Controllability, Observability and Disturbance Attenuation by Boundary Control of Repetitive Processes with Smoothing

T-P Azevedo-Perdicoúlis and G. Jank

***Abstract*— In this paper, we present an explicit representation of solutions for a specific class of linear repetitive processes with smoothing. This representation then is used to obtain direct criteria for controllability and observability properties of this class of discrete time 2–D systems with delays. We not only consider classical controllability properties, where control is obtained by choosing the inhomogeneity appropriately, but also controllability of the system by steering the system through boundary data control. From the point of view of technical applications, for instance in high pressure gas network modelling (see [1]), it seems to be more reliable to consider boundary data controls. Therefore in this paper we emphasise boundary control properties of the system. A disturbed optimal boundary control problem with a quadratic criterion is also solved.**

Keywords : Boundary control, Controllability, Disturbed games, Observability, Optimal control, Repetitive processes, 2D–systems.

I. INTRODUCTION

Repetitive processes (RP) are a distinct class of two dimensional 2–D systems of both systems theoretic and applications interest. For more information see [10].

In [5] there was obtained a RP model for a gas distribution network stemming from a discretization approach in time and space variables and in [1] a repetitive model with smoothing was used to model a high pressure gas network. In the latter article the controls used were compressor stations and offtakes, hence controlling boundary data of the net. For that reason in this paper we emphasise boundary control properties of the system. We consider here a linear discrete model with interpass smoothing that has been introduced in [2]

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + \sum_{\ell=0}^{\alpha-1} B_{\ell p} y_k(\ell), \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + \sum_{\ell=0}^{\alpha-1} D_{\ell p} y_k(\ell), \end{aligned} \tag{1}$$

This research was supported in part by Qatar National Research Fund (NPRP 08-345-1-067).

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with boundary conditions

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \\ y_0(p) &= f(p), \\ &\text{for} \quad k = 0, 1, \dots \text{ and } 0 \le p \le \alpha - 1. \end{aligned} \tag{2}$$

where $\alpha < \infty$ denotes the pass length and k may run in a finite or infinite time horizon. Also on pass k, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the pass profile vector, and $u_k(p) \in \mathbb{R}^r$ are the control inputs.

The matrices in equation (1) are of the following dimensions: $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times r}, B_{\ell p} \in \mathbb{R}^{n\times m}, 0 \le \ell, p \le \alpha - 1$, $C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times r}, D_{\ell p} \in \mathbb{R}^{m\times m}, 0 \le \ell, p \le \alpha - 1$.

So, in this model the α time steps per pass are indexed with $p \in [0, \alpha - 1] \cap \mathbb{Z}$ and the pass number is indexed by $k \in [0, T] \cap \mathbb{Z}$.

Set $\mathcal{I} = [0, T] \times [0, \alpha - 1] \cap \mathbb{Z} \times \mathbb{Z}$. The state $x_{(.)}(.) \in \ell^{2,n}(\mathcal{I})$, the pass profile $y_{(.)}(.) \in \ell^{2,n}(\mathcal{I})$ and the controls are admissible if $u_{(.)}(.) \in \ell^{2,r}(\mathcal{I})$. Here $\ell^{2,\nu}(\mathcal{I})$ denotes the Hilbert space of ν-dimensional sequences defined on $\mathcal{I}$ with the standard scalar product.

In [8], [10] and related articles cited therein, controllability, reachability, observability matters are considered, but, the criteria obtained are based on an implicit representation of the solutions, since they are defined by some recursions. Also general algebraic methods, i.e. module theoretic or behavioral approaches could be used to obtain controllability results. To our knowledge, however, no results for boundary data control are available by these methods. Controllability properties by boundary data control in continuous time 2–D systems are also obtained in [6]and [7].

In this article, therefore, we exploit a direct representation formula which strictly separates the influences of the boundary data from the influence of the control parameters. This eventually allows for studying the sole influence of boundary data on the system dynamics.

