Infinite Structure for Infinite-Dimensional Systems: A Directional Approach

Petteri Laakkonen and Seppo Pohjolainen

***Abstract*— In this article structure at infinity of infinite-dimensional linear time invariant systems with finite-dimensional input and output spaces is discussed. It is shown that by appropriately restricting the paths approaching infinity and under certain majorization conditions a diagonal form that describes the behavior at infinity can be found. This diagonal form is a generalization of the Smith-McMillan form at infinity. It is then used to simplify certain solvability conditions of a regulation problem. Examples on time-delay and distributed parameter systems are given.**

I. INTRODUCTION

For finite-dimensional systems the structure at infinity is well-understood. As it describes the behavior of the system at high frequencies, it is natural that it has an important role in control theory. For example, solvability of the decoupling problem and exact model matching problem can be characterized by using structure at infinity [5], [8]. The structure at infinity and finite frequencies also determine the asymptotic behavior of the root loci [2].

For finite-dimensional systems the structure at infinity can be defined by using the Smith-McMillan form over the Euclidean domain of proper rational functions [17]. The proof of the fact that every rational function can be brought to the Smith-McMillan form at infinity leans heavily on the capability to majorize the elements of a rational matrix by one of the elements. The asymptotic behavior of rational matrices does not depend on the path infinity is approached, because the rate of convergence or divergence of functions meromorphic at infinity does not depend on the path.

Only little is known about the structure at infinity of infinite-dimensional systems. However, the high frequency behavior of transfer functions is still important. For example, a robustly regulating controller for signals generated by an infinite-dimensional exosystem with spectrum diverging to infinity along the imaginary axis uses the high frequency behavior of the transfer function at the exosystem's spectrum points [6]. Thus the asymptotic behavior of the transfer function along the imaginary axis is crucial in this problem.

Many difficulties when considering the structure at infinity of an infinite-dimensional system are due to the fact that transfer functions are not usually meromorphic. Their behavior might change drastically when infinity is approached along different paths [3]. If the transfer function of an infinite-dimensional system is analytic at infinity the behavior at hight frequencies can be described as in the finite-dimensional case [9].

It is obvious that some assumptions must be made in order to generalize the concept of transmission zeros and structure at infinity for infinite-dimensional systems. The above discussion indicates that there is no way to give a general description, that captures high-frequency behavior of a transfer function no matter how the infinity is approached. On the other hand, the knowledge of the high-frequency behavior along some restricted set of paths is sufficient in some problems. Thus a sensible approach is to restrict the way infinity can be approached. The advantage of restricting the consideration to some certain paths is that one can (possibly) majorize elements of a transfer function by certain elements of the transfer function. This in turn enables the algorithm that brings the transfer function to a diagonal form that defines asymptotic behavior. An approach along these lines was used by Malabre and Rabah for time-delay systems [10], [11], [12]. By considering two separate sets of paths they gave two structures at infinity which they called strong and weak structures.

The main contribution of this paper is to give a definition of structure at infinity that is suitable for a class of distributed parameter systems. The definition given here extends the definitions given by Malabre and Rabah [9], [10], [11], [12] and is suitable for a more general class of systems. To show the applicability of the definition a robust regulation problem is considered and some solvability conditions are given in terms of the structure at infinity. The results clearly shows a connection between infinite structure and existence of a robust controller. Examples on time-delay systems and distributed parameter systems are presented to illustrate the theory.

The article has the following structure: In Section II the basic notations and definitions are given. In Section III a diagonal form, that describes the asymptotic behavior of a matrix with complex functions as its elements, is presented and it is used to give a definition of structure at infinity. In Section IV the directed structure is used to simplify solvability conditions of a robust regulation problem with an infinite-dimensional system and an infinite-dimensional exosystem [6]. In Section V we briefly sum up the contributions of this paper and present some directions for future work.

II. PRELIMINARIES

The complex plane is denoted by $\mathbb{C}$. Euclidean domain of all proper rational functions is denoted by $\mathbb{C}_{pr}(s)$. The degree

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function in $\mathbb{C}_{pr}(s)$ is defined as $\delta(f) = \deg(d) - \deg(n)$, where $f = \frac{n}{d} \in \mathbb{C}_{pr}(s)$ and $\deg(n)$ and $\deg(d)$ are the degrees of the polynomials n and d respectively.

If S is a set, then the set of all $n \times m$ matrices with elements in S is denoted by $S^{n\times m}$. A rational matrix $G(s)$ is called bi-proper, if $G(s) \in \mathbb{C}_{pr}^{n\times n}(s)$ and $G^{-1}(s) \in \mathbb{C}_{pr}^{n\times n}(s)$. Notation $\|\cdot\|$ is used for vector and matrix norms.

Let M be an $n\times m$-matrix. An $i\times i$-minor of M is denoted by $|M|^i_{\mathbf{r},\mathbf{c}}$, where the multi-indices $\mathbf{r} = (r_1, r_2, \ldots, r_i)$ and $\mathbf{c} = (c_1, c_2, \ldots, c_i)$, where $1 \le r_j < r_{j+1} \le n$ and $1 \le c_j < c_{j+1} \le m$, define the rows and columns selected respectively.

The set of all continuous paths diverging to infinity in the complex plane is denoted by

$$\mathbf{P} = \{p : [0,\infty) \to \mathbb{C} \,|\, |p(\alpha)| \to \infty \text{ as } \alpha \to \infty, \\ p \text{ continuous}\}.$$

Let $P \subseteq \mathbf{P}$. The set of all complex functions $f : \mathbb{C} \supseteq \mathcal{D}(f) \to \mathbb{C} \cup \{\infty\}$ such that $p(\alpha) \in \mathcal{D}(f)$ for all $p \in P$ and $\alpha \in [0,\infty)$ is denoted by $\mathbf{F}_P$.

Let $a \in \mathbb{C} \cup \{\infty\}$. For the next definition it is convenient to define $\frac{a}{0} = \infty$ for $a \neq 0$, $\frac{0}{0} = 1$, $\frac{a}{\infty} = 0$ for $a \neq \infty$ and $\frac{\infty}{\infty} = 1$.

Definition 1: Let $P \subseteq \mathbf{P}$ and $f, g \in \mathbf{F}_P$. If

$$\forall p \in P : \exists \rho \ge 0 : \sup_{\alpha\in[\rho,\infty)} \left|\frac{g(p(\alpha))}{f(p(\alpha))}\right| < \infty$$

it is said that f majorizes g along the paths in P or simply f majorizes g along P and is denoted by $f \geq_P g$.

