A Stochastic Paradox for Reflected Brownian Motion?

Erik I. Verriest

***Abstract*— This paper is pedagogical in nature. It is shown that a stochastic integral with respect to a Wiener process does not always yield a martingale. Consequently, ignoring the $\mathrm{d}w$ term in the calculation of expectations, as might be done in a first 'back-of-the-envelope' approach, can lead to false results. We develop a model for Brownian motion constrained to $x > 0$, for which the moments can be computed exactly. There is no paradox if one solves the problem correctly, however a "blind" application of the Itô calculus yields to the paradox that the variance may become negative. In order to put the student back on track, we close the paper by stating some sufficient conditions that guarantee that the stochastic integral is a true martingale.**

I. Introduction

The objective of this paper is to point out to the student that when one uses the Itô calculus, one should be careful about the interpretation of the Itô differentials, and handle the process of exchanging the order of taking mathematical expectations and differentiation (or integration) with care.

Typically, one of the first examples the student may encounter is the scalar linear system with additive noise, in Itô form given by the equation

$$\mathrm{d}x = ax + b\mathrm{d}w \tag{1}$$

Applying the Itô differential rule, it is derived that

$$\mathrm{d}x^2 = (2ax^2 + b^2)\mathrm{d}t + 2bx\mathrm{d}w. \tag{2}$$

Taking expectations gives respectively

$$\mathbf{E}\,\mathrm{d}x = a\mathbf{E}\,x\mathrm{d}t$$

and

$$\mathbf{E}\,\mathrm{d}x^2 = (2a\mathbf{E}\,x^2 + b^2)\mathrm{d}t.$$

one may conclude that $\frac{\mathrm{d}\mathbf{E}\,x}{\mathrm{d}t} = a\mathbf{E}\,x$ and $\frac{\mathrm{d}\mathbf{E}\,x^2}{\mathrm{d}t} = 2a\mathbf{E}\,x^2 + b^2$ leading to the well known mean and variance differential equations.

$$\dot{m} = am \tag{3}$$

$$\dot{P} = 2aP + b^2. \tag{4}$$

At this point, the student may expect that $\mathbf{E}\,\mathrm{d}f(x) = \mathrm{d}\,\mathbf{E}\,f(x)$, and proceed accordingly in similar problems.

The problem of reflected Brownian motion shows otherwise. First we build a suitable model for reflected Brownian motion in Section II and show some of its properties regarding the exit behavior and show that the process is well defined in Sections III and IV. We compute (carelessly, I should say) the variance in V and find that it gets negative: a paradox! Then we resolve the paradox in Section VI.

E.I. Verriest is with the School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0250, USA erik.verriest@ece.gatech.edu

II. A Model for Reflected Brownian Motion

In this section we build a model to approximate a reflected Brownian motion by a process which is mathematically more tractable. It is assumed that the process starts deterministically at $x_0 > 0$. If $x(t)$ is large, it should behave as a pure Brownian motion, but when $x(t)$ approaches zero, a potential function should push it back towards the positive side. Let us choose $\frac{1}{x}$ for this potential.

This process obeys the Itô equation

$$\mathrm{d}x = \frac{1}{x}\mathrm{d}t + \mathrm{d}w. \tag{5}$$

This does not quite satisfy the Lipschitz conditions, but something can be done by taking (ϵ, ∞) as the domain of validity, and using various bounding arguments.
For now, I shall neglect that issue, and proceed determining some properties. Taking expectations of the above equation yields

$$\mathrm{d}\,\mathbf{E}\,x = \mathbf{E}\left(\frac{1}{x}\right)\mathrm{d}t. \tag{6}$$

Next, we derive the Itô differential for the process $1/x$.

$$\mathrm{d}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\mathrm{d}x + \frac{1}{2}\frac{2}{x^3}\mathrm{d}t.$$

Thus

$$\mathrm{d}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\left(\frac{1}{x}\mathrm{d}t + \mathrm{d}w\right) + \frac{1}{x^3}\,\mathrm{d}t = -\frac{1}{x^2}\,\mathrm{d}w.$$

Now, taking expectations yields

$$\mathrm{d}\left(\frac{1}{x}\right) = 0 \tag{7}$$

The initial condition is $\left(\frac{1}{x(0)}\right) = \frac{1}{x_0}$. Hence, for all $t > 0$ we get

$$\mathbf{E}\left(\frac{1}{x(t)}\right) = \frac{1}{x_0}. \tag{8}$$

Consequently,

$$\frac{\mathrm{d}\mathbf{E}\,x}{\mathrm{d}t} = \frac{1}{x_0} \tag{9}$$

from which

$$\mathbf{E}\,x(t) = x_0 + \frac{1}{x_0}t. \tag{10}$$

Intuitively, this seems plausible, the effect of the nonlocal potential gives a drift towards the right (increasing x). We'll sharpen our intuition about this process by determining the average exit time and exit probabilities in the next sections.

