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Riesz basis for strongly continuous groups.

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***Abstract*— Given a Hilbert space and the generator of a strongly continuous group on this Hilbert space. If the eigenvalues of the generator have a uniform gap, and if the span of the corresponding eigenvectors is dense, then these eigenvectors form a Riesz basis (or unconditional basis) of the Hilbert space. Furthermore, we show that none of the conditions can be weakened.**

I. Introduction and main results

We begin by introducing some notation. By H we denote the Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and by A we denote an unbounded operator from its domain $D(A) \subset H$ to H.

If A is self-adjoint and has a compact resolvent operator, then it has an orthonormal basis of eigenvectors. Unfortunately, even a slight perturbation of A can destroy the self-adjointness of A and so also the orthonormal basis property of the eigenvectors. However, in general the (normalized) eigenvectors, $\{\phi_n\}_{n\in\mathbb{N}}$ will still form a Riesz basis, i.e., their span is dense in H and there exist (positive) constants m and M such that

$$m \sum_{n=1}^{N} |\alpha_n|^2 \leq \left\| \sum_{n=1}^{N} \alpha_n \phi_n \right\|^2 \leq m \sum_{n=1}^{N} |\alpha_n|^2 \qquad (1)$$

for every sequence $\{\alpha_n\}_{n=1}^N$. If A processes a Riesz basis of eigenvectors, then many system theoretic properties like stability, controllability, etc. are easily checkable, see e.g. [3].

Since this Riesz-basis property is so important there is an extensive literature on this problem. We refer to the book of Dunford and Schwartz [4], where this problem is treated for discrete operators, i.e., the inverse of A is compact. They apply these results to differential operators. Riesz spectral properties for differential operators is also the subject of Mennicken and Moller [11] and Tretter [13]. In these references no use is made of the fact that for many differential operators the abstract differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \qquad (2)$$

on the Hilbert space H has a unique solution for every initial condition x_0, i.e., A is the infinitesimal generator of a C_0-semigroup. In [15] this property is used. The property will also be essential in our paper. In many applications the differential operator A arises from a partial differential equation, for which it is known that (2) has a solution. Hence the assumption that A generates a C_0-semigroup is not strong. However, for our result the semigroup property does not suffice, we need that A generates a group, i.e., (2) possesses a unique solution forward and backward in time. Since we also assume that the eigenvalues lie in a strip parallel to the imaginary axis, the group condition is not very restrictive. For more information on groups and semigroups, we refer the reader to [3], [5].

The approach which we take to prove our result is different to the one taken in [4], [11], [13], and [15]. We use the fact that every generator of a group has a bounded $\mathcal{H}_\infty$-calculus on a strip. This means that to every complex valued function f bounded and analytic in a strip parallel to the imaginary axis, there exists a bounded operator $f(A)$. For more detail we refer to [8]. Note that the $\mathcal{H}_\infty$-calculus extends the functional calculus of Von Neumann [14] for self-adjoint operators.

We formulate our main results.

Theorem 1.1: Let A be the infinitesimal generator of the C_0-group $(T(t))_{t\in\mathbb{R}}$ on the Hilbert space H. We denote the eigenvalues of A by λ_n (counting with multiplicity), and the corresponding (normalized) eigenvectors by $\{\phi_n\}$. If the following two conditions hold,

1) The span of the eigenvectors form a dense set in H;
2) The point spectrum has a *uniform gap*, i.e.,

$$\inf_{n\neq m} |\lambda_n - \lambda_m| > 0. \qquad (3)$$

Then the eigenvectors form a Riesz basis on H. □

From this result, we have some easy consequences.

