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Boundary Oracles for Control-Related Matrix Sets

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***Abstract*— This paper presents closed-form solutions for the problem of finding the points of intersection of a 1D line and the boundary of typical matrix sets encountered in control; specifically, those defined by linear matrix inequalities. This procedure is referred to as boundary oracle; it is the key technical component of various random walk algorithms exploited within the randomized approach to control and optimization. In the paper, several such oracles are devised and generalized to robust formulations where the coefficients of matrix inequalities are subjected to uncertainties.**

I. INTRODUCTION

A randomized approach to solution of "hard" control and optimization problems in an approximate way is currently in wide use in the theory and practice, see [16] and the references therein. The overall idea of this approach is to soften the originally deterministic formulation by allowing for a small risk of violation of the desired property of the control system. As a result, a solution can often be obtained easily yet at the expense of neglecting low-probability events. Equally importantly, with this approach, solution of "traditional" problems can as well be obtained for high-dimensional data and/or robust statements of the problem.

A wide class of randomized methods is based on sampling the required system parameters randomly in their feasible domains in order to form the associated probabilistic performance index. One of the popular sampling procedures, the Hit-and-Run (HR) algorithm introduced in [15] and later extensively analyzed by L. Lovász and co-authors (e.g., see [7]), has been applied since recently to the uniform generation of points inside feasible domains, probabilistic solution of wide classes of optimization problems [1], [3], randomized control system design [10], etc.

A cornerstone of the HR-algorithm is computation of the intersection points of a line and the boundary of the feasible domain; this is referred to as *boundary oracle* (BO), [12]. Often, BOs can be implemented as a straightforward 1D search; however, since this operation is to be performed frequently, it is of importance to devise a "closed-form" solution to speed up the algorithm. Such *numerical* operations as computing the roots of polynomials or the eigenvalues of matrices, solving reasonably-sized semidefinite programs (SDPs) and other similar procedures can be considered closed-form solutions, since they are fast and numerically stable as implemented in the standard MATLAB environment.

In this note, we present the construction of efficient BOs for several generic matrix sets encountered in control problems, specifically, those defined by linear matrix inequalities (LMIs). These include LMI constraints in the standard form, Lyapunov-like inequalities, and quadratic matrix inequalities. Importantly, due attention is paid to robust formulations, where the matrix coefficients are subject to uncertainty.

The emphasis on LMIs is made because of the wide spread of this apparatus in systems theory; see the classical book [2]. Notably, since the LMI sets are convex, only two intersection points are to be found.

The setup in all the problems considered below is the following.

—Input for the BO: An implicitly specified feasible domain $\mathcal{D}$ (in the vector or matrix space); a feasible point $x \in \operatorname{int}\mathcal{D}$, and a direction y.

—Output of the BO: The minimum $\underline{\lambda}$ and maximum $\overline{\lambda}$ values of $\lambda \in \mathbb{R}$ such that $x + \lambda y \in \mathcal{D}$ for all $\lambda \in [\underline{\lambda}, \overline{\lambda}]$.

II. PRELIMINARIES

A. Recalling Hit-and-Run

We briefly recall the HR procedure aimed at generating random points in many-dimensional domains. For convex bodies, this iterative scheme in its simplest version formulates as follows.

Let x^0 be an initial point in the interior of a closed convex bounded set $\mathcal{D} \subset \mathbb{R}^n$ and let x^j be the point obtained at the jth step of the algorithm. A random direction $y \in \mathbb{R}^n$ is generated uniformly on the unit sphere in $\mathbb{R}^n$ and the 1D-line $x^j + \lambda y$ is considered. The points $\underline{x}^j$ and $\overline{x}^j$ of its intersection with the boundary of $\mathcal{D}$ are computed by means of BO. The next-step point x^{j+1} is then generated randomly uniformly on the chord $[\underline{x}^j, \overline{x}^j]$ and the process continues from this new point. This is schematically illustrated in Fig. 1.