If $f \geq_P g$ and $g \geq_P f$ notation $f =_P g$ is used. If $f \geq_P g$, but $f \neq_P g$ then notation $f >_P g$ is used. If $P = \{p\}$, then notations $\geq_p$ and $=_p$ can be used instead of $\geq_P$ and $=_P$. Some direct consequences of Definition 1 are listed in the following lemma.

Lemma 2: Let $P \subset \mathbf{P}$ and f_1, f_2, f_3 and f_4 be complex functions. The relation $\geq_P$ has the following properties:

1) *If $f_1 \geq_P f_2$ and $f_1 \geq_P f_3$, then $f_1 \geq_P f_2 + f_3$.*
2) *If $f_1 \geq_P f_2$ and $f_3 \geq_P f_4$, then $f_1 f_3 \geq_P f_2 f_4$.*

Remark 3: Relation $=_P$ is an equivalence relation in the set of all complex functions. Denotation $[f]_P$ is used for the equivalence class of a function f. Furthermore, it can be shown that $\geq_P$ defines a partial order in the set of equivalence classes and $[0]_P$ is the smallest element respect to this order. ▲

Let X and Y be linear spaces. The set of all bounded operators from X to Y is denoted by $\mathcal{L}(X,Y)$ and $\mathcal{L}(X,X) = \mathcal{L}(X)$. Denotation $\Sigma(A,B,C,D)$ is an abbreviation for a linear system of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X, \tag{1a}$$

$$y(t) = Cx(t) + Du(t), \qquad t \geq 0, \tag{1b}$$

where $A : \mathcal{D}(A) \to X$ is a generator of a C_0-semigroup in a Hilbert space X and $B \in \mathcal{L}(U,X)$, $C \in \mathcal{L}(X,Y)$ and $D \in \mathcal{L}(U,Y)$. Input and output spaces are $U = \mathbb{C}^m$ and $Y = \mathbb{C}^n$ respectively. Transfer function of a system $\Sigma(A,B,C,D)$ is defined to be

$$G(s) = C(sI - A)^{-1}B + D,\ s \in \rho(A), \tag{2}$$

where $\rho(A)$ is the resolvent set of the operator A.

As the results of Section IV are related to those presented in [6] the definition of transfer function adopted here is the most convenient. Other definitions for transfer functions are also possible [18]. It should be noted that the theory presented in Section III-A is valid for all matrices with complex functions as elements and thus it is not important what definition is used.

Time-delay systems of the form presented below are considered as examples and notation $\Sigma_D(A_0, A_1, B, C)$ is used for them.

$$\begin{aligned} &\dot{x}(t) = A_0x(t) + A_1x(t-\gamma) + Bu(t), \quad t \geq 0,\\ &x(0) = x_0 \in \mathbb{C}^n,\\ &x(\theta) = f(\theta), \qquad\qquad -\gamma \leq \theta < 0,\\ &y(t) = Cx(t), \end{aligned}$$

where $\gamma > 0$, $x(t) \in \mathbb{C}^n$, $A_0, A_1 \in \mathcal{L}(\mathbb{C}^n)$, $f \in \mathrm{L}_2([-\gamma,0],\mathbb{C}^n)$, $B \in \mathcal{L}(\mathbb{C}^m,\mathbb{C}^n)$, $C \in \mathcal{L}\left(\mathbb{C}^n,\mathbb{C}^k\right)$ and $u \in \mathrm{L}_2([0,\tau],\mathbb{C}^m)$, for all $\tau > 0$.

Transfer function of a time-delay system $\Sigma_D(A_0, A_1, B, C)$ is given as [4, Lemma 4.3.9]

$$G(s) = C\left(sI - A_0 - A_1e^{-s\gamma}\right)^{-1}B. \tag{4}$$

Definition 4: Let $G : \mathcal{D}(A) \subseteq \mathbb{C} \to \mathbb{C}^{n\times m}$ be a transfer function of a linear system $\Sigma(A,B,C,D)$. If $\operatorname{rank} G(s) = k$ almost everywhere in its domain, then it is said that the normal rank of G is k and this is denoted by $\operatorname{nrank}(G(s)) = k$. If $\operatorname{rank} G(s_0) < \operatorname{nrank}(G(s))$, then it is said that the system $\Sigma(A,B,C,D)$ has a transmission zero at $s_0 \in \mathbb{C}$.

III. DIRECTED STRUCTURE AT INFINITY

A. The structure for matrices with complex function elements

Let $G(s)$ be a rational matrix. It is shown in [17], that $G(s)$ can be written as

$$G(s) = V(s)\begin{bmatrix}\Lambda(s) & 0\\ 0 & 0\end{bmatrix}U(s),$$

where $V(s)$ and $U(s)$ are bi-proper matrices and $\Lambda(s)$, the Smith-McMillan form at infinity, is a diagonal matrix with diagonal elements of the form s^{q_i}, $q_i \in \mathbb{Z}$. Bi-proper matrices are uniformly bounded with uniformly bounded inverses near infinity. For matrices with complex function elements the above form does not exist in general. One reason for this is that the behavior at infinity of a complex function cannot be described in a path independent way. Thus, we consider only some restricted set of paths and make the following definition.

Definition 5: Let $P \subseteq \mathbf{P}$ and $G(s) \in \mathbf{F}_P^{n\times m}$. The matrix valued function $G(s)$ is said to have a simple structure along the paths in P, if it can be written as

$$G(s) = V(s)\begin{bmatrix}\Lambda_P(s) & 0\\ 0 & 0\end{bmatrix}U(s), \tag{5}$$

where $\Lambda_P(s) = \operatorname{diag}(q_1(s), \ldots, q_r(s))$ is a diagonal matrix with non-zero elements satisfying

$$q_1 \geq_P q_2 \geq_P \cdots \geq_P q_r,$$

zero blocks may be non-existent and for all $p \in P$ there exists $M_p > 0$ such that matrices $V(p(\alpha))$ and $U(p(\alpha))$ satisfy the following conditions:

1) *$U(p(\alpha))$ and $V(p(\alpha))$ are invertible for all $\alpha \geq M_p$*
2) *$U(p(\alpha))$, $U^{-1}(p(\alpha))$, $V(p(\alpha))$ and $V^{-1}(p(\alpha))$ are uniformly bounded on $[M_p, \infty)$.*

In the above definition $q_i(s)$ are general complex functions that define the behavior of $G(s)$ as $s \to \infty$ along the paths in P. We now define the structure at infinity for matrices with simple structure along a set of paths.