Remark 1.2: Under the conditions of Theorem 1.1 we have

- Since we are counting the eigenvalues with multiplicity, we see from (3) that all eigenvalues are simple.
- If the operator possesses a Riesz basis of eigenvectors, then the spectrum equals the closure of the point spectrum. Using (3), we see that the spectrum of A, $\sigma(A)$, is pure point spectrum, and $\sigma(A) = \{\lambda_n\}_{n\in\mathbb{N}}$.
- It is easy to see that for $\lambda \in \rho(A)$, the finite-range operator $\sum_{n=1}^N \frac{1}{\lambda-\lambda_n}\phi_n$ converges uniformly to $(\lambda I - A)^{-1}$. Thus under the conditions of Theorem 1.1, the resolvent operator of A is compact, i.e., A is discrete.

□

If the eigenvalues do not satisfy (3), then Theorem 1.1 need not to hold. A simple counter example is given next.

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Example 1.3: Let H be a Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. Define A as

$$A\sum_{n\in\mathbb{N}}\alpha_n e_n = \sum_{k\in\mathbb{N}}(ki\alpha_{2k-1}+\alpha_{2k})e_{2k-1} + (k+\frac{1}{k})i\alpha_{2k}e_{2k} \tag{4}$$

with domain

$$D(A)=\left\{x=\sum_{n=1}^{\infty}\alpha_n e_n\in H \mid \sum_{n=1}^{\infty}|n\alpha_n|^2<\infty\right\}. \tag{5}$$

Hence the operator A is block diagonal, i.e.,

$$A=\operatorname{diag}\left[\begin{array}{cc} ki & 1 \\ 0 & (k+\frac{1}{k})i\end{array}\right]. \tag{6}$$

Using this structure, it is easy to see that A generates a strongly continuous group on H, and that its eigenvalues are given by $\{ki,(k+\frac{1}{k})i; k\in\mathbb{N}\}$, with the normalized eigenvectors $\phi_{2k-1}=e_{2k-1}$ and $\phi_{2k}=\frac{1}{\sqrt{k^2+1}}(kie_{2k-1}+e_{2k})$, $k\in\mathbb{N}$.

Since

$$\inf_k \|i\phi_{2k-1}-\phi_{2k}\|=0$$

we have that the eigenvectors do not from a Riesz basis. □

If the eigenvalues have an uniform gap, then one can associate to every eigenvalue λ_n the spectral projection P_n. Wermer [16] proved the following relation between the Riesz basis and the spectral projections..

Lemma 1.4: The (simple) normalized eigenvectors $\{\phi_n\}$ form a Riesz basis if and only if there exists a M_2 such that for every subset $\mathbb{J}$ of $\mathbb{N}$ there holds

$$\|\sum_{n\in\mathbb{J}}P_n\|\le M_2. \tag{7}$$

II. Functional calculus for groups.

We begin by introducing some notation. Since $(T(t))_{t\in\mathbb{R}}$ is a group, there exists a ω_0 and M_0 such that $\|T(t)\|\le M_0e^{\omega_0|t|}$.

For $\alpha>0$ we define a strip parallel to the imaginary axis by $S_\alpha:=\{s\in\mathbb{C}\mid -\alpha<\operatorname{Re}(s)<\alpha\}$.

By $\mathcal{H}^\infty(S_\alpha)$ we denote the linear space of all functions from S_α to $\mathbb{C}$ which are analytic and (uniformly) bounded on S_α. The norm of a function in $\mathcal{H}^\infty(S_\alpha)$ is given by

$$\|f\|_\infty=\sup_{s\in S_\alpha}|f(s)|. \tag{8}$$

In Haase [7], [8] it is shown that the generator of the group, A, has a $\mathcal{H}^\infty(S_\alpha)$-calculus for $\alpha>\omega_0$. This we explain in a little bit more detail.