ISBN 978-963-311-370-7

III. Expected Exit Time

Consider the closed interval $[\beta, B]$, with $0 < \beta < B < \infty$. The expected first passage time is given as the solution to the equation [1]

$$\frac{1}{x}u'(x) + \frac{1}{2}u''(x) = -1 \tag{11}$$

with the boundary conditions $u(\beta) = u(B) = 0$. We get

$$\begin{aligned} xu''(x) + 2u'(x) &= -2x \\ (xu'(x) + u(x))' &= -2x \\ xu'(x) + u(x) - \beta u'(\beta) - u(\beta) &= \beta^2 - x^2. \end{aligned}$$

The left hand side contains

$$[xu(x)]' = xu'(x) + u(x).$$

Thus

$$\begin{aligned} xu(x) - \beta u(\beta) - [\beta u'(\beta) + u(\beta)](x - \beta) \\ = -\frac{1}{3}(x^3 - \beta^3) - \beta^2(x - \beta) \end{aligned}$$

With $u(\beta) = 0$, we get

$$xu(x) = -\frac{1}{3}(x^3 - \beta^3) - \beta^2(x - \beta) + \beta u'(\beta)(x - \beta).$$

From $u(B) = 0$, it follows that

$$\beta u'(\beta) = \frac{1}{3}(B^2 + B\beta + \beta^2) - \beta^2.$$

Hence,

$$u'(\beta) = \frac{B - \beta}{\beta}$$

and thus finally

$$u(x) = \frac{(B - x)(x - \beta)(B + x + \beta)}{3x} \tag{12}$$

or, letting $\tau_{[\beta,B]}$ be the exit time for the process $x(t)$, starting at $x(0) = x$, from the interval $[\beta, B]$, of course with $\beta < x < B$

$$\mathbf{E}\tau_{[\beta,B]} = \frac{(B - x)(x - \beta)(B + x + \beta)}{3x} \tag{13}$$

Also this seems intuitively clear. If x is either β or B, exit is instantaneous. Taking the limit for $\beta \to 0$,

$$\mathbf{E}\tau_{[0,B]} = \frac{B^2 - x^2}{3}. \tag{14}$$

If $B \to \infty$, the exit time approaches infinity, no matter how small x is. This reinforces the intuition that exit at zero is not possible.

IV. Exit Probabilities

Consider again the closed interval $[\beta, B]$, with $0 < \beta < B < \infty$. The probability of exit at B is given by the solution to the equation [1]

$$\frac{1}{x}v'(x) + \frac{1}{2}v''(x) = 0 \tag{15}$$

with the boundary conditions $v(\beta) = 0$ and $v(B) = 1$. We get

$$\begin{aligned} (xv'(x) + v(x))' &= 0 \\ xv'(x) + v(x) &= A. \end{aligned}$$

for some constant A. The left hand side is

$$[xv(x)]' = A$$

Thus

$$xv(x) = Ax + a,$$

where a is another constant. With $v(\beta) = 0$, we get $a = -A\beta$, and from $Bv(B) = B = AB + a = A(B - \beta)$, it follows that

$$A = \frac{B}{B - \beta}$$

and thus finally

$$v(x) = \frac{B}{B - \beta}\,\frac{x - \beta}{x}. \tag{16}$$

It follows that

$$Pr\,(\text{exit at } B) = \frac{B}{B - \beta}\,\frac{x - \beta}{x}. \tag{17}$$

If $\beta \to 0$, then $Pr\,(\text{exit at } B) \to 1$. Hence the probability of the process crossing over $x = 0$ is zero. Also this seems intuitively plausible.