Definition 6: Let $P \subseteq \mathbf{P}$ and $G(s) \in \mathbf{F}_P^{n\times m}$. If $G(s)$ has simple structure at infinity along P and $\Lambda_P(s)$ is as in (5), *then we say that $\{[q_1]_P, \ldots, [q_r]_P\}$ is the structure at infinity of the matrix $G(s)$ along P. The functions q_i, $i = 1, \ldots, r$, are called structural functions.*

By definition the structure at infinity along some set of paths is a set of equivalence classes, where the same equivalence class can occur multiple times. If there is no risk of confusion we use only notation f instead of notation $[f]_P$. For the defined structure at infinity to make sense it must be unique. Next the uniqueness is shown.

Theorem 7: Let $P \subseteq \mathbf{P}$. For all matrices with complex function elements in $\mathbf{F}_P$ and simple structure along P, the structure at infinity along P is unique.

Proof: Assume that the diagonal matrices $\Lambda_1(s) = \operatorname{diag}(\lambda_1(s), \ldots, \lambda_r(s))$ and $\Lambda_2 = \operatorname{diag}(\sigma_1(s), \ldots, \sigma_r(s))$ are as in (5), so that for $i = 1, 2$

$$G(s) = V_i(s)L_i(s)U_i(s) = V_i(s)\begin{bmatrix}\Lambda_i(s) & 0\\ 0 & 0\end{bmatrix}U_i(s),$$

where $U_i(s)$ and $V_i(s)$ satisfy the condition 1) and 2) in Definition 5. Write $V(s) = V_1^{-1}(s)V_2(s)$ and $U(s) = U_2(s)U_1^{-1}(s)$. Matrices $V(s)$ and $U(s)$ are uniformly bounded near infinity along the paths in P and $L_1(s) = V(s)L_2(s)U(s)$.

By using the Binet-Cauchy formula [13, Theorem 1.3] and noting that $L_2(s)$ have non-zero elements only on the diagonal it can be seen, that

$$\begin{aligned}|L_1(s)|^i_{\mathbf{r},\mathbf{r}} &= |V(s)L_2(s)U(s)|^i_{\mathbf{r},\mathbf{r}}\\ &= \sum_{\mathbf{h},\mathbf{v}} |V(s)|^i_{\mathbf{r},\mathbf{h}}\,|L_2(s)|^i_{\mathbf{h},\mathbf{v}}\,|U(s)|^i_{\mathbf{v},\mathbf{r}}\\ &= \sum_{\mathbf{h}} |V(s)|^i_{\mathbf{r},\mathbf{h}}\,|L_2(s)|^i_{\mathbf{h},\mathbf{h}}\,|U(s)|^i_{\mathbf{h},\mathbf{r}}\,. \qquad (6)\end{aligned}$$

Since $U(p(\alpha))$ and $V(p(\alpha))$ with $p \in P$ are uniformly bounded near infinity, it follows by using Lemma 2 that $1 \geq_P |V(s)|^i_{\mathbf{r},\mathbf{h}}\,|U(s)|^i_{\mathbf{h},\mathbf{r}}$, for all $\mathbf{h}$ and $\mathbf{r}$. Set $r = (1, 2, \ldots, k)$, where $k \leq r$. By using Lemma 2 and the assumption, that $\sigma_i \geq_P \sigma_{i+1}$ for all $i = 1, \ldots, r-1$, to (6) we get

$$\prod_{j=1}^{k} \sigma_j(s) \geq_P \prod_{j=1}^{k} \lambda_j(s). \qquad (7)$$

Similarly we can show that

$$\prod_{j=1}^{k} \lambda_j(s) \geq_P \prod_{j=1}^{k} \sigma_j(s). \qquad (8)$$

From (7) and (8) follows, that $\lambda_j =_P \sigma_j$ for $j = 1, \ldots, r$, which completes the proof. ■

Two questions arises naturally. Firstly, when is a matrix of simple form along a set of paths? Secondly, if a matrix is of simple form, how can the diagonal form (5) and the structure at infinity be found? Next we consider the second question and at the same time the first one is answered partially.

The definition of the structure at infinity along some set of paths is a simple generalization of the structure of rational functions. For rational functions the Smith-McMillan form always exists and it can be found by an algorithm based on the fact, that among the elements of a given matrix there is an element of the highest degree. It is easily verified that is f and g are rational functions and $p \in \mathbf{P}$ is chosen arbitrarily, then $\delta(f) \geq \delta(g)$ if and only if $g \geq_p f$. This shows, that for rational functions it is possible to find a description of asymptotic behavior not depending the way infinity is approached, which is not possible for general complex functions. By restricting the set of paths we may be able to find a majorizing function from a given set of functions. This is illustrated by the next example.

Example 8: Consider the functions $f_1(s) = e^{-s}$ and $f_2(s) \equiv 1$ and the paths $p_1(\alpha) = \alpha$, $p_2(\alpha) = -\alpha$ and $p_3(\alpha) = \alpha i$. Let the set $P_1 \subset \mathbf{P}$ be $P_1 = \{p_1, p_2\}$.

Now $f_1 \not\geq_{P_1} f_2$ and $f_2 \not\geq_{P_1} f_1$, but $f_2 >_{p_1} f_1$ and $f_1 >_{p_2} f_2$. This shows, that by restricting the set of paths we are able to find a majorizing term. Furthermore, $f_3 := f_1 + f_2 >_{P_1} f_i$ for $i = 1, 2$, which shows, that unlike the sum of rational functions, the sum of general complex functions can actually strictly majorize all the summed functions.

The definition of majorization takes accumulation of poles and zeros at infinity into account, thus enabling us to compare the behavior at infinity. To illustrate this consider the functions $f_i(s)$, $i = 1, 2, 3$, along the path p_3. By definition $f_1 =_{p_3} f_2 >_{p_3} f_3$, so the only real difference between the functions f_1, f_2 and f_3 from the majorization point of view when asymptotic behavior is considered along p_3 is that f_3 has an accumulation point of zeros at infinity. ♦

Because majorization is used in the construction of the Smith-McMillan form in the finite-dimensional case and the majorization can be possibly achieved by using a restricted set of paths we can use an algorithm similar to the one presented in [17] to find a diagonal form describing the asymptotic behavior of a matrix with complex function elements. Example 8 hints, that a majorizing term does not necessarily exist, so the existence must be assumed.