II. REPRESENTATION OF SOLUTIONS

Due to space restriction and being sufficient to present the essential ideas we begin with a simplified version of model (1). Since we want to start by studying controllability properties, we do exclude the output, i.e. we set $n = m, D = 0, D_{\ell p} = 0, 0 \le \ell, p \le \alpha - 1$, $C = I_n$ the n-dimensional identity matrix and, consequently $y_k(p) = x_k(p)$. So (1) and

(1) reduce to:

$$x_{k+1}(p+1) \quad = \quad Ax_{k+1}(p) + Bu_{k+1}(p) + \sum_{\ell=0}^{\alpha-1} B_{\ell p} x_k(\ell), \tag{3}$$

with boundary conditions

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \qquad (4)\\ x_0(p) &= f(p), \\ &\text{for} \quad k = 0,1,\dots \ \text{ and } \ 0 \le p \le \alpha - 1. \end{aligned}$$

Defining now the stacked vectors

$$X_k := \begin{pmatrix} x_k(1) \\ \vdots \\ x_k(\alpha-1) \end{pmatrix},$$

$$U_k := \begin{pmatrix} u_k(0) \\ \vdots \\ u_k(\alpha-2) \end{pmatrix},$$

$$F_0 := \begin{pmatrix} f(1) \\ \vdots \\ f(\alpha-1) \end{pmatrix}$$

and the block matrices

$$\mathcal{A} = \begin{pmatrix} I_n & 0 & 0 & \dots & 0 \\ A & I_n & 0 & \dots & 0 \\ A^2 & A & I_n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^{\alpha-3} & A^{\alpha-4} & A^{\alpha-5} & \dots & 0 \\ A^{\alpha-2} & A^{\alpha-3} & A^{\alpha-4} & \dots & I_n \end{pmatrix}$$

$$\mathcal{B}_0 = \begin{pmatrix} B_{1,0} & \dots & B_{\alpha-1,0} \\ B_{1,1} & \dots & B_{\alpha-1,1} \\ \vdots & \dots & \vdots \\ B_{1,\alpha-2} & \dots & B_{\alpha-1,\alpha-2} \end{pmatrix}$$

with dimensions $(\alpha-1)n \times (\alpha-1)n$, and $\mathcal{B} = \operatorname{diag}(B,\dots,B) \in \mathbb{R}^{(\alpha-1)n\times(\alpha-1)r}$. Furthermore, we define the blocked column matrices

$$\mathbb{A} = \begin{pmatrix} A \\ A^2 \\ \vdots \\ A^{(\alpha-1)} \end{pmatrix} \text{ and } \mathbb{B} = \begin{pmatrix} B_{00} \\ B_{01} \\ \vdots \\ B_{0\ \alpha-2} \end{pmatrix}.$$

With these definitions, after a short calculation, we obtain from (3) the following recursive expression along the time steps.

$$X_{k+1} = \mathbb{A}d_{k+1} + \mathcal{A}\mathbb{B}d_k + \mathcal{A}\mathcal{B}U_{k+1} + \mathcal{A}\mathcal{B}_0 X_k, \quad X_0 = F_0. \tag{5}$$

Notice that with an output of type

$$y_{k+1}(p) = Cx_{k+1}(p), \tag{6}$$

where $C \in \mathbb{R}^{m\times n}$ and then clearly $B_{\ell p} \in \mathbb{R}^{n\times m}, 0 \le \ell, p \le \alpha - 1$, we obtain, by the same procedure and replacing $B_{\ell p}$ in (5) by the $n \times n$ matrix $B_{\ell p}C$, the following:

$$X_{k+1} = \mathbb{A}d_{k+1} + \mathcal{A}\mathbb{B}\mathcal{C}d_k + \mathcal{A}\mathcal{B}U_{k+1} + \mathcal{A}\mathcal{B}_0\mathcal{C}X_k, \quad X_0 = F_0. \tag{7}$$