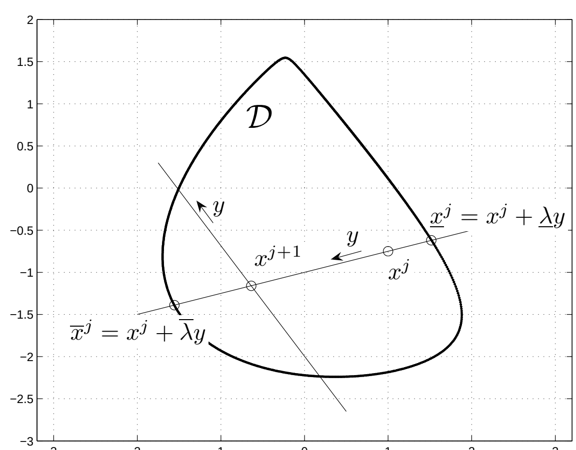

Fig. 1. The HR scheme

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Under certain conditions, the random point x^N generated in such a way has asymptotically uniform distribution on $\mathcal{D}$. Originally, this method was devised for numerical computation of multi-dimensional integrals over convex domains. Since recently, there have appeared other applications of this procedure. For instance, from these sampled points, the center of gravity of $\mathcal{D}$ can be evaluated for further use in optimization schemes based on cutting-plane methods, [1], [3], [13]. Alternatively, in control problems, the HR-points can be used to "representatively" sweep feasible domains $\mathcal{D}$ which do not admit simple characterization.

Among the attractive features of the HR procedure is its simplicity, fast *practical* convergence for so-called isotropic sets, generalizations to nonconvex sets and non-uniform distributions, etc. On the other hand, the method suffer certain drawbacks such as pessimistic theoretical estimates of the rate of convergence, poor performance for "narrow" sets, and others. Since our goal in this paper is to study not the overall behavior of the HR-procedure, but rather its BO component, we prefer not to dwell deep into the details; the interested readers are referred to [15], [7] and [5], [11]. In particular, the two latter papers provide novel modifications of the standard HR scheme combined with barrier functions method leading to strongly polynomial convergence to the uniform distribution for "arbitrary" convex sets.

It is also worth noting that independently of its use in HR-like algorithms, BOs can be thought of as tools for approximate description of the boundary of convex sets. Indeed, such a description can be obtained by fixing a ("deep") feasible point x^0 and generating sufficiently many random directions y^i.

B. Illustration: Static output feedback stabilization

To illustrate by a typical control problem, let us consider the linear time invariant system given by

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{1}$$

where the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{k\times m}$ are known, the pair (A, B) is controllable, and let $\mathcal{D} \subset \mathbb{R}^{k\times m}$ be the (nonempty) set of all matrices K of the stabilizing controllers $u = Ky$. The goal is to optimize certain "engineering" performance index over the feasible set $\mathcal{D}$; for instance, overshoot, damping time, etc. Noting that direct optimization over all static output controllers is hard, a randomized alternative might be to sample the set $\mathcal{D}$ via the HR-algorithm and optimize the index "manually" by simply computing its values at the sampled points. Such a semi-heuristic approach to optimal design is simple and often leads to better controllers than those obtained by regular techniques; e.g., see [6].

Hence, in the HR implementation, assuming that a certain feasible $K \in \mathcal{D}$ is given and letting $L \in \mathbb{R}^{k\times m}$ be a matrix direction, the BO formulates as finding the critical values of $\lambda \in \mathbb{R}$ retaining the stability of the $n \times n$ matrix $A + B(K + \lambda L)C$. This is readily doable by denoting the stable $A + BKC \doteq A_0$ and the increment $BLC \doteq A_1$, and computing the eigenvalues of certain combinations of Kronecker sums of A_0, A_1, [4]. This result is presented below for completeness of the exposition.

Lemma 1 ([4]): Consider $A_0, A_1 \in \mathbb{R}^{n\times n}$, where A_0 is Hurwitz stable. Then the minimal and the maximal values of the parameter $\lambda \in \mathbb{R}$ retaining the stability of the matrix $A_0 + \lambda A_1$ are given by

$$\underline{\lambda} = \begin{cases} \dfrac{1}{\min\limits_{\lambda_i<0} \lambda_i}, & \\ -\infty, & \text{if all} \quad \lambda_i > 0; \end{cases} \tag{2}$$

$$\overline{\lambda} = \begin{cases} \dfrac{1}{\max\limits_{\lambda_i>0} \lambda_i}, & \\ +\infty, & \text{if all} \quad \lambda_i < 0, \end{cases} \tag{3}$$

where $\min$ and $\max$ are taken over the real eigenvalues λ_i of the matrix $-(A_0 \oplus A_0)^{-1}(A_1 \oplus A_1)$, and $A \oplus A$ stands for the Kronecker sum defined by $A \otimes I + I \otimes A$, where $\otimes$ is the Kronecker product of matrices. □

The matrix chord $[K + \underline{\lambda}L;\ \ K + \overline{\lambda}L]$ thus obtained belongs to the set of output stabilizing controllers, and the next HR point is then selected randomly on this chord (note that, generally speaking, the set $\mathcal{D}$ in this problem is nonconvex and not connected, and Lemma 1 gives only the first intersections).