Next an algorithm that constructs a diagonal form describing the behavior at infinity of a matrix with complex function elements is presented. An assumption that there is at least one term that majorizes all the other terms in the non-diagonalized part is made in every step. The assumption is clarified when describing the algorithm. The algorithm uses elementary row and column operations. Notation $G'(s) = (q_{ij}(s))$ denotes the matrix during the algorithm received by using necessary row and column operations on $G(s)$. Note that $G'(s)$ changes after each operation. The step 0 is to set $G'(s) = G(s)$ and the general step is as follows.

Step p**:** At the beginning of this step $G'(s)$ is in the following form

$$G'(s) = \begin{bmatrix} D(s) & 0 \\ 0 & G_p(s) \end{bmatrix}, \tag{9}$$

where $D = \operatorname{diag}(q_{11}(s), \ldots, q_{p-1,p-1}(s))$.

Assumption 9: There exists an element $q_{i'j'}$, with $i', j' \geq p$, such that $q_{i',j'} \geq_P q_{ij}$ for all $p \leq i \leq n$ and $p \leq j \leq m$.

To begin, bring the majorizing element by changing rows and columns to position (p,p). Now q_{pp} majorizes all the elements in the p. row and column. Subtract the p. row multiplied by $\frac{q_{ip}(s)}{q_{pp}(s)}$ from i. one. After this operation $q_{ip}(s) = 0$. In this way it is possible to zero out all the elements in the p. row except the p. one. Similarly, all the elements in the p. column except the p. one can be zeroed out.

End of the algorithm: If $G_p(s)$ at the beginning of the p. step is a zero matrix or $p > \min\{m, k\}$, then the algorithm ends.

For every elementary operation made during the algorithm there exists a matrix representation. Matrices, corresponding to an interchanging of rows or columns, are constant matrices. Assumption 9 guarantees that every matrix corresponding to an elementary row or column operation that adds a row or a column to another one is uniformly bounded at infinity and has an uniformly bounded inverse near infinity along the paths in P. It is therefore clear that there exist matrices $U(s)$ and $V(s)$ such that they are bounded and boundedly invertible along the paths in P and

$$G(s) = V(s) \begin{bmatrix} \operatorname{diag}(q_{ii}(s))_{i=1}^{r} & 0 \\ 0 & 0 \end{bmatrix} U(s). \tag{10}$$

By using properties 1) and 2) of Lemma 2 it is easy to show, that $q_{ii} \geq q_{i+1,i+1}$ in (10). We now have the following theorem.

Theorem 10: Let $P \subseteq \mathbf{P}$ and $G(s) \in \mathbf{F}_P^{n\times m}$. If Assumptions 9 hold and the above algorithm is used, then the structural functions of $G(s)$ at infinity along P are $q_{ii}(s)$, $i = 1, \ldots, r$, in (10).

Remark 11: If Assumptions 9 hold, then it is possible to define the structure at infinity by using minors as in the case of rational matrices [7], [16]. This method is a direct generalization, where one uses majorization instead of the degree function δ. ▲

Example 12: Consider the matrix

$$G(s) = \begin{bmatrix} \frac{e^{-s}-1}{g(s)} & \frac{e^{-s}+1}{g(s)} & \frac{(e^{-s}+s)(e^{-s}-1)}{(s+1)g(s)} \\ \frac{s+1}{g(s)} & \frac{-(e^{-s}+1)}{(s+1)g(s)} & \frac{s(e^{-s}-1)}{(s+1)g(s)} \\ 0 & \frac{1}{s+1} & 0 \end{bmatrix},$$

where $g(s) = (s+1)^2 + e^{-s} - 1$.

As we are interested in the behavior along the imaginary axis, we choose

$$P = \{p \in \mathbf{P} : p(\alpha) = \rho(\alpha)i, \text{ where } \rho(\alpha) \text{ is real continuous and } |\rho(\alpha)| \to \infty \text{ as } \alpha \to \infty\}. \tag{11}$$

It is easily verified that $\frac{1}{s+1}$ majorizes all the other elements in $G(s)$ along the paths in P. After the operations suggested by the algorithm presented in the proof of Theorem 10 the transfer function is brought to the form

$$G'(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{s+1}{g(s)} & \frac{s(e^{-s}-1)}{(s+1)g(s)} \\ 0 & \frac{e^{-s}-1}{g(s)} & \frac{(e^{-s}+s)(e^{-s}-1)}{(s+1)g(s)} \end{bmatrix}.$$

The element $\frac{s+1}{g(s)}$ majorizes all the elements of the submatrix received by deleting the first row and the first column. Thus, the diagonal form

$$\Lambda_P'(s) = \operatorname{diag}\left(\frac{1}{s+1}, \frac{s+1}{g(s)}, \frac{e^{-s}-1}{(s+1)^2}\right)$$

is obtained. Multiplying the diagonal elements by appropriate functions satisfying $f =_P 1$ the diagonal matrix can be simplified to be

$$\Lambda_P(s) = \operatorname{diag}\left(s^{-1}, s^{-1}, s^{-2}(e^{-s}-1)\right).$$

This reveals that at infinity a part of $G(s)$ has rational structure, but another part has structure that is a combination of the periodically behaving function $e^{-s}-1$ and the rational function s^{-2}. Note that the periodic behavior of the function $e^{-s}-1$ is essential when considering the behavior at infinity, because the zeros of $e^{-s}-1$ lying on the imaginary axis have an accumulation point at infinity. ♦

The next example shows, that it is possible to find a diagonal form as in (5) even if Assumption 9 fails at some step. This is possible, because the sum of two functions can strictly majorize the elements of the sum, as was seen in Example 8.

Example 13: Let $p(\alpha) = \alpha i$, $\alpha \in [0, \infty)$. Consider the matrix

$$H(s) = \begin{bmatrix} e^{-s} - 1 & e^{-s} + 1 \\ \frac{1}{s} & e^{-s} + 1 \end{bmatrix}.$$

There is no element satisfying Assumption 9. We subtract the second column from the first one to have

$$H'(s) = \begin{bmatrix} -2 & e^{-s} + 1 \\ \frac{1}{s} - (e^{-s}+1) & e^{-s} + 1 \end{bmatrix}.$$

Now -2 majorizes all the other functions and by using the algorithm we can show, that the structure at infinity along p is $\left\{1, (e^{-s}+1)\left((1-e^{-s}) + \frac{1}{s}\right)\right\}$. ♦

B. The structure for linear systems

So far matrices with complex functions as elements have been considered. Next, transfer functions of linear systems of the form (1) are discussed and the structure at infinity along $P \subset \mathbf{P}$ for a linear system is defined.

