symmetric;
- (iii) $L_\Psi|_{\mathfrak{B}}$ is nondegenerate.

We now show how Hamiltonianity is reflected in the state-space equations of a behavior; the result of Theorem 1 will be instrumental in reaching this conclusion. For simplicity of exposition we give the proof for the case in which the characteristic polynomial of the behavior has no root in zero.

Theorem 5: Let \mathfrak{B} be an autonomous Hamiltonian system whose characteristic polynomial $\chi_{\mathfrak{B}}(\xi)$ has no root at zero. Then $\deg \chi_{\mathfrak{B}}(\xi)$ is even, i.e. there exists a nonzero integer n such that $\deg \chi_{\mathfrak{B}}(\xi) = 2n$. Define

$$J := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix};$$

then there exists a state representation (6) of \mathfrak{B} such that the matrix A satisfies $A^\top J + JA = 0$.

Proof: We proceed by reducing to the scalar case: let $U(\xi)\Delta(\xi)V(\xi) = R(\xi)$ be the Smith form of $R(\xi)$, and observe that another kernel representation of \mathfrak{B} is given by $V(-\xi)^\top \Delta(\xi)V(\xi)$. Now define

$$\mathfrak{B}' := \{w' \mid w' = V \left(\frac{d}{dt} \right) w, w \in \mathfrak{B}\} = V \left(\frac{d}{dt} \right) \mathfrak{B};$$

since V is unimodular, there exists a one-one correspondence between the trajectories of \mathfrak{B} and those of \mathfrak{B}' . Now denote the diagonal elements of Δ with $\delta_i \in \mathbb{R}[\xi]$, $i = 1, \dots, w$, and observe that $w' \in \mathfrak{B}'$ if and only if $w'_i \in \ker \delta_i \left(\frac{d}{dt} \right)$, $i = 1, \dots, w$. Under the assumption on $\chi_{\mathfrak{B}}(\xi)$ having no root at zero, it follows from Theorem 3.4 of [5] that the δ_i are all even.

We now construct a state representation for $\ker \delta_i \left(\frac{d}{dt} \right)$, $i = 1, \dots, w$ with the required property on the state matrix; from this a state representation for \mathfrak{B}' , and ultimately for \mathfrak{B} , will be obtained satisfying the claim of the Theorem.

Using the result of Theorem 1, we define $\Psi'_i(\zeta, \eta)$ from

$$\delta_i(-\zeta) - \delta_i(\eta) = (\zeta + \eta)\Psi'_i(\zeta, \eta);$$

since δ_i is even, it follows that

$$(\zeta + \eta)\Psi'_i(\zeta, \eta) = \delta_i(\zeta) - \delta_i(\eta),$$

from which it is easily seen that Ψ'_i is skew-symmetric (as a two-variable polynomial matrix), and consequently that its coefficient matrix is also skew-symmetric. This argument can be repeated for all $i = 1, \dots, n$, thus obtaining a skew-symmetric matrix

$$\Psi'(\zeta, \eta) := \text{diag} (\Psi'_i(\zeta, \eta))_{i=1, \dots, w}.$$

A canonical factorization of the coefficient matrix of $\Psi'(\zeta, \eta)$ yields a minimal state variable for \mathfrak{B}' : in particular, we can take a skew-symmetric factorization

$$\Psi'(\zeta, \eta) = M'(\zeta)^\top J M'(\eta),$$

where the dimension of the matrix J is $2n = \sum_{i=1}^w \text{deg } \delta_i$, thus proving the first claim of the Theorem.

It follows from the one-one correspondence between trajectories of \mathfrak{B}' and of \mathfrak{B} established by the polynomial differential operator $V \left(\frac{d}{dt} \right)$ that $M(\xi) := M'(\xi)V(\xi)^{-1}$ is a minimal state map for \mathfrak{B} . To the state variable induced by $M \left(\frac{d}{dt} \right)$ corresponds a state representation (6); we now show that the matrix A satisfies $A^\top J + JA = 0$.

Define first $\Psi(\zeta, \eta) := M(\zeta)^\top J M(\eta)$ and $R'(\xi) := V(-\xi)^\top \Delta(\xi)V(\xi)$; and note that $R'(-\xi)^\top = R'(\xi)$. Moreover, note that

$$(\zeta + \eta)\Psi(\zeta, \eta) = R'(\zeta)^\top - R'(\eta).$$

Consequently, for all trajectories (w_i, x_i) , $i = 1, 2$ in the full behavior of this state representation it holds that

$$\frac{d}{dt} L_\Psi(w_1, w_2) = \frac{d}{dt} x_1^\top J x_2 = 0;$$

since $\frac{d}{dt} x_1^\top J x_2 = x_1^\top (A^\top J + JA) x_2 = 0$, the claim follows from the minimality of the state variable x . ■

The result of Theorem 5 is of course well-known in the classical state-space approach; however, it is important to remark that our result is obtained in a *representation-free* context, in which no a priori state-space representation is given. We can repeat here the same consideration made at the end of section IV: our conclusions on state representation are deduced from first principles, and intrinsic.

Remark 6: It is interesting to note that the two-variable polynomial matrix Ψ appearing in the proof of Theorem 5 is precisely the skew-symmetric matrix used in the proof of Theorem 3.4 of [5].

VI. CONCLUSIONS

In this paper we have proposed a novel approach to the computation of state maps based on the calculus of BDF. The main result of this paper is Theorem 1, from which an algorithm has been derived for the computation of minimal state maps. We have also shown how from the insight given by Theorem 1 follow some interesting conclusions which relate, in an intrinsic way, internal and external symmetries (see section IV); and autonomous Hamiltonian systems and properties of their state-space representations (section V).

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