The matrices to be tested in Lemma 1 are of the squared size $n^2 \times n^2$ so that for very high dimensions, execution times may be too large (e.g., for n being as small as $n = 35$, the computations via the formulae above require more than 15 seconds on a standard PC). More importantly, the MATLAB memory limitations can be exceeded. Hence, for high dimensions, one has to recourse to the direct one-dimensional search.

In what follows, we present some other typical control-related matrix sets that admit efficient boundary oracles, which are also based on computing the eigenvalues but the dimensionality issues are not crucial.

III. GENERIC LMI SETS

We first consider LMIs in the standard form:

$$F_0 + \sum_{i=1}^{n} x_i F_i \doteq F(x) \preceq 0, \tag{4}$$

where the (known) real $m \times m$ matrices F_i, $i = 0, \dots, n$, are symmetric, $x = (x_1, \dots, x_n)^{\mathsf{T}} \in \mathbb{R}^n$ is the vector variable, and $\preceq$ stands for negative semidefiniteness. The feasible set associated with this LMI constraint is

$$\mathcal{D} = \{x \in \mathbb{R}^n : F(x) \preceq 0\},$$

which is obviously convex (if nonempty).

To exclude trivialities, we assume that the set $\mathcal{D}$ is nonempty; then, letting $x^0 \in \mathrm{int}\mathcal{D}$ and $y \in \mathbb{R}^n$ be a feasible point and a direction, respectively, the goal is to find the critical values $\underline{\lambda}, \overline{\lambda}$ of $\lambda \in \mathbb{R}$ retaining the sign-definiteness of the matrix $F(x^0 + \lambda y)$ over the segment $[\underline{\lambda}, \overline{\lambda}]$. We have

$$F(x^0 + \lambda y) = F_0 + \sum_{i=1}^{n}(x_i^0 + \lambda y_i)F_i \doteq A_0 + \lambda A_1,$$

where it is denoted $F(x^0) \doteq A_0 \prec 0$ and $\sum_{i=1}^{n} y_i F_i \doteq A_1 = A_1^{\mathsf{T}}$. Hence, BO reduces to finding the critical values of $\lambda \in \mathbb{R}$ retaining the sign-definiteness of the matrix pencil $A_0 + \lambda A_1$. This is doable via computing the generalized eigenvalues of the pair A_0, A_1 as shown below.

Lemma 2 ([12]): Let $A_0 \prec 0$ and $A_1 = A_1^{\mathsf{T}}$, then the minimal and maximal values of the parameter $\lambda \in \mathbb{R}$ retaining the negative semidefiniteness of the matrix $A_0 + \lambda A_1$ are given by

$$\underline{\lambda} = \begin{cases} \max\limits_{\lambda_i < 0} \lambda_i, & \\ -\infty, & \text{if all} \quad \lambda_i > 0; \end{cases} \tag{5}$$

$$\overline{\lambda} = \begin{cases} \min\limits_{\lambda_i > 0} \lambda_i, & \\ +\infty, & \text{if all} \quad \lambda_i < 0, \end{cases} \tag{6}$$

where λ_i are the generalized eigenvalues of the pair of matrices A_0 and $-A_1$, i.e., $A_0 e_i = -\lambda_i A_1 e_i$. □

Indeed, the loss of sign-definiteness is equivalent to the loss of nonsingularity, i.e., to the existence of nonzero $e \in \mathbb{R}^m$ such that $(A_0 + \lambda A_1)e = 0$, which completes the proof.