Choose ω_1 and ω such that $\omega_0<\omega_1<\alpha<\omega$. Furthermore, let $\Gamma=\gamma_1\oplus\gamma_2$ with $\gamma_1=-\omega_1-ir$, $\gamma_2=\omega_1+ir$, $r\in\mathbb{R}$. For $f\in\mathcal{H}^\infty(S_\alpha)$ and $x\in D(A^2)$ we define

$$f(A)x=\frac{1}{2\pi i}\int_\Gamma\frac{f(z)}{z^2-\omega^2}(zI-A)^{-1}dz\cdot(A^2-\omega^2)x. \tag{9}$$

For $\lambda\notin\overline{S_\alpha}$ we have that

$$\left(\frac{1}{\lambda-\cdot}\right)(A)x=(\lambda I-A)^{-1}x$$

Furthermore, the operator defined in (9) extends to a bounded operator on H, and

$$\|f(A)\|\le c\|f\|_\infty, \tag{10}$$

with c independent of f.

In the following lemma we show that this functional calculus behaves like one would expect from the functional calculus of von Neumann and Dunford.

Lemma 2.1: Let A be the infinitesimal generator of a group and let ϕ_n be an eigenvector for the eigenvalue λ_n. Then for every $f\in\mathcal{H}^\infty(S_\alpha)$ there holds that

$$f(A)\phi_n=f(\lambda_n)\phi_n,\qquad n\in\mathbb{N}. \tag{11}$$

Proof: Since ϕ_n is an eigenvector, it is an element of $D(A^2)$ and so we may use equation (9). Hence

$$\begin{aligned}f(A)\phi_n &= \frac{1}{2\pi i}\int_\Gamma\frac{f(z)}{z^2-\omega^2}(zI-A)^{-1}dz\cdot(A^2-\omega^2)\phi_n\\ &= \frac{1}{2\pi i}\int_\Gamma\frac{f(z)}{z^2-\omega^2}(zI-A)^{-1}dz\,(\lambda_n^2-\omega^2)\phi_n\\ &= (\lambda_n^2-\omega^2)\frac{1}{2\pi i}\int_\Gamma\frac{f(z)}{z^2-\omega^2}(zI-\lambda_n)^{-1}\phi_n dz.\end{aligned}$$

Since f is bounded on S_α, we see that the integrand converges quickly to zero for $|z|$ large. Hence we may apply Cauchy residue theorem. The only pole within the contour is λ_n, and so we obtain

$$f(A)\phi_n=(\lambda_n^2-\omega^2)\frac{f(\lambda_n)}{\lambda_n^2-\omega^2}\phi_n=f(\lambda_n)\phi_n.$$

This shows equation (11).

Hence we have proved the assertion. □

III. Interpolation sequences

For the proof of Theorem 1.1 we need the following interpolation result, see [6, Theorem VII.1.1].

Theorem 3.1: Consider the sequence μ_n which satisfies $\beta_1>\operatorname{Re}(\mu_n)>\beta_2>0$ and $\inf_{n\neq m}|\mu_n-\mu_m|>0$. Then for every bounded sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of complex numbers there exists a function g holomorphic and bounded in the right-half plane $\mathbb{C}_+:=\{s\in\mathbb{C}\mid\operatorname{Re}(s)>0\}$ such that

$$g(\mu_n)=\alpha_n. \tag{12}$$

Furthermore, there exists an M independent of g and $\{\alpha_n\}_{n\in\mathbb{N}}$ such that

$$\sup_{\operatorname{Re}(s)>0}|g(s)|\le M\sup_{n\in\mathbb{N}}|\alpha_n|. \tag{13}$$

A sequence $\{\mu_n\}_{n\in\mathbb{N}}$ satisfying the conditions of the above theorem is called an *interpolation sequence*.

Now we have all the ingredients for the proof of Theorem 1.1.

IV. Proof of Theorem 1.1

Proof of Theorem 1.1: Let α be the positive number defined at the beginning of Section II. We define the complex numbers μ_n as

$$\mu_n = \lambda_n + \alpha \qquad n \in \mathbb{N}. \tag{14}$$

By the conditions on α and λ_n, we see that $\{\mu_n\}_{n\in\mathbb{N}}$ satisfies the conditions of Theorem 3.1.