Solving iteratively equation (7) yields

Theorem 1: With the previously introduced notations the solution of equation (1) with an output equation (6) is given by

$$\begin{aligned} X_k \quad &= \quad (\mathcal{A}\mathcal{B}_0\mathcal{C})^k F_0 \qquad (8)\\ &+ \sum_{s=0}^{k-1} (\mathcal{A}\mathcal{B}_0\mathcal{C})^s (\mathbb{A}d_{k-s} + \mathcal{A}\mathbb{B}\mathcal{C}d_{k-s-1} + \mathcal{A}\mathcal{B}U_{k-s}), \\ &k = 1,2,\dots. \end{aligned}$$

Proof: The proof is obtained by mathematical induction. ■

Equation (8) is an explicit representation of the solution of (1)–(2) in terms of the pass profile X_k. From this representation we also could obtain a representation for the profile vectors $y_k(p), 0 \le p \le \alpha - 1$, along each pass. This would be necessary if we want to obtain "point controllability" conditions as in [10]. Because of space limitations in this article we restrict our considerations to "pass controllability" (to be defined later) which then contains point controllability.

Remark 1: Notice that if $\{d_k\}_{k\in\mathbb{N}} \in \ell^{2,n(\alpha-1)}$, $\{U_k\}_{k\in\mathbb{N}} \in \ell^{2,m(\alpha-1)}$, and $||\mathcal{A}\mathcal{B}_0\mathcal{C}||_2 < 1$ the solution exists also on an infinite time horizon.

Proof: From our assumptions there exists $M \ge 0$ such that $|\mathbb{A}d_{k-s} + \mathcal{A}\mathbb{B}\mathcal{C}d_{k-s-1} + \mathcal{A}\mathcal{B}U_{k-s}| \le M$, for all $k, s \in \mathbb{N}$. With this we estimate $|X_k|_2 \le ||\mathcal{A}\mathcal{B}_0\mathcal{C}||_2^k |F_0|_2 + \frac{M}{1-||\mathcal{A}\mathcal{B}_0\mathcal{C}||_2}$. This proves $\{X_k\}_{k\in\mathbb{N}} \in \ell^{2,n(\alpha-1)}$. ■

III. Controllability

From (8) we now deduce first some controllability criteria. Since system (3)–(4) is quasi one dimensional in the pass profile vectors, we also expect to obtain classical results.

Definition 1: System (3)–(4) is called *(completely) pass controllable* in $k_0, k_0+1, \dots, k_1,\ k_0, k_1 \in \{0,1,2,\dots,\}$ if for any boundary condition in (4) and any given terminal pass vector $X_f \in \mathbb{R}^{n(\alpha-1)}$ there exists a sequence of control vectors $U_k, k = k_0, k_0+1, \dots, k_1$, such that $X_{k_1} = X_f$.

We obtain the following necessary and sufficient criterion for pass controllability.

Theorem 2: System (3)–(4) is completely pass controllable on $0,1,\dots,T$, if and only if the grammian matrix

$$G_T := \sum_{s=0}^{T-1} (\mathcal{A}\mathcal{B}_0)^s \mathcal{A}\mathcal{B}\mathcal{B}^* \mathcal{A}^* (\mathcal{B}_0^* \mathcal{A}^*)^s > 0, \tag{9}$$

where "* " denotes the transpose of a matrix.

Notice that considering the discrete time interval $0,1,\dots,T$, instead of $k_0, k_0+1, \dots, k_0+T = k_1$ is in a time invariant system context no restriction.

Proof: In (8) we set $m = n$ and $\mathcal{C} = Id_{n(\alpha-1)}$, then the proof is the same as in classical systems and is therefore omitted here. ■

Next, we obtain controllability results on boundary control. Therefore in the sequel we assume $U_k, k = 1, 2, \ldots$ in (3) to be given and fixed or, equivalently $U_k = 0, k = 1, 2, \ldots$. First we deal with pass controllability if we try to steer the system with boundary data $\{d_k\}_{k\in\mathbb{N}}$.