This result can be thought of as a particularization of Lemma 1 to symmetric matrices, for which stability is equivalent to negative definiteness. The complete analogy becomes clear by noting that since A_0 is invertible, (usual) eigenvalues of the matrix $-A_0^{-1}A_1$ can be taken (cf. the matrix $-(A_0 \oplus A_0)^{-1}(A_1 \oplus A_1)$ in Lemma 1) instead of the generalized eigenvalues of the pair $(A_0, -A_1)$. As a result, formulae (5)–(6) transform into (2)–(3).

Note that unlike Lemma 1, Kronecker operations need not be performed so that the dimensionality issues are not crucial, and for $n = 35$, computations via formulae (5)–(6) are executable in about a millisecond.

Another comment relates to the availability of a feasible point $x^0 \in \mathrm{int}\mathcal{D}$. Such a point can be found by solving the auxiliary semidefinite program

$$\min \alpha \quad \text{s.t.} \quad F(x) \preceq \alpha I. \tag{7}$$

If the solution $\{\tilde{x}, \tilde{\alpha}\}$ of this problem is such that $\tilde{\alpha} > 0$, the original problem (4) is infeasible; otherwise we take $x^0 = \tilde{x}$.

A. Robust Setup

Assume now that the matrix coefficients F_i in (4) are subject to independent additive norm-bounded symmetric uncertainties $\Delta_i \in \mathbb{R}^{m \times m}$, $\|\Delta_i\| \le \varepsilon_i$, $i = 0, \ldots, n$, where $\|\cdot\|$ is the spectral norm and $\varepsilon_i \ge 0$ are given numbers. Such uncertainties will be referred to as admissible. We then arrive at the uncertain LMI

$$(F_0 + \Delta_0) + \sum_{i=1}^{n} x_i(F_i + \Delta_i) \doteq F(x, \Delta) \preceq 0$$

in the vector variable x. This relation is required to hold robustly for all admissible matrix perturbations Δ_i; i.e., the robustly feasible domain for the uncertain LMI is

$$\mathcal{D}^{rob} = \{x \in \mathbb{R}^n \colon F(x, \Delta) \preceq 0 \quad \text{for all admissible} \quad \Delta_i\},$$

which is also convex (if nonempty). The symmetricity of Δ_i's is assumed in order to preserve the symmetricity of the matrix coefficients.

Similarly to the non-robust case, assume that $x^0 \in \mathrm{int}\mathcal{D}^{rob}$ is given and $y \in \mathbb{R}^n$ is a direction; then the goal is to find the critical values $\underline{\lambda}^{rob}$ and $\overline{\lambda}^{rob}$ such that for all $\lambda \in [\underline{\lambda}^{rob}, \overline{\lambda}^{rob}]$, the uncertain LMI $F(x + \lambda y, \Delta) \preceq 0$ holds for all admissible uncertainties Δ_i. The structure of this problem is illustrated in Fig. 2 for an LMI with $n = 2$ variables x_1, x_2, randomly generated symmetric $F_i \in \mathbb{R}^{3\times3}$, $i = 0, 1, 2$, and certain levels ε_i of uncertainty leading to a nonempty $\mathcal{D}^{rob}$.

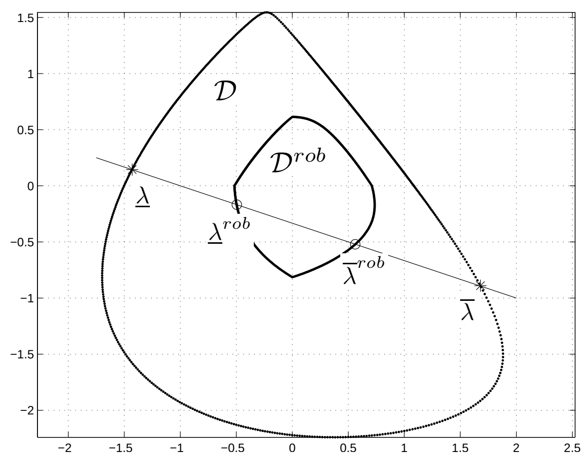

Fig. 2. Feasible and robustly feasible LMI domains

The robust BO reduces to solving a simple nonlinear equation in one scalar variable as shown in the lemma below.