V. Variance Paradox

In this section the variance for this approximation to the reflected Brownian motion is computed. Since the process starts deterministically at x_0, the initial variance is zero. Consider now the process $x^2(t)$. With the Itô-rule,

$$\begin{aligned} \mathrm{d}x^2 &= 2x\,\mathrm{d}x + \mathrm{d}t \\ &= 2x\left(\frac{1}{x}\mathrm{d}t + \mathrm{d}w\right) + \mathrm{d}t \\ &= 3\mathrm{d}t + 2x\mathrm{d}w. \end{aligned} \tag{18}$$

Taking expectations

$$\mathrm{d}\,\mathbf{E}\,x^2 = 3\mathrm{d}t.$$

Integrating, this yields explicitly

$$\mathbf{E}\,x^2(t) = x_0^2 + 3t.$$

We also have the deterministic equation

$$\mathbf{E}\,x(t)^2 = x_0^2 + 2t + \frac{1}{x_0^2}t^2.$$

The variance is obtained from

$$\begin{aligned}\mathrm{Var}_{x(t)} &= \mathbf{E}\,x^2(t) - (\mathbf{E}\,x(t))^2 \\ &= x_0^2 + 3t - (x_0^2 + 2t + \frac{1}{x_0^2}t^2) \\ &= t - \frac{t^2}{x_0^2}. \end{aligned} \tag{19}$$

For $t > x_0^2$, *this variance is negative*, a paradox! What went wrong?

VI. Resolution of the Paradox

Consider the d-dimensional Brownian motion, where the coordinates of the sample paths are independent standard one-dimensional Brownian motions. It is known [2, p.59] that if $r(t) = \|x(t)\|$, then

$$\Pr\{\, r(t) \in \mathrm{d}r \,|\, \mathcal{R}_s\} = g(t-s, r(s), r)\,\mathrm{d}r,$$

where

$$g(t,a,b) = t^{-1}\exp\left(-\frac{a^2+b^2}{2t}\right)(ab)^{1-\frac{d}{2}} I_{\frac{d}{2}-1}\left(\frac{ab}{t}\right) b^{d-1}$$

and $I_{(d/2)-1}$ is the modified Bessel function. The probability is conditioned on the filtration $\mathcal{R}_s = \{r(\tau)\,|\,\tau \le s\}$. This $g(t,a,b)$ is also the fundamental solution of

$$\frac{\partial u}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial b^2} + \frac{d-1}{b}\frac{\partial u}{\partial b}\right),$$

with boundary condition

$$\lim_{b\downarrow 0} b^{d-1}\frac{\partial u}{\partial b} = 0.$$

The corresponding Itô equation for the process $r(t)$ is

$$\mathrm{d}r(t) = \frac{d-1}{2r(t)}\mathrm{d}t + \mathrm{d}w(t).$$

Note that for $d = 3$, the three-dimensional Brownian motion, this gives

$$\mathrm{d}r(t) = \frac{1}{r(t)}\mathrm{d}t + \mathrm{d}w(t), \tag{20}$$

which is precisely the equation we were investigating. Equation (20) and its connection with the Bessel process provides now a different avenue to resolve the paradox described in section V.

Consider thus the three-dimensional Brownian motion (BM), starting at the point with coordinates $(x_0, 0, 0)$, where $x_0 = r_0$. Let $r(t)$ be the distance of the BM from the origin. The stochastics of $r(t)$, the distance of the Brownian particle from the origin, is identical to the 1-D dynamics on the half-line for the particle in a $1/x$ potential.

But this 3-dimensional problem is easily solved by elementary probability [3]. The easiest way is to obtain first the density of r^2. From the properties of 1-D Brownian motion, we know that $x(t)$ is normally distributed with mean r_0 and variance t (standard case). Likewise, both $y(t)$ and $z(t)$ are normally distributed with mean zero and variance t. These three processes are independent. Hence $u(t) = y^2(t) + z^2(t)$ has a χ-square distribution with two degrees of freedom. The density is

$$f_{u(t)}(u) = \frac{1}{2t}\exp\left(-\frac{u}{2t}\right)H(u), \tag{21}$$

where H is the Heaviside step function. The random variable $v = x^2(t)$ has density

$$f_{v(t)}(v) = \frac{1}{2\sqrt{v}}\left(f_x(\sqrt{v}) + f_x(-\sqrt{v})\right)H(v). \tag{22}$$