Definition 2: System (3)–(4) is *completely pass-boundary-controllable* on $k_0, k_0+1, \ldots, k_0+T = k_1$ if for any boundary condition $d_0 = f(0)$ and $f(1), f(2), \ldots, f(\alpha-1)$ in (4), subsumed into F_0, and any given terminal pass vector $X_f \in \mathbb{R}^{n(\alpha-1)}$ there exists a sequence of control boundary data $d_k, k = k_0, k_0+1, \ldots, k_1$, such that $X_{k_1} = X_f$.

Theorem 3: System (3)–(4) is completely pass-boundary-controllable on $0, 1, \ldots, T$ if and only if the boundary grammian matrix

$$\Gamma_T := \sum_{s=1}^{T-1} (\mathcal{A}\mathcal{B}_0)^{s-1}\mathcal{A}[\mathcal{B}_0\mathbb{A}+\mathbb{B}][\mathcal{B}_0\mathbb{A}+\mathbb{B}]^*\mathcal{A}^*(\mathcal{B}_0^*\mathcal{A}^*)^{s-1} > 0. \tag{10}$$

Proof: In (8) we set $m = n$ and $\mathcal{C} = Id_{n(\alpha-1)}$, then from (8) we obtain:

$$\begin{aligned}\tilde{X}_k &= X_k - (\mathcal{A}\mathcal{B}_0)^k F_0 - \mathbb{A}d_k - (\mathcal{A}\mathcal{B}_0)^{k-1}\mathcal{A}\mathbb{B}d_0 \\ &= \sum_{s=1}^{k-1}(\mathcal{A}\mathcal{B}_0)^{s-1}\mathcal{A}[\mathcal{B}_0\mathbb{A}+\mathbb{B}]d_{k-s} \\ &= \sum_{s=1}^{k-1}\Phi_{s-1}d_{k-s},\end{aligned}$$

where $\Phi_{s-1} = (\mathcal{A}\mathcal{B}_0)^{s-1}\mathcal{A}[\mathcal{B}_0\mathbb{A}+\mathbb{B}]$. Now we set $d_{k-s} := \Phi_{s-1}^* L_k$, and obtain from (11) $\tilde{X}_k = \sum_{s=1}^{k-1}\Phi_{s-1}\Phi_{s-1}^* L_k$.

The expression on the left hand side of this identity is given for $k = T$, since $X_T = X_f$, and if the boundary grammian $\Gamma_T = \sum_{s=1}^{T-1}\Phi_{s-1}\Phi_{s-1}^*$ is positive definite then its inverse exists and we obtain $L_T = \Gamma_T^{-1}\tilde{X}_T$ and the boundary controls, steering the system to X_f then are $d_{T-s} = \Phi_{s-1}^* L_T, s = 1, 2, \ldots, T-1$, hence the system is controllable.

If, on the other hand, the system (3)–(4) is pass-boundary-controllable then there exists for any given X_f or equivalently for any given $\tilde{X}_T$ a sequence of boundary control data $d_{T-s}, s = 1, \ldots, T-1$, such that

$$\tilde{X}_T = \sum_{s=1}^{T-1}\Phi_{s-1}d_{T-s}.$$

If $\tilde{X}_T \neq 0$ were in the kernel of $\Gamma_T \geq 0$, i.e. $\Gamma_T\tilde{X}_T = 0$ then

$$\tilde{X}_T^*\Gamma_T\tilde{X}_T = \tilde{X}_T^*\sum_{s=1}^{T-1}\Phi_{s-1}\Phi_{s-1}^*\tilde{X}_T = \sum_{s=1}^{T-1}|\tilde{X}_T^*\Phi_{s-1}|^2 = 0,$$

henceforth $\tilde{X}_T^*\Phi_{s-1} = 0, s = 1, \ldots, T-1$. This implies $|\tilde{X}_T|^2 = \sum_{s=1}^{T-1}\tilde{X}_T^*\Phi_{s-1}d_{T-s} = 0$ and hence a contradiction, which proves that $\Gamma_T > 0$. ■