Lemma 3 ([12]): Given $x^0 \in \mathrm{int}\mathcal{D}^{rob}$ and $y \in \mathbb{R}^n$, compose the following functions in the scalar variable λ:

$$\phi(\lambda) = \left\| \Big(F_0 + \sum_{i=1}^{n}(x_i^0 + \lambda y_i)F_i \Big)^{-1} \right\|,$$

$$\varepsilon(\lambda) = \frac{1}{\varepsilon_0 + \sum\limits_{i=1}^{n} |x_i^0 + \lambda y_i| \, \varepsilon_i}.$$

The minimal and maximal values of λ retaining the negative definiteness of the matrix $F(x^0 + \lambda y, \Delta)$ under all admissible perturbations Δ are given by the two solutions of the equation $\phi(\lambda) = \varepsilon(\lambda)$ on the segment $[\underline{\lambda}, \, \overline{\lambda}]$ (5)–(6). □

The idea of the proof (see [12]) is based on the same consideration; namely, the loss of robust sign-definiteness is equivalent to the loss of nonsingularity for some admissible Δ_is. Technically, the result on the symmetric radius of nonsingularity (an analog of the result in [9]) is used and the fact that Δ_is are independent.

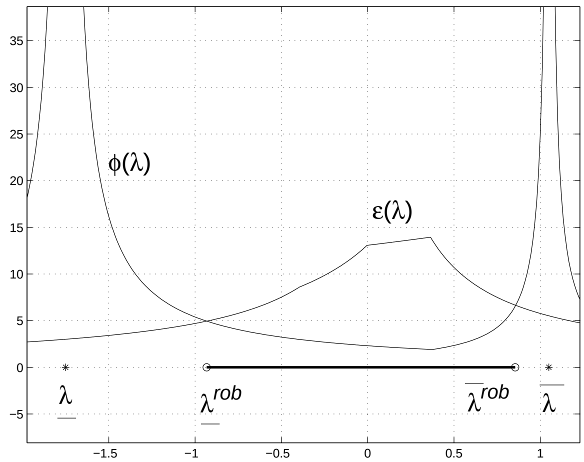

Fig. 3. The robust boundary oracle

Figure 3 depicts the functions $\phi(\lambda)$ and $\varepsilon(\lambda)$ and the resulting segments for λ in a typical LMI with four uncertain matrices.

Applications of the BOs considered in this section to solution of semidefinite programs (in particular, in the robust statement) are illustrated in [13], [3].

IV. QUADRATIC STABILIZATION

In control applications, LMI constraints are often formulated in terms of a matrix (rather than vector) variables; the simplest one has the form

$$AX + XA^{\mathsf T} + G \prec 0, \qquad X \succ 0, \tag{8}$$

where $G = G^{\mathsf T}$ and A are given and $X = X^{\mathsf T}$ is the $n \times n$ matrix variable.

For instance, this formulation is typical to quadratic stabilization. Indeed, it is well known that stabilizing state-feedback controllers $u = Kx$ for system (1) can be obtained in the form $K = -B^{\mathsf T}X^{-1}$ from the solutions of the LMI above with $G = -2BB^{\mathsf T}$, and $V(x) = x^{\mathsf T}X^{-1}x$ is verified to be a quadratic Lyapunov function for the closed-loop system.

This naturally gives rise to the following feasible domain in the matrix space:

$$\mathcal{D} = \{X \in \mathbb{R}^{n\times n} : AX + XA^{\mathsf T} + G \preceq 0,\ X \succeq 0\}, \tag{9}$$

for which the BO is immediately formulated using Lemma 2.

Indeed, given a feasible $X \in \operatorname{int}\mathcal{D}$ and a direction $Y = Y^{\mathsf T}$, we have

$$A(X + \lambda Y) + (X + \lambda Y)A^{\mathsf T} + G = A_0 + \lambda A_1,$$

where it is denoted $AX + XA^{\mathsf T} + G \doteq A_0 \prec 0$ and $AY + YA^{\mathsf T} \doteq A_1$. Then, using Lemma 2, find the segment of sign-definiteness of the pencil $A_0 + \lambda A_1$, denote it by Λ_A. Next, we have to account for the sign-definiteness of the pencil $X + \lambda Y$, denote the respective segment by Λ_X, which is found in the same way. Finally, the feasible segment of variation of the parameter λ is taken as $\Lambda = \Lambda_A \cap \Lambda_X$.

Hence, the boundary oracle for the set (9) reduces to doubly computing the generalized eigenvalues of the pairs of $n \times n$ matrices.