Let $\mathbb{J}$ be a subset of $\mathbb{N}$. Since the eigenvalues satisfy (3), we conclude by Theorem 3.1 there exists a function $g_{\mathbb{J}}$ bounded and analytic in $\mathbb{C}_+$ such that

$$g_{\mathbb{J}}(\mu_n) = \begin{cases} 1, & \text{if } n \in \mathbb{J} \\ 0, & \text{if } n \notin \mathbb{J}. \end{cases} \tag{15}$$

Furthermore, see (13)

$$\sup_{s\in\mathbb{C}_+} |g_{\mathbb{J}}(s)| \le M. \tag{16}$$

Given this $g_{\mathbb{J}}$ we define $f_{\mathbb{J}}$ as

$$f_{\mathbb{J}}(s) = g_{\mathbb{J}}(s+\alpha), \qquad s \in S_\alpha. \tag{17}$$

Then using the properties of $g_{\mathbb{J}}$ we have that $f_{\mathbb{J}} \in \mathcal{H}_\infty(S_\alpha)$,

$$f_{\mathbb{J}}(\lambda_n) = \begin{cases} 1, & \text{if } n \in \mathbb{J} \\ 0, & \text{if } n \notin \mathbb{J}, \end{cases} \tag{18}$$

and there exists a $M > 0$ independent of $f_{\mathbb{J}}$ such that

$$\|f_{\mathbb{J}}\|_\infty \le M. \tag{19}$$

Next we identify the operator $f_{\mathbb{J}}(A)$. Combining (11) with (18) gives

$$f_{\mathbb{J}}(A)\phi_n = \begin{cases} \phi_n, & \text{if } n \in \mathbb{J} \\ 0, & \text{if } n \notin \mathbb{J}. \end{cases}$$

Since $f_{\mathbb{J}}(A)$ is a linear operator, we obtain that

$$f_{\mathbb{J}}(A)\left(\sum_{n=1}^{N} \alpha_n\phi_n\right) = \sum_{n\in\mathbb{J}\cap\{1,\cdots,N\}} \alpha_n\phi_n. \tag{20}$$

By assumption the span of $\{\phi_n\}_{n\in\mathbb{N}}$ is dense in H. Furthermore, $f_{\mathbb{J}}(A)$ is a bounded operator. So we conclude that $f_{\mathbb{J}}$ is the spectral projection associated to the spectral set $\{\lambda_n \mid n \in \mathbb{J}\}$.

Combining (10) with (19) we have that these spectral projections are uniformly bounded. Since the eigenvalues are simple, this implies that the (normalized) eigenvectors form a Riesz basis, see [16]. □

V. Closing remarks

A natural question is the following: *If A is the infinitesimal generator of a group and A has only point spectrum with multiplicity one, is the span over all eigenvectors dense in H?*

In general the answer to this question is negative. On page 665 of Hille and Phillips [9] one may find an example of a generator of a group without any spectrum.

However, there are some interesting cases for which the answer is positive. If A generates a bounded group, i.e., $\sup_{t\in\mathbb{R}} \|T(t)\| < \infty$, then A is similar to a skew-adjoint operator, and so there is a complete spectral measure, see [2], [17]. Another interesting situation is the following. Since A generates a group, it can be written as $A = A_0 + Q$, where A_0 generates a bounded group, and Q is a bounded linear operator, see [7]. If A_0 has only point spectrum which satisfies (3), then by Theorem XIX.5.7 of [4] we know that condition 1. holds for A.

When calculating the eigenvalues of a differential operator, one normally finds that these eigenvalues are the zeros of an entire function. If this function has its zeros in a strip parallel to the imaginary axis, and on the boundary of this strip the function is bounded and bounded away from zero, then its zeros can be decomposed into finitely many interpolation sequences, see Proposition II.1.28 of [1]. They name this class of entire function *sine type functions*, but in Levin [10] this name is restricted to a smaller class of functions. The general case with multiple eigenvalues can be found in [18].

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