There is one more possibility of boundary control of system (3)–(4) namely by steering it with data on the initial pass, i.e. by choosing appropriately $x_0(1) = f(1), x_0(2) = f(2), \ldots, x_0(\alpha-1) = f(\alpha-1)$, while keeping the other boundary data fixed. We begin with

Definition 3: System (3)–(4) is *completely pass controllable by initial pass control* if for any boundary condition $d_0 = f(0), d_1, \ldots, d_T)$ in (4), and any given terminal pass vector $X_f \in \mathbb{R}^{n(\alpha-1)}$ there exists a sequence of control boundary data $f(1), f(2), \ldots, f(\alpha-1)$, subsumed in F_0 such that $X_T = X_f$.

There is a simple criterion for this type of controllability.

Theorem 4: System (3)–(4) is completely pass controllable by initial pass control if $\mathcal{A}\mathcal{B}_0 \in \mathbb{R}^{n(\alpha-1)\times n(\alpha-1)}$ has full rank.

Proof: Again we arrange equation (8) such that everything known at time step T appears on the left, hence

$$\begin{aligned}\tilde{X}_T &= X_f - \mathbb{A}d_T - (\mathcal{A}\mathcal{B}_0)^{T-1}\mathcal{A}\mathbb{B}d_0 && (11)\\ &\quad - \sum_{s=1}^{T-1}(\mathcal{A}\mathcal{B}_0)^{s-1}\mathcal{A}[\mathcal{B}_0\mathbb{A}+\mathbb{B}]d_{T-s} && (12)\\ &= (\mathcal{A}\mathcal{B}_0)^T F_0 && (13)\end{aligned}$$

and we obtain the desired boundary data at initial pass as $F_0 = (\mathcal{A}\mathcal{B}_0)^{-T}\tilde{X}_T$. ■

IV. Observability

In this section, we discuss some observability properties of (3)–(4) with the output equation (6). Due to space limitations we discuss only boundary observability.

Definition 4: System (3) with output (6) is *pass- boundary-observable* in {0,1, ...,T}, if for all $t_1 \in \mathbb{N}, 0 < t_1 \leq T$ and for all controls $U_k, 0 < k \leq t_1$ and boundary data d_0, F_0, for any two trajectories $X_k, \tilde{X}_k, 0 < k \leq t_1$, belonging to the same input $U_k, 0 < k \leq t_1$, from

$$\mathcal{C}X_k = \mathcal{C}\tilde{X}_k, 0 < k \leq t_1$$

it follows necessarily that $X_k = \tilde{X}_k, 0 < k \leq t_1$.

Remark 2: Set $\hat{X}_k := X_k - \tilde{X}_k, k = 1, \ldots, T$, then pass-boundary-observability is equivalent to the condition that

$$\mathcal{C}\hat{X}_k = 0,\ \ 0 < k \leq t_1 \quad implies \quad \hat{X}_k = 0,\ \ 0 < k \leq t_1, \tag{14}$$

where $\hat{X}_k$ is any solution of the homogenous equation, i.e. with $U_k = 0,\ \ 0 < k \leq t_1$ and with $d_0 = 0, F_0 = 0$.

We obtain the following sufficient pass-boundary-observability criterion.

Theorem 5: System (3)–(4) with output (6) is pass-boundary-observable in $\{0, 1, \ldots, T\}$ if $\operatorname{rank}\mathcal{C}\mathbb{A} = n$.