From the HR point of view, generation of matrix directions Y can be performed uniformly on the unit ball in the Frobenius norm $\|Y\|_F = 1$ followed by symmetrization $Y + Y^{\mathsf T} \to Y$.

A. Robust Setup

The BO proposed above admits a generalization to the robust case where the matrix A in (8) is subjected to the structured uncertainty of the form

$$A_\Delta = A + M\Delta N, \tag{10}$$

where $\Delta \in \mathbb{R}^{p\times q}$ is a perturbation matrix bounded in the spectral norm $\|\Delta\| \le 1$, and $M \in \mathbb{R}^{n\times p}$, $N \in \mathbb{R}^{q\times n}$ are fixed "frame" matrices which specify the uncertainty structure. This model of uncertainty is quite general and appears naturally in many control problems; e.g., it may specify certain selected entries of A that may be subjected to perturbations.

The Lyapunov inequality (8) with the matrix A changed for A_Δ is associated with finding a common quadratic Lyapunov function $V(x) = x^{\mathsf T}X^{-1}x$ for the uncertain state space system and to robust quadratic stabilization.

These problems lead to the robustly feasible set

$$\mathcal{D}^{rob} = \left\{X \succeq 0 :\ A_\Delta X + XA_\Delta^{\mathsf T} + G \preceq 0\ \forall\ \|\Delta\| \le 1\right\}, \tag{11}$$

and we are aimed at constructing a boundary oracle for this set. As above, given $X \in \mathcal{D}$ and a direction $Y = Y^{\mathsf T}$, the goal is to find min and max values of λ such that $X + \lambda Y \in \mathcal{D}^{rob}$ for all $\|\Delta\| \le 1$.

In problems of this sort, the main tool for dealing with uncertainty of the form (10) is the so-called *Petersen's lemma* which establishes necessary and sufficient conditions of robust sign-definiteness of a matrix.

Lemma 4 ([8]): Let $U = U^{\mathsf T} \in \mathbb{R}^{n\times n}$ and the two nonzero matrices $P \in \mathbb{R}^{n\times p}$, $Q \in \mathbb{R}^{q\times n}$ be known. Then the inequality

$$U + P\Delta Q + (P\Delta Q)^{\mathsf T} \preceq 0$$

holds for all $\Delta \in \mathbb{R}^{p\times q}$, $\|\Delta\| \le 1$, if and only if there exists $\varepsilon > 0$ such that

$$U + \varepsilon PP^{\mathsf T} + \frac{1}{\varepsilon}Q^{\mathsf T}Q \preceq 0. \tag{12}$$

□

Notably, the lemma reduces checking the robust sign-definiteness to a scalar problem.

By taking the Schur complement, inequality (12) can be rewritten as an LMI in the scalar variable ε:

$$\begin{pmatrix} U + \varepsilon PP^{\mathsf T} & Q^{\mathsf T} \\ Q & -\varepsilon I \end{pmatrix} \preceq 0.$$

We also note that the quantity

$$\gamma_{\max} = \max\{\gamma :\ U + P\Delta Q + (P\Delta Q)^{\mathsf T} \preceq 0\ \forall\ \|\Delta\| \le \gamma\} \tag{13}$$

referred to as the *radius of robust sign-definiteness* of the uncertain matrix $U + P\Delta Q + (P\Delta Q)^{\mathsf{T}}$ can be efficiently computed by solving a respective semidefinite program in the two scalar variables; see [14].

Back to the construction of BO for the set (11), we need to find the maximum and minimum values of λ such that the inequality

$$(A+M\Delta N)(X+\lambda Y) + (X+\lambda Y)(A+M\Delta N)^{\mathsf{T}} + G \preceq 0$$

holds robustly for all $\|\Delta\| \leq 1$.

Denoting

$$U(\lambda) = AX + XA^{\mathsf{T}} + G + \lambda(AY + YA^{\mathsf{T}})$$

and

$$P = M, \quad Q(\lambda) = N(X + \lambda Y),$$

the desired inequality writes

$$U(\lambda) + P\Delta Q(\lambda) + \big(P\Delta Q(\lambda)\big)^{\mathsf{T}} \preceq 0.$$

By Petersen's lemma, for this inequality to hold robustly against all $\|\Delta\| \leq 1$ for some fixed λ, it is necessary and sufficient that there exist $\varepsilon > 0$ such that

$$\begin{pmatrix} U(\lambda) + \varepsilon PP^{\mathsf{T}} & Q^{\mathsf{T}}(\lambda) \\ Q(\lambda) & -\varepsilon I \end{pmatrix} \preceq 0.$$

Noting that both $U(\lambda), Q(\lambda)$ are affine functions of λ, the inequality above is an LMI in λ, ε; hence, to find the critical values of λ, we minimize and maximize the variable λ subject to this LMI constraint. It now remains to take into account the sign-definiteness of the matrix $X + \lambda Y$. We thus arrived at the following *robust Lyapunov* BO.