In finite-dimensional case one can always define the transfer function in many equivalent ways and it has a unique closed form expression as a rational matrix. It has been shown in [18] that in the infinite-dimensional case the different definitions can lead to different transfer functions. One crucial difference between different definitions is that they may lead to different domains of transfer functions. As the paths should be contained in the domain, one may be able to define structure along some set $P \subset \mathbf{P}$ when using one definition while the structure along P is not defined when using another one. Here we have defined $G(s)$ to be (2) so there is no confusion as long as $P \subset \mathbf{P}$ is selected so that all the paths in P lies in $\rho(A)$.

In general the definition of the transfer function adopted here is only an abstract operator valued function that has no closed form expression. From the theoretical point of view the absence of the closed form is not a restriction. One can consider the transfer function as an $n \times m$-matrix with complex function elements $g_{ij}(s)$, where $g_{ij}(s) = e_i^T G(s) e_j$, where e_i and e_j are the i. and j. natural basis vectors of $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. This leads to the following definition.

Definition 14: Let the transfer function of a linear system $\Sigma(A, B, C, D)$ be $G(s)$ and $P \subset \mathbf{P}$ be such a set of paths that if $p \in P$, then $p(\alpha) \in \rho(A)$ for all $\alpha \in [0, \infty)$. If $G(s)$ has simple structure at infinity along P, then the directed structure of $\Sigma(A, B, C, D)$ at infinity along P is defined to be the structure of $G(s)$ at infinity along P.

Remark 15: If $P = \mathbf{P}$ and the system $\Sigma(A, B, C, D)$ has only bounded operators, then Definition 14 gives the same structure as the definition given in [9].

Let

$$P_s = \{p \in \mathbf{P} \,|\, \mathrm{Re}(p(\alpha)) \to \infty \text{ as } \alpha \to \infty\}$$

and

$$P_w = \{p \in \mathbf{P} \,|\, \mathbb{R} \ni p(\alpha) \to \infty \text{ as } \alpha \to \infty\}.$$

Consider a delay system $\Sigma_D(A_0, A_1, B, C)$. If $P = P_s$ ($= P_w$), then the structure given by Definition 14 is essentially the same as the strong (weak) structure defined in [11]. ▲

Remark 16: The definition of structure given here is especially suitable for those systems that have a transfer function with a closed form expression, that is a combination of fractional and exponential terms, because it is relatively easy to verify the majorization conditions for them along some set of paths. Obviously delay systems, but also several other distributed parameter systems [1], are of this type. ▲

Example 17: Let P be as in (11) and consider the system $\Sigma_D(A_0, A_1, B, C)$, where

$$A_0 = \begin{bmatrix} -1 & -1 & 1 & -1 \\ -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The transfer function of $\Sigma_D(A_0, A_1, B, C)$ is $G(s)$ presented in Example 12. The structure at infinity of $\Sigma_D(A_0, A_1, B, C)$ along P is $\{s^{-1}, s^{-1}, s^{-2}(e^{-s} - 1)\}$. ♦

Example 18: A metal bar is heated with two heaters and temperature is measured along two intervals as shown in Figure 1. The resulting system is written as in (1) where $D = 0$ and the linear operators A, B and C are defined as follows.

$$Ax(z) = \frac{d^2x}{dz^2}(z) - x(z), \text{with}$$

$$\mathcal{D}(A) = \left\{ h \in \mathbf{L}_2(0,1) \,\middle|\, h, \frac{dh}{dz} \text{ are absolutely continuous}, \frac{d^2h}{dz^2} \in \mathbf{L}_2(0,1) \text{ and } \frac{dh}{dz}(0) = \frac{dh}{dz}(1) = 0 \right\},$$

$$Bu = 4\left[\mathbf{1}_{[\frac{2}{7},\frac{3}{7}]}(z), \mathbf{1}_{[\frac{6}{7},1]}(z)\right] u,$$

$$y = Cx(z) = \int_0^1 \begin{bmatrix} \mathbf{1}_{[0,\frac{1}{7}]}(z) \\ \mathbf{1}_{[\frac{4}{7},\frac{5}{7}]}(z) \end{bmatrix} x(z) dz$$

where the state space, the input space and the output space are $X = \mathbf{L}_2(0,1)$ and $U = Y = \mathbb{C}^2$ respectively and the function $\mathbf{1}_{[a,b]}(z)$ is defined as

$$\mathbf{1}_{[a,b]}(z) = \begin{cases} 1, & z \in [a,b] \\ 0, & z \notin [a,b] \end{cases}.$$

Note that the term $-x(z)$ in A represents the heat transfer to the environment. The operator A is a generator of a C_0-semigroup and B and C are bounded operators. Spectrum of A consists of eigenvalues $\lambda_n = -1 - (n\pi)^2$, $n = 0, 1, \ldots$. The system is exponentially stable.

The transfer function is

$$G(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix},$$

where

$$g_{11}(s) = \frac{7\sinh\left(\frac{1}{7}\sqrt{s+1}\right)}{(s+1)^{\frac{3}{2}}\sinh\left(\sqrt{s+1}\right)} \cdot \left(\sinh\left(\frac{4}{7}\sqrt{s+1}\right) - \sinh\left(\frac{5}{7}\sqrt{s+1}\right)\right),$$

$$g_{12}(s) = \frac{7\sinh^2\left(\frac{1}{7}\sqrt{s+1}\right)}{(s+1)^{\frac{3}{2}}\sinh\left(\sqrt{s+1}\right)},$$

$$g_{21}(s) = \frac{-7\left(\sinh\left(\frac{4}{7}\sqrt{s+1}\right) - \sinh\left(\frac{5}{7}\sqrt{s+1}\right)\right)^2}{(s+1)^{\frac{3}{2}}\sinh\left(\sqrt{s+1}\right)} + \frac{7}{(s+1)^{\frac{3}{2}}}\left(\sinh\left(\frac{1}{7}\sqrt{s+1}\right) + \sinh\left(\frac{3}{7}\sqrt{s+1}\right) - 2\sinh\left(\frac{2}{7}\sqrt{s+1}\right)\right),$$

$$g_{22}(s) = -g_{11}(s).$$

Next the behavior of the transfer function is considered in a closed sector Ω containing the imaginary axis and the positive real axis, see Figure 2.