Proof: Denoting the output $Y_k := \mathcal{C}X_k$ and accordingly $\hat{Y}_k := \mathcal{C}\hat{X}_k,\ k = 1,\dots,T$ then from representation (8) after some manipulations we eventually infer

$$\hat{Y}_k = \mathcal{C}\mathbb{A}(d_k - \tilde{d}_k) + \sum_{s=1}^{k-1} \mathcal{C}(\mathcal{A}\mathcal{B}_0\mathcal{C})^{s-1}\mathcal{A}(\mathbb{B}\mathcal{C} + \mathcal{B}_0\mathcal{C}\mathbb{A})(d_{k-s} - \tilde{d}_{k-s}), \quad (15)$$

where d_k denotes the boundary data from X_k and $\tilde{d}_k$ that of $\tilde{X}_k$. Setting $\hat{Y}_k = 0, k = 1, 2, \dots, t_1$, we obtain:
for $k = 1 : 0 = \mathcal{C}\mathbb{A}(d_1 - \tilde{d}_1)$, hence $d_1 - \tilde{d}_1 = 0$.
$k = 2 : 0 = \mathcal{C}\mathbb{A}(d_2 - \tilde{d}_2) + \mathcal{C}\mathcal{A}(\mathbb{B}\mathcal{C} + \mathcal{B}_0\mathcal{C}\mathbb{A})\underbrace{(d_1 - \tilde{d}_1)}_{=0}$,

hence it follows $d_2 - \tilde{d}_2 = 0$.

Continuing this procedure, we obtain $d_k - \tilde{d}_k = 0, k = 1,\dots,t_1$. This yields $\hat{X}_k = 0, k = 1,\dots,t_1$, hence the boundary data $d_k, k = 1,\dots,t_1$, are uniquely defined by a measured output profile $Y_k, k = 1,\dots,t_1$ which proves the theorem. ∎

V. Disturbance Attenuation by Boundary Control

Based on the representation formula (8) we obtain in this section solvability conditions for a disturbed optimal control problem with boundary controls along the passes. The disturbance is assumed to occur across the whole system. This means that we are given the following dynamics with disturbances $w_{k+1}(p)$ and boundary data control d_k:

$$x_{k+1}(p+1) = Ax_{k+1}(p) + Ew_{k+1}(p) + \sum_{\ell=0}^{\alpha-1} B_{\ell p}x_k(\ell), \quad (16)$$

with boundary data

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad (17)\\ x_0(p) &= f(p),\\ &\text{for} \quad k = 0, 1, \dots, T \ \text{ and } \ 0 \le p \le \alpha - 1. \end{aligned}$$

which has the solution, adapted from (8)

$$\begin{aligned} X_k = {} & (\mathcal{A}\mathcal{B}_0)^k F_0 + \mathbb{A}d_k + (\mathcal{A}\mathcal{B}_0)^{k-1}\mathcal{A}\mathbb{B}d_0 \\ & + \sum_{s=1}^{k-1} (\mathcal{A}\mathcal{B}_0)^{s-1}[\mathcal{A}\mathcal{B}_0\mathbb{A} + \mathcal{A}\mathbb{B}]d_{k-s} \\ & + \sum_{s=0}^{k-1} (\mathcal{A}\mathcal{B}_0)^s \mathcal{A}\mathcal{E}W_{k-s}), \end{aligned}$$

with $k = 1, 2, \dots, T$, $W_k = (w_k(0)^*, \dots, w_k(\alpha-2)^*)^*$ is the disturbance term and the other entries are as previously defined. With this definitions we further define the following quadratic criterion in $[0, T]$

$$2J(d, W) = X_T^* K_f X_T + \sum_{k=1}^{T-1} (X_k^* Q_k X_k + d_k^* Q_{b,k} d_k) + \sum_{k=0}^{T} W_k^* P_k W_k. \quad (18)$$