With

$$f_{x(t)}(x) = \frac{1}{2\pi t}\exp\left(-\frac{x^2}{2t}\right), \tag{23}$$

this gives

$$f_{v(t)}(v) = \frac{\exp\left(-\frac{(\sqrt{v}-r_0)^2}{2t}\right)}{\sqrt{v}\sqrt{2\pi t}\left(1+\mathrm{erf}\left(\frac{r_0}{\sqrt{2t}}\right)\right)}H(v). \tag{24}$$

Independence of $u(t)$ and $v(t)$ results in the convolution formula for the density of their sum $r^2(t) = u(t) + v(t)$. Thus,

$$\begin{aligned} f_{r^2(t)}(\rho) &= \int_0^\rho f_{v(t)}(\rho - s) f_{u(t)}(s)\,\mathrm{d}s \\ &= \frac{1}{2^{3/2}r_0\sqrt{\pi t}}\left(\exp\left(-\frac{(\sqrt{\rho}-r_0)^2}{2t}\right) + \right. \\ &\qquad \left. - \exp\left(-\frac{(\sqrt{\rho}+r_0)^2}{2t}\right)\right) H(\rho) \end{aligned} \tag{25}$$

With this density, the expected value of $r^2(t)$ is readily computed

$$\mathbf{E}\,r^2(t) = \int_0^\infty \rho f_{r^2(t)}(\rho)\,\mathrm{d}\rho = r_0^2 + 3t. \tag{26}$$

Likewise, we obtain the expected value of $r(t)$ by

$$\begin{aligned} \mathbf{E}\,r(t) &= \int_0^\infty \sqrt{\rho} f_{r^2(t)}(\rho)\,\mathrm{d}\rho \\ &= \frac{1}{r_0\sqrt{\pi}}\left(r_0\sqrt{2t}\exp(-\frac{r_0^2}{2t}) + \right. \\ &\qquad \left. +\sqrt{\pi}(r_0^2 + t)\mathrm{erf}\left(\frac{r_0}{\sqrt{2t}}\right)\right). \end{aligned} \tag{27}$$

Finally, the inverse distance, $1/r(t)$ has average value

$$\mathbf{E}\left(\frac{1}{r(t)}\right) = \int_0^\infty \frac{1}{\sqrt{\rho}} f_{r^2(t)}(\rho)\,\mathrm{d}\rho = \frac{1}{r_0}\mathrm{erf}\left(\frac{r_0}{\sqrt{2t}}\right). \tag{28}$$

Unless $t = 0$, this is strictly less than $1/r_0$. Figure 1 displays $\mathbf{E}\left(\frac{r_0}{r(t)}\right)$ and it can be seen that the naive prediction (equality to 1) only holds approximately for small values of t and x_0.

It is now easily verified that for the process we considered in Section II, we get indeed,

$$\mathrm{d}\,\mathbf{E}\,x = \mathbf{E}\left(\frac{1}{x}\right)\mathrm{d}t, \tag{29}$$