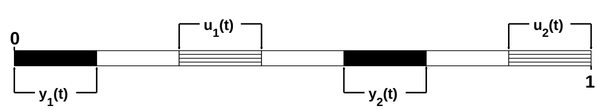

Fig. 1. A heated metal bar.

If $s = a + bi$, $s \in \Omega$ and $a < 0$, then for some $M > 0$ $M|a| \leq |b|$ and thus

$$\mathrm{Re}\left(\sqrt{s+1}\right) = \sqrt{\sqrt{(a+1)^2+b^2}+a+1} \geq \sqrt{-(\sqrt{M^2+1}-1)a} \to \infty,$$

as $a \to -\infty$. It is obvious that if $a \to +\infty$ or a is bounded and $b \to \infty$, also then $\mathrm{Re}\left(\sqrt{s+1}\right) \to \infty$. So whenever $s \in \Omega$ and $|s| \to \infty$, then $\mathrm{Re}\left(\sqrt{s+1}\right) \to \infty$.

Now it is easily seen, that $\left|\sinh\left(d\sqrt{s+1}\right)\right|$, where $d > 0$, behaves approximately as $\left|e^{d\sqrt{s+1}}\right|$ at infinity for $s \in \Omega$.

Let $P \subset \mathbf{P}$ be the set of all paths $p : [0,\infty) \to \Omega$ approaching infinity. Using the above observations it can be show that

$$g_{11}(s) =_P g_{22}(s) =_P g_{21}(s) =_P s^{-\frac{3}{2}}e^{-\frac{1}{7}\sqrt{s}} \geq_P s^{-\frac{3}{2}}e^{-\frac{5}{7}\sqrt{s}} =_P g_{12}(s).$$

and that the structure at infinity along P is $\{s^{-\frac{3}{2}}e^{-\frac{1}{7}\sqrt{s}}, s^{-\frac{3}{2}}e^{-\frac{1}{7}\sqrt{s}}\}$.

The exponential decay rate $e^{-d\sqrt{s}}$, $d > 0$, of g_{ij} is a consequence of the gap of d units between the j. heater and the i. interval of measurement. If there would be a heater connected to, but not overlapping, some interval of measurement, then $s^{-\frac{3}{2}}$ would be a structural function. If a heater overlaps an interval of measurement s^{-1} would be a structural function. ♦

IV. AN APPLICATION

In this section the (f_k, ω_k)-regulation problem or shortly the (f_k, ω_k)-RP presented by Immonen and Pohjolainen in [6] is considered and the solvability conditions they gave are then given here in terms of the directed structure. The results of this section show that the asymptotic behavior of transfer function plays a significant role in the robust regulation.

A. The (f_k, ω_k)-regulation problem

For easy reference we present the (f_k, ω_k)-RP here shortly and extend it to the MIMO-case. Consider the system (1)

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + d(t), \qquad & x(0) &= x_0 \in X \\ y(t) &= Cx(t) + Du(t), & t &\geq 0, \end{aligned}$$

with a disturbance term d. The pair (A, B) is assumed to be exponentially stabilizable. Here we choose for simplicity $Y = U = \mathbb{C}^n$. The signal generator is of the form

$$\begin{aligned} \dot{w}(t) &= S_F w(t), \qquad & w(0) &= w_0 \in W_F, \\ y_{ref}(t) &= Qw(t), & t &\geq 0, \\ d(t) &= Ew(t), & E &\in \mathcal{L}(W_F, X). \end{aligned}$$

If $n = 1$, then the operators S_F and Q and the Hilbert space W_F are as in [6]. If $n > 1$, then S_F and Q are block diagonal operators and W_F a direct sum defined by using the operators and the Hilbert space of the case $n = 1$ respectively.

The signal generator generates exactly the reference signals in the class

$$\{u : \mathbb{R} \to \mathbb{C}^n \ : \ u(t) = \sum_{k\in I} a_k e^{i\omega_k t} \text{ for all } t \in \mathbb{R}, \sum_{k\in I} |f_k|^2 \|a_k\|^2 < \infty \ \ and \ \ (a_k)_{k\in I} \subset \mathbb{C}^n\},$$

where $I \subseteq \mathbb{Z}$, sequences $(\omega_k)_{k\in I}$ and $(f_k)_{k\in I}$ are fixed sequences of real numbers that satisfy $\omega_k = \frac{2\pi k}{p}$, with $p > 0$ fixed, and $f_k \geq 1$ for all $k \in I$ and $(f_k^{-1})_{k\in I} \in \ell^2$.

The (f_k, ω_k)-RP is to find a feedback control law

$$u(t) = Kx(t) + Lw(t) \tag{12}$$

such that $K \in \mathcal{L}(X, U)$, $L \in \mathcal{L}(W_F, U)$, $A + BK$ is the generator of an exponentially stable C_0-semigroup $T_{A+BK}(t)$ on X and the tracking error

$$\begin{aligned} e(t) &= y(t) - y_{ref}(t) \\ &= (C + DK)x(t) + (DL - Q)w(t) \to 0, \end{aligned}$$

as $t \to \infty$ for all initial conditions $x_0 \in X$ and $w_0 \in W_F$.

B. Solvability conditions in terms of the structure at infinity

In this section new conditions for the solvability of $(f_k, \omega_k) - RP$ are given in terms of the structure at infinity. It is assumed that the transfer function of the stabilized plant $G_c(s) = (C + DK)R(s_0, A + BK)B + D$ is invertible, i.e. $\mathrm{nrank}\,(G_c(s)) = n$. Set $h_d(k,l) = (C + DK)R(i\omega_k, A + BK)E\varphi_{kl}$, where e_l are the natural basis vectors of $\mathbb{C}^n$, φ_{kl} are the orthogonal basis vectors of W_F, $k \in I$ and $1 \leq l \leq n$. Define

$$L = \sum_{\substack{k\in I \\ 1\leq l\leq n}} G_c^{-1}(i\omega_k)(e_l - h_d(k,l))\langle\cdot, \varphi_{kl}\rangle. \tag{13}$$

First we need the following lemma given in [6] in SISO-case. The proof of the result in MIMO-case is an easy (but lengthy) modification of the proof by Immonen and Pohjolainen.