where here and in the sequel we use the following simplifying notations: $\mathcal{X} = (X_1^*, X_2^*, \dots, X_{T-1}^*)^* = \text{vec}\{X_k\}_{k=1,\dots,T-1} \in \mathbb{R}^{n(\alpha-1)(T-1)}$, $d = \text{vec}\{d_k\}_{k=1,\dots,T-1} \in \mathbb{R}^{n(T-1)}$, $\mathcal{W} = \text{vec}\{W_k\}_{k=1,\dots,T} \in \mathbb{R}^{n(\alpha-1)T}$ and the symmetric matrices $\mathcal{Q} = \text{diag}(Q_1, Q_2, \dots, Q_{T-1}) \in \mathbb{R}^{n(\alpha-1)(T-1)\times n(\alpha-1)(T-1)}$, $\mathcal{Q}_b = \text{diag}(Q_{b,1}, Q_{b,2}, \dots, Q_{b,(T-1)})) \in \mathbb{R}^{n(T-1)\times n(T-1)}$ $\mathcal{P} = \text{diag}(P_1, P_2, \dots, P_{T-1}) \in \mathbb{R}^{n(\alpha-1)(T-1)\times n(\alpha-1)(T-1)}$.

With these notations the criterion can be written as:

$$\begin{aligned} 2J(d, \mathcal{W}) &= X_T^* K_f X_T + d^*\mathcal{Q}_b d + \mathcal{X}^*\mathcal{Q}\mathcal{X} + \mathcal{W}^*\mathcal{P}\mathcal{W} \\ &= \langle X_T, K_f X_T\rangle + \langle d, \mathcal{Q}_b d\rangle \\ &\quad + \langle \mathcal{X}, \mathcal{Q}\mathcal{X}\rangle + \langle \mathcal{W}, \mathcal{P}\mathcal{W}\rangle, \quad (19) \end{aligned}$$

where the scalar products $\langle . \rangle$ have to be interpreted in the appropriate euclidean spaces. Thus, the solution written in compact notation is

$$\mathcal{X} = \varphi_0 + \mathcal{D}_b d + \mathcal{D}\mathcal{W}, \quad (20)$$

where the vector φ_0 and the linear operators $\mathcal{D}_b, \mathcal{D}$ have the following matrix representations

$$\begin{aligned} \varphi_0 &= \text{vec}\left((\mathcal{A}\mathcal{B}_0)^k\right)_{k=1,\dots,T-1} F_0 \\ &\quad + \text{vec}\left(\mathcal{A}\mathcal{B}_0)^k\right)_{k=0,\dots,T-2} \mathcal{A}\mathbb{B}d_0, \quad (21) \end{aligned}$$

$$\mathcal{D}_b = \mathbb{A} + \mathcal{D}_b' \quad (22)$$

with $\mathcal{D}_b'$ defined as:

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \mathcal{A}[\mathcal{B}_0\mathbb{A} + \mathbb{B}] & 0 & 0 & \cdots & 0 & 0 \\ \mathcal{A}\mathcal{B}_0\mathcal{A}[\mathcal{B}_0\mathbb{A} + \mathbb{B}] & \mathcal{A}[\mathcal{B}_0\mathbb{A} + \mathbb{B}] & 0 & \cdots & 0 & 0 \\ (\mathcal{A}\mathcal{B}_0)^2\mathcal{A}[\mathcal{B}_0\mathbb{A} + \mathbb{B}] & \mathcal{A}\mathcal{B}_0\mathcal{A}[\mathcal{B}_0\mathbb{A} + \mathbb{B}] & \mathcal{A}[\mathcal{B}_0\mathbb{A} + \mathbb{B}] & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (\mathcal{A}\mathcal{B}_0)^{T-3}\mathcal{A}[\mathcal{B}_0\mathbb{A} + \mathbb{B}] & \cdots & \cdots & \cdots & \mathcal{A}[\mathcal{B}_0\mathbb{A} + \mathbb{B}] & 0 \end{pmatrix}$$

$$\mathcal{D} = \begin{pmatrix} \mathcal{A}\mathcal{E} & 0 & \cdots & 0 \\ (\mathcal{A}\mathcal{B}_0)^2\mathcal{A}\mathcal{E} & \mathcal{A}\mathcal{E} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (\mathcal{A}\mathcal{B}_0)^{T-2}\mathcal{A}\mathcal{E} & \cdots & \cdots & \mathcal{A}\mathcal{E} \end{pmatrix}. \quad (23)$$