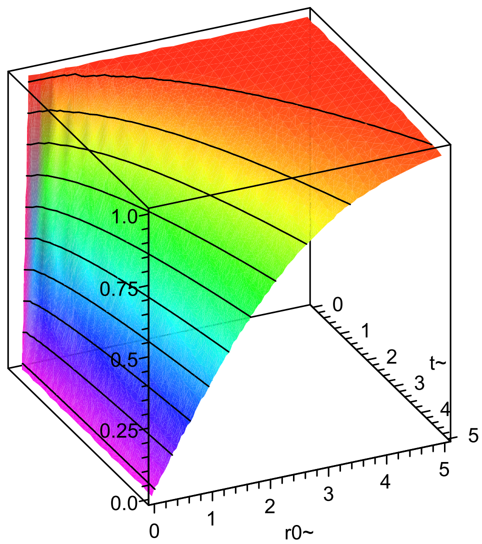

Fig. 1. The process $\left\langle \frac{r_0}{r(t)} \right\rangle$

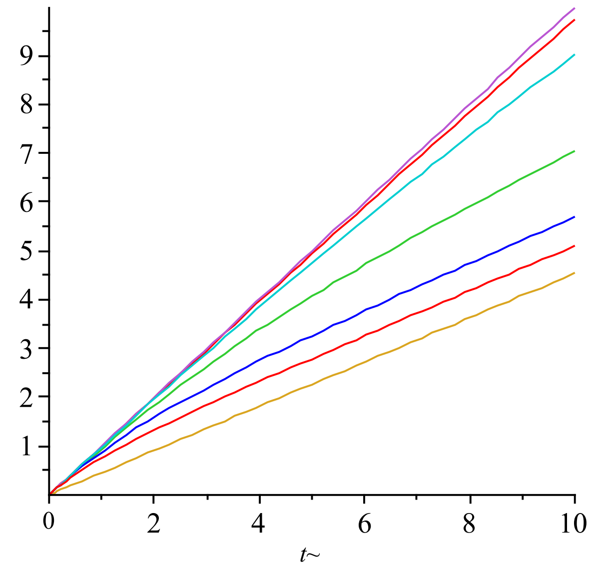

Fig. 2. Variance of $r(t)$

but

$$\mathrm{d}\,\mathbf{E}\left(\frac{1}{x}\right) = -\frac{1}{\sqrt{2\pi t^3}}\exp\left(-\frac{x_0^2}{2t}\right)\,\mathrm{d}t, \tag{30}$$

which is not zero!

Finally, the correct expression of the variance follows from

$$\mathrm{Var}(r(t)) = \langle r^2(t)\rangle - \langle r(t)\rangle^2.$$

Figure 2 displays the variance for different values of x_0 as function of time. For $x_0 \to 0$, the variance has slope $3-8/\pi$, whereas for $x_0 \to \infty$ we get slope 1 as expected (standard Brownian motion). The curves in the figure correspond to $x_0 = 0.1$, 2, 3, 5, 10, 20 and 100.

The figure below shows the average position as function of time together with the 1-sigma bounds for the uncertainty ($\sigma = \sqrt{t}$) for initialization at $x_0 = 0$, 10 and 20.

VII. Further Thoughts

The origin of the problem is that in the differential Itô equation we get correctly

$$\mathbf{E}\,\mathrm{d}\left(\frac{1}{x}\right) = 0. \tag{31}$$

However, this does not imply that

$$\mathrm{d}\,\mathbf{E}\left(\frac{1}{x}\right) = 0.$$

The integrated form is:

$$\mathbf{E}\left(\frac{1}{x(t)}\right) = \frac{1}{x_0} - \mathbf{E}\left(\int_0^t \frac{\mathrm{d}w(s)}{x(s)^2}\right).$$

More detailed arguments, invoking stopping times, show that $1/x(\min(t,\tau_k))$ approaches $1/x(t)$ a.e. as $k \to +\infty$, however the convergence is neither monotone nor dominated

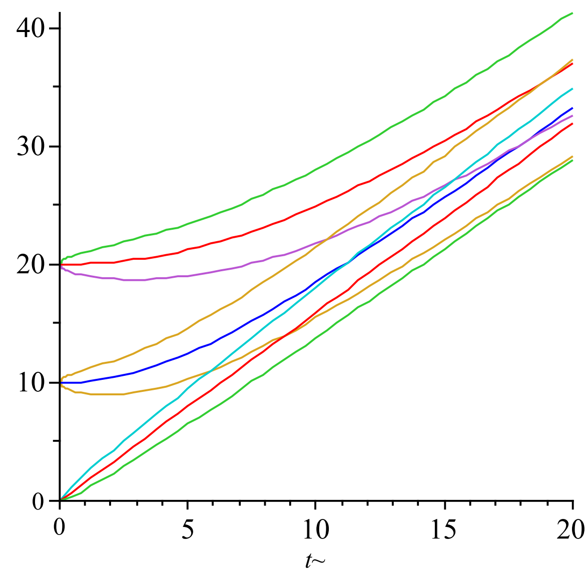

Fig. 3. Evolution of mean and uncertainty (1-σ bounds)

so the only thing one can do to pass to the limit in the integral is to use Fatou's Lemma to get

$$\begin{aligned}\mathbf{E}\left(\frac{1}{x(t)}\right) &= \mathbf{E}\left(\lim_{k\to+\infty}\frac{1}{x(\min(t,\tau_k))}\right)\\ &\le \liminf_{k\to+\infty}\mathbf{E}\left(\frac{1}{x(\min(t,\tau_k))}\right) = 1/x_0.\end{aligned}$$