Theorem 1: Let $X \in \mathcal{D}^{rob}$ (11)–(10) and $Y = Y^{\mathsf{T}} \in \mathbb{R}^{n\times n}$. Then the minimum and maximum values of $\lambda \in \mathbb{R}$ such that $X + \lambda Y \in \mathcal{D}^{rob}$ are given by the solutions of the two semidefinite programs

$$\min(\max)\, \lambda \quad \text{s.t.}$$

$$\begin{pmatrix} U(\lambda)+\varepsilon MM^{\mathsf{T}} & XN^{\mathsf{T}}+\lambda YN^{\mathsf{T}} \\ NX+\lambda NY & -\varepsilon I \end{pmatrix} \preceq 0, \ X + \lambda Y \succeq 0$$

in the scalar variables λ, ε, where it is denoted $U(\lambda) = AX + XA^{\mathsf{T}} + G + \lambda(AY + YA^{\mathsf{T}})$. □

These two problems are well-posed SDPs in just two scalar variables λ, ε, and the dimension of the block matrix is at most $2n \times 2n$. Using the MATLAB toolboxes SeDuMi and Yalmip, the computations on a standard PC takes slightly more than a second for $n = 35$.

We conclude this section with the three comments.

The first issue is the availability of a feasible point $X \in \mathcal{D}^{rob}$. This point can be found by solving an auxiliary SDP

$$\min\, \mu \quad \text{s.t.}$$

$$\begin{pmatrix} U + \varepsilon MM^{\mathsf{T}} & XN^{\mathsf{T}} \\ NX & -\varepsilon I \end{pmatrix} \preceq \mu I, \ X \succeq 0$$

(where it is denoted $U = AX + XA^{\mathsf{T}} + G$), in the matrix variable X and two scalar variables μ, ε. Likewise (7), if the solution $(\tilde{X}, \tilde{\mu}, \tilde{\varepsilon})$ is such that $\tilde{\mu} < 0$, then take $X = \tilde{X}$.

Second, if the LMI constraints in the auxiliary SDP above are incompatible, one might decrease the level of uncertainty and check again if $\mathcal{D}^{rob}$ is nonempty. However, a guaranteed acceptable level of uncertainty

$$\max\{\gamma : \exists X \succeq 0 : A_\Delta X + XA_\Delta^{\mathsf{T}} + G \preceq 0 \ \ \forall \ \ \|\Delta\| \leq \gamma\}$$

referred to as the *radius of robust quadratic stabilizability* can be computed *a priory* by solving an appropriate SDP of a similar form; cf. (13).

Third, uncertainty of the same form entering the B matrix in (1) can as well be taken into account along the same lines although at the expense of enlarging the dimension of the LMI constraints because of the presence of quadratic terms. This enlargement will be better explained in the next section.

V. LINEAR QUADRATIC REGULATORS

Yet another type of matrix sets typical to control applications is given by

$$\mathcal{D} = \{X \in \mathbb{R}^{n\times n} : XRX + AX + XA^{\mathsf{T}} - G \preceq 0, \ X \succ 0\}, \quad (14)$$

where $R \succ 0$, $G \succeq 0$, and A are given real $n \times n$ matrices; this set is defined by the positive definite solutions of the quadratic matrix Riccati *inequality*.

Indeed, design of linear stabilizing controllers $u = Kx$ for system (1) that minimize the integral quadratic functional leads to the matrix Riccati *equation* with $G = BS^{-1}B^{\mathsf{T}}$, whose unique solution $X \succ 0$ yields the optimal controller $K = -S^{-1}B^{\mathsf{T}}X^{-1}$. This is a classical linear quadratic regulation (LQR) problem, where $R \succ 0$ and $S \succ 0$ are weighting matrices entering the functional.