Lemma 19: Let L be as in (13) *and K to exponentially stabilize the pair (A, B). Assume that $i\omega_k$ is not a finite transmission zero of the closed loop transfer function for any $k \in I$. Then the control law $u(t) = Kx(t) + Lw(t)$ solves the (f_k, ω_k)-RP if and only if*

$$\sum_{\substack{k\in I \\ 1\leq l\leq n}} \left\|(G_c^{-1}(i\omega_k)(e_l - H_d(k,l))\right\|^2 |f_k|^{-2} < \infty.$$

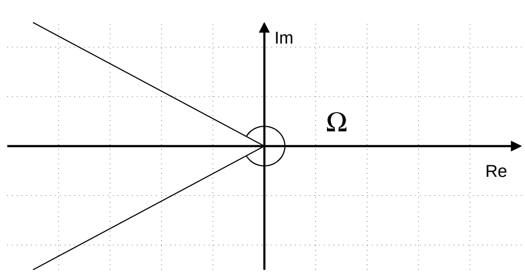

Fig. 2. A closed sector not containing the negative real axis.

In SISO-case it is obvious that the faster $G_c^{-1}(i\omega_k)$ approaches zero as $k \to \infty$ the faster the terms f_k must diverge towards infinity. Thus, the behavior of the closed loop transfer function $G_c(s)$ imposes some obvious restrictions for the class of reference signals the presented controller can regulate. Next this idea in terms of the directed structure at infinity for MIMO-systems is considered. Because all the values $i\omega_k$ have no real part and $|i\omega_k| \to \infty$, for the rest of this section P is chosen to be as in (11).

Theorem 20: Let the assumptions of Lemma 19 to be satisfied. Let K be an exponentially stabilizing operator for the pair (A, B) and L be as in (13). *Furthermore assume that $G_c(s)$ has the directed structure $\{q_1, \dots, q_n\}$ at infinity along P.*

(i) *If $g : \mathbb{C} \to \mathbb{C}$ is such that $q_n \geq_P g$ for all $j = 1, \dots, n$ and*

$$\sum_{\substack{k\in I\\ 1\leq l\leq n}} |g(i\omega_k)|^{-2} \|e_l - H_d(k,l)\|^2 |f_k|^{-2} < \infty, \quad (14)$$

then the control law $u(t) = Kx(t) + Lw(t)$ solves the (f_k, ω_k)-RP.

(ii) *If $h : \mathbb{C} \to \mathbb{C}$ is such that $h \geq_P q_1$ for all $j = 1, \dots, n$ and $u(t) = Kx(t)+Lw(t)$ solves the (f_k, ω_k)-RP, then*

$$\sum_{\substack{k\in I\\ 1\leq l\leq n}} |h(i\omega_k)|^{-2} \|e_l - H_d(k,l)\|^2 |f_k|^{-2} < \infty. \quad (15)$$

Proof: First we show that (i) holds. Let $p \in P$. It follows from the sub-multiplicativity of the matrix norm and the uniform boundedness of inverses that

$$\begin{aligned} &\|g(p(\alpha))G_c^{-1}(p(\alpha))\| \\ &\quad\leq \|U^{-1}(p(\alpha))\| \left\| \operatorname{diag}\left(\frac{g(p(\alpha))}{q_j(p(\alpha))}\right)_{j=1}^n \right\| \|V^{-1}(p(\alpha))\| \\ &\quad< M_0 \end{aligned}$$

for some $M_0 > 0$ and for all α sufficiently large. Thus there exists some $M > 0$ such that for all $k \in I$

$$\left\|g(i\omega_k)G_c^{-1}(i\omega_k)\right\| < M.$$

Furthermore

$$\begin{aligned} &\left\|G_c^{-1}(i\omega_k)(e_l - H_d(k,l))\right\| \\ &\quad\leq \left\|G_c^{-1}(i\omega_k)\right\| \|(e_l - H_d(k,l))\| \\ &\quad= |g^{-1}(i\omega_k)| \left\|g(i\omega_k)G_c^{-1}(i\omega_k)\right\| \|(e_l - H_d(k,l))\| \\ &\quad\leq |g^{-1}(i\omega_k)| M \|(e_l - H_d(k,l))\|, \end{aligned}$$

so

$$\begin{aligned} &\sum_{\substack{k\in I\\ 1\leq l\leq n}} \left\|G_c^{-1}(i\omega_k)(e_l - H_d(k,l))\right\|^2 |f_k|^{-2} \\ &\quad\leq M \sum_{\substack{k\in I\\ 1\leq l\leq n}} |g^{-2}(i\omega_k)| \|(e_l - H_d(k,l))\|^2 |f_k|^{-2}. \end{aligned}$$

The result follows from Lemma 19.

Now we show that (ii) holds. Let $p \in P$. It is easily verified, that if a matrix valued function $M(s)$ has a bounded inverse in some set $s \in \Omega$, then there exists $\rho > 0$ such that $\|M(s)a\| \geq \rho \|a\|$ for $s \in \Omega$. Because $h \geq_P q_j(s)$ for all $j = 1, \dots, n$, $h^{-1}(p(\alpha))G(p(\alpha))$ is bounded for sufficiently large α. Thus, $h(p(\alpha))G^{-1}(p(\alpha))$ has a bounded inverse and for sufficiently large $k_0 \in \mathbb{N}$

$$\begin{aligned} &\sum_{\substack{k\in I\\ |k|>k_0\\ 1\leq l\leq n}} \left\|G_c^{-1}(i\omega_k)(e_l - H_d(k,l))\right\|^2 |f_k|^{-2} \\ &\quad\geq \rho \sum_{\substack{k\in I\\ |k|>k_0\\ 1\leq l\leq n}} |h^{-2}(i\omega_k)| \|(e_l - H_d(k,l))\|^2 |f_k|^{-2}. \end{aligned}$$

■

Remark 21: If the closed loop transfer function is analytic at infinity, this happens for example, if the system is finite-dimensional, one has $q_k(s) = s^{-p}$, $p \in \mathbb{N}$, and $q_j(s) \geq_P q_n(s)$, for all $j = 1, \dots, n$. Then the above solvability condition (14) can be written in the remarkably simple form

$$\sum_{\substack{k\in I\\ 1\leq l\leq n}} |k^{2p}| \|e_l - H_d(k,l)\|^2 |f_k|^{-2} < \infty.$$

One can make a similar simplification in (15) as well. ▲

Example 22: Consider the (f_k, ω_k)-RP with the time-delay system presented in Example 17. Assume, that there is no disturbance term, i.e. $E = 0$. The system is already exponentially stable. Because the role of the state feedback K in the feedback control law (12) is to stabilize the system, it is now possible to take $K = 0$ and simply ask when the proposed controller $u = Lw$, where L is as in (13), solves the (f_k, ω_k)-RP.