So one gets only inequality instead of the equality $\mathbf{E}\,(1/x(t)) = 1/x_0$. Hence interchanging expectation and integration is not implied, as only Fatou's lemma can be used to provide the bound

$$\mathbf{E}\left(\frac{1}{x(t)}\right) \leq \frac{1}{x_0}. \tag{32}$$

Also we note that (5) does not satisfy a Lipshitz condition since

$$\left|\frac{1}{x} - \frac{1}{y}\right|$$

cannot be bounded by $C|x-y|$ for any constant C. Hence this process may not be an Itô diffusion. We note that the three dimensional Brownian motion is (trivially) an Itô diffusion. Applying the Itô-rule to the map

$$\phi : w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \to \|w\| = \sqrt{w_1^2 + w_2^2 + w_3^2}$$

is dubious, since ϕ is not C^2 at the origin. Formal application gives

$$\mathrm{d}\|w\| = \frac{1}{\|w\|}\mathrm{d}t + \sum_{i=1}^{3} \frac{w_i \mathrm{d}w_i}{\|w\|}.$$

Now, $\sum_{i=1}^{3} \frac{w_i \mathrm{d}w_i}{\|w\|}$ has the same finite-dimensional distributions as a one dimensional Brownian motion $\mathrm{d}\overline{w}$ [4, p. 135]. Since $[w_1, w_2, w_3]$ never hits the origin, (5) is an Itô diffusion. It can be shown that the process $u(w) = \frac{1}{\|w\|}$ is a positive supermartingale [5, p.33]. If $\{T_n\}$ is an increasing sequence of stopping times, with $T_n \to \infty$, then the process $u(w(t \wedge T_n))$ is a martingale, implying that $u(w(t))$ is a local martingale. Since $\mathbf{E}\left(u(w(t))\right) \to 0$ as $t \to \infty$, and $\mathbf{E}\, u(w(0)) = \frac{1}{|x_0|} \neq 0$, it follows that $u(w(t))$ cannot be a martingale.

We recall the following result [4].
Let w be a Wiener process (Brownian motion). Define $\mathcal{F}_t$ to be the σ-algebra generated by the random variables $w(s)$, for $s \leq t$. Let $f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$ be such that

i) $(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable , where $\mathcal{B}$ denotes the σ-algebra on $[0,\infty)$

ii) $f(t,\omega)$ is $\mathcal{F}_t$-adapted

iii) $\mathbf{E} \int_s^t f(\tau,\omega)^2 \, \mathrm{d}\tau < \infty$

then the integral $\int_s^t f(\tau,\omega)\mathrm{d}w(t,\omega)$ in the Itô-sense is a martingale.

In fact the conditions can be relaxed [4, p.31]: Condition (ii) may be replaced by
(ii)’: There exists an increasing family of σ-algebras $\mathcal{H}_t$, such that $w(t)$ is a martingale with respect to $\mathcal{H}_t$ and $f(t)$ is $\mathcal{H}_t$-adapted.

Condition (iii) can be relaxed to:
(iii)’

$$\Pr\left\{\int_0^t f(s,\omega)^2 \, \mathrm{d}s < \infty \quad \text{for all } t \geq 0\right\} = 1.$$

VIII. Conclusions

We have illustrated by way of example, that it is worthwhile to check the intricate details and conditions involved to guarantee that the differential rule and interchanging integrals (integration and expectation) are applicable. We have shown that carelessness leads to the paradoxical situation of resulting in a process with a negative variance. The paradox is resolved by developing properties of the three dimensional Brownian motion, and its related Bessel process. For this approximation of the reflected Brownian motion by a repulsive potential, exit probabilities and expected exit times from an interval were calculated. Sufficient conditions were presented under which the result of the Itô integral is a true martingale.

Acknowledgement The author is indebted to Prof. Andrzej Swiech (Math, GT), for valuable discussions on the paradox expressed in section V and providing the impetus to unravel the paradox. In addition, the reviewers’ comments, which helped to improve this note, are gratefully acknowledged.

References

[1] Z. Schuss, *Theory and Application of Stochastic Differential Equations* Wiley 1980.

[2] K. Itô and H.P. McKean, Jr. *Diffusion Processes and their Sample Paths*. Academic Press, 1965.

[3] A. Papoulis, *Probability, Random Variables and Stochastic Processes.* McGraw-Hill, 1965

[4] B. Øksendal, *Stochastic Differential Equations*, Fourth Ed. Springer-Verlag 1995.

[5] P. Protter, *Stochatic Integration and Differential Equations*, Springer-Verlag 1990.