Likewise the above, to construct the boundary oracle for the set (14), consider $X \in \operatorname{int}\mathcal{D}$ and a direction $Y = Y^{\mathsf{T}}$ and find the critical values of λ.

By taking the Schur complement, we re-write the Riccati inequality in the equivalent form:

$$\begin{pmatrix} AX + XA^{\mathsf{T}} - G & XR^{1/2} \\ R^{1/2}X & -I \end{pmatrix} \preceq 0.$$

Then, with X changed for $X + \lambda Y$, this inequality writes

$$A_0 + \lambda A_1 \preceq 0,$$

where

$$A_0 = \begin{pmatrix} AX + XA^{\mathsf{T}} - G & XR^{1/2} \\ R^{1/2}X & -I \end{pmatrix} \prec 0$$

and

$$A_1 = \begin{pmatrix} AY + YA^{\mathsf{T}} & YR^{1/2} \\ R^{1/2}Y & \mathbf{0} \end{pmatrix}$$

so that we are within the setup of Lemma 2. Here, $\mathbf{0}$ is the zero matrix of the appropriate dimension.

Finally, similarly to the Lyapunov BO, the segment of sign-definiteness of the pencil $X + \lambda Y$ is to be computed via Lemma 2 and the feasible segment of variation of the parameter λ is then taken as the intersection of the two segments thus obtained.

Hence, the BO for the set specified by the quadratic matrix inequality reduces to the (nonrobust) BO in Sections III, IV at the expense of doubling the dimension.

A. Robust Setup

We now consider the robust version of the Riccati BO where the matrix A contains uncertainty of the same structured form (10). The respective robustly feasible domain

$$\mathcal{D}^{rob} = \{X \in \mathbb{R}^{n\times n} \colon XRX + A_\Delta X + XA_\Delta^\mathsf{T} - G \preceq 0,\ X \succ 0\} \tag{15}$$

is associated with solutions of a robust version of the LQR problem.

It is now clear from the above that the robust modification of the Riccati BO may be obtained by using both the Schur complement (to convert quadratic inequalities into linear ones) and Petersen's lemma (to get rid of the uncertainty).

Ommiting the straightforward but bulky and not insightful manipulations, we present this result in its final form.

Theorem 2: Let $X \in \mathrm{int}\mathcal{D}^{rob}$ (15)–(10). The boundary oracle for the set $\mathcal{D}^{rob}$ is given by the solutions of the following two SDPs in the scalar variables λ, ε:

$$\min(\max)\,\lambda \quad \text{s.t.}$$

$$\begin{pmatrix} A_1(\varepsilon) + \lambda A_2 & \lambda Z \\ \lambda Z^\mathsf{T} & -I \end{pmatrix} \preceq 0, \quad X + \lambda Y \succeq 0,$$

where it is denoted

$$A_1(\varepsilon) = \begin{pmatrix} XRX + AX + XA^\mathsf{T} + \varepsilon MM^\mathsf{T} & XN^\mathsf{T} \\ NX & -\varepsilon I \end{pmatrix};$$

$$A_2 = \begin{pmatrix} XRY + YRX + AY + YA^\mathsf{T} & YN^\mathsf{T} \\ NY & \mathbf{0} \end{pmatrix};$$

$$Z = \begin{pmatrix} YR^{1/2} \\ \mathbf{0} \end{pmatrix}.$$

□

Hence, the *robust Riccati* BO reduces to solving two semidefinite programs in two scalar variables, where the LMI constraints are of the size at most $4n \times 4n$.

VI. CONCLUSIONS AND FUTURE WORK

We presented the construction of boundary oracles for several LMI sets typically encountered in control problems. The core of these oracles is computation of the generalized eigenvalues of a pair of matrices or solving low-dimensional SDPs in two scalar variables. The computational effort is minor thus making these BOs efficient for use in various random walk methods. Preliminary computational experience testifies to reasonable performance of randomized algorithms based on the BOs of Section III; e.g., see [3].

The problems considered above are not exceptions; many other typical problems also admit BOs, e.g., sets described by linear algebraic inequalities, stable polynomials, positive polynomials, to name just a few. We also mention that the robust BOs proposed in this paper can as well be formulated for the affine matrix uncertainty setup.

Among the most challenging directions for future research is the design of efficient versions of the HR algorithm over matrix sets.

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