The assumptions of Theorem 20 hold if there are no transmission zeros at frequencies $i\omega_k = i\frac{2\pi k}{p}$, $k \in I \subseteq \mathbb{Z}$. The time-delay system has finite transmission zero on the imaginary axis if and only if $e^{-s} - 1 = 0$, i.e. $s = i2\pi l$, $l \in \mathbb{Z}$.

Assume that $\omega_k \neq 2\pi l$, $l \in \mathbb{Z}$. Now Theorem 20 gives the following sufficient solvability condition

$$\begin{aligned} &\sum_{k\in I} \left|(i\omega_k)^{-2}(e^{-i\omega_k} - 1)\right|^{-2} |f_k|^{-2} < \infty \\ &\quad\Leftrightarrow \sum_{k\in I} \frac{k^4}{\left|(e^{\frac{-2\pi k i}{p}} - 1)f_k\right|^2} < \infty. \end{aligned}$$

To show the importance of the role that finite transmission zeros have when defining the infinite structure, set $p = 2\pi$ and consider an infinite set $I \subset \mathbb{Z}$ of elements $k \in \mathbb{Z}$ that satisfy $|2\pi l - k| < \frac{\rho}{k}$ for some $l \in \mathbb{Z}$ and $\rho > 0$. Such a set I exists, because there are infinitely many pairs of integers (k, l) such that $\left|\frac{1}{2\pi} - \frac{l}{k}\right| < \frac{1}{k^2}$ [15, Corollary 5.4.].

Because the minimum distance of $i\omega_k$ to transmission zeros approach zero as $k \to \infty$, we see that $\lim_{k\to\pm\infty} e^{i\omega_k} - 1 = 0$. The rate of which ω_k can approach the finite transmission zeros is not known exactly. It is believed that the irrationality measure of π is 2. So far the best approximation of irrationality measure is $7.6063\dots$ due Salikhov [14]. If

α is greater than or equal to the irrationality measure of π, then a sufficient condition for solvability would be

$$\sum_{k\in I} \frac{k^{2+2\alpha}}{|f_k|^2} < \infty.$$

♦

Example 23: Consider the system in Example 18. The system is exponentially stable and there are no transmission zeros on the imaginary axis. Because both of the structural functions are $s^{-\frac{3}{2}}e^{-\frac{1}{7}\sqrt{s}}$ Theorem 20 states that the sum condition

$$\sum_{\substack{k\in I\\ 1\leq l\leq n}} k^3 \left| e^{\frac{2}{7}\sqrt{i\omega_k}} \right| \|e_l - H_d(k,l)\|^2 \, |f_k|^2 < \infty$$

is necessary and sufficient for the control law $u = Lw$ to solve the (f_k, ω_k)-RP. ♦

V. Concluding Remarks

A diagonal form that describes the behavior of transfer functions of infinite-dimensional systems at infinity was presented. The diagonal form gives so called directed structure because the way infinity can be approached was restricted. Two examples were given to illustrate the theory. These examples show that the structure of a time-delay system along imaginary axis has rational and periodic elements and that the structure of a heat equation has quotients of fractional powers and exponential structural elements. It was also noted that when considering the structure at infinity the poles and transmission zeros with an accumulation point at infinity must be taken into account.

The earlier definitions of the structure at infinity for infinite-dimensional systems with analytic transfer function [9] and for time-delay systems [10], [11], [12] are included in this more general theory. The directed structure approach was used to simplify the solvability conditions of a regulation problem and a connection between robust regulation problem and structure at infinity was found. These results show that the directed structure approach makes sense and is applicable.

Future work includes further investigation of the applicability of the directed structure. For example, connection between directed structure and root-locus is to be studied. Generalization of the results to more general systems, i.e. for systems that do not have simple structure, is also under consideration.

References

[1] Anatoliy G. Butkovskiy. *Green's Functions and Transfer Functions Handbook*. Ellis Horwood Ltd., 1982.

[2] C. I. Byrnes and P. K. Stevens. The McMillan and Newton polygons of a feedback system and the construction of root loci. *International Journal of Control*, 35, 1982.

[3] Ruth Curtain and Kirsten Morris. Transfer functions of distributed parameter systems: A tutorial. *Automatica*, 45, 2009.

[4] Ruth F. Curtain and Hans J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer-Verlag, New York, 1995.

[5] J. Descusse and J. M. Dion. On the structure at infinity of linear square decoupled systems. *IEEE Transactions on Automatic Control*, 27, 1982.

[6] E. Immonen and S. Pohjolainen. What periodic signals can an exponentially stabilizable linear feedforward control system asymptotically track? *SIAM Journal of Control and Optimization*, 44, 2006.

[7] Thomas Kailath. *Linear Systems*. Prentice-Hall, Englewood Cliffs, N.J., 1980.

[8] M. Malabre and V. Kučera. Infinite structure and exact model matching problem: A geometric approach. *IEEE Transactions on Automatic Control*, 29, 1984.

[9] M. Malabre and R. Rabah. Zeros at infinity for infinite dimensional systems. *Proceedings of the International Symposium MTNS-89 Vol. 1: Realization and Modelling in System Theory Series: Progress in Systems and Control Theory, Vol. 3*, 1990.

[10] M. Malabre and R. Rabah. Structure at infinity, model matching and disturbance rejection for linear systems with delays. *Kybernetika*, 29, 1993.

[11] M. Malabre and R. Rabah. Structure at infinity of linear delay systems with application to the disturbance decoupling problem. *Kybernetika*, 35, 1999.

[12] M. Malabre and R. Rabah. Weak structure at infinity and row-by-row decoupling for linear delay systems. *Kybernetika*, 40, 2004.

[13] H. H. Rosenbrock. *State-Space and Multivariable Theory*. Thomas Nelson and Sons Ltd., 1970.

[14] V. Kh. Salikhov. On the irrationality measure of π. *Russian Mathematical Surveys*, 63, 2008.

[15] Judith D. Sally and Paul J., Jr. Sally. *Roots to Research: A Vertical Development of Mathematical Problems*. American Mathematical Society, 2007.

[16] A. I. G. Vardulakis. *Algebraic Analysis and Synthesis Methods*. John Wiley & Sons, 1991.

[17] A. I. G. Vardulakis, D. N. J. Limebeer, and K. Karcanias. Structure and Smith-MacMillan form of a rational matrix at infinity. *International Journal of Control*, 35, 1982.

[18] H. Zwart. Transfer functions for infinite-dimensional systems. *Systems & Control Letters*, 52, 2004.