With these notations, next we define a disturbed optimal boundary control where the deterministic disturbances are acting on the system. The disturbed or worst case optimal control guarantees to minimize the cost criterion for any unknown disturbance input. The problem to be dealt with here is closely related with a robust controller design problem, known as "$\mathcal{H}^\infty$−optimal control problem", where one studies a family of optimal control problems depending on a positive parameter λ, the attenuation level. Since in practice it turned out that this problem is difficult to solve one can study a weaker problem, the "disturbance attenuation" problem. In this case one does not consider a family of problems, but just a min-max game with fixed attenuation level λ. Therefore we define the following optimization problem:

Definition 5: We define a disturbed optimal boundary control of (18) under the constraints (16) in two steps:

1. Given a boundary control vector d then $\hat{\mathcal{W}} = \hat{\mathcal{W}}(d)$ is called a " worst case " disturbance with respect to d if

$$J(d, \hat{\mathcal{W}}(d)) \geq J(d, \mathcal{W}), \quad \text{for all} \quad \mathcal{W} \in \mathbb{R}^{n(\alpha-1)T}, \tag{24}$$

2. $\hat{d}$ is called an optimal boundary control of the disturbed system if

$$J(\hat{d}, \hat{\mathcal{W}}(\hat{d})) \leq J(d, \hat{\mathcal{W}}(d)) \quad \text{for all} \quad d \in \mathbb{R}^{n(T-1)}. \tag{25}$$

In other words the optimal boundary control $\hat{d}$ of the disturbed system is given as $\hat{d} = \arg\left[\min_d \max_{\mathcal{W}} J(d, \mathcal{W})\right]$.

In order to find a solution of this problem we first obtain conditions for existence of a unique worst case disturbance to any boundary control.

Theorem 6: To any fixed boundary control vector $d \in \mathbb{R}^{n(T-1)}$ there exists a unique worst case disturbance $\hat{\mathcal{W}}(d) \in \mathbb{R}^{n(\alpha-1)T}$ if and only if

$$\mathcal{P} + \mathcal{D}^*\mathcal{Q}\mathcal{D} < 0. \tag{26}$$

In this case the worst case disturbance is given by

$$\hat{\mathcal{W}}(d) = -(\mathcal{P} + \mathcal{D}^*\mathcal{Q}\mathcal{D})^{-1}(\mathcal{D}^*\mathcal{Q}(\mathcal{D}_b d + \varphi_0)). \tag{27}$$

Proof: Inserting (20) into (19), after a short calculation yields the quadratic functional in $\mathcal{W}$

$$\begin{aligned} 2J(d, \mathcal{W}) &= \langle \mathcal{W}, (\mathcal{P} + \mathcal{D}^*\mathcal{Q}\mathcal{D})\mathcal{W} \rangle \qquad (28) \\ &\quad + 2\langle \mathcal{W}, \mathcal{D}^*\mathcal{Q}(\varphi_0 + \mathcal{D}_b d) \rangle + J_0, \end{aligned}$$

where J_0 denotes those remaining terms not depending on $\mathcal{W}$. From classical theory of quadratic functionals we infer that this functional has a unique maximum if and only if (26) holds. This unique maximum, as solution of a parametric optimization problem, then is given by (27). ■

If we now assume that to any boundary control exists a unique worst case disturbance the final optimization problem to minimize the performance criterion can be solved. We obtain:

Theorem 7: If $\mathcal{P} + \mathcal{D}^*\mathcal{Q}\mathcal{D} < 0$ then there exists a unique disturbed optimal control of the system (16) if and only if

$$\mathcal{Q}_b - \mathcal{D}_b^*\mathcal{Q}\mathcal{D}(\mathcal{P} + \mathcal{D}^*\mathcal{Q}\mathcal{D})^{-1}\mathcal{D}^*\mathcal{Q}\mathcal{D}_b > 0 \tag{29}$$

This disturbed optimal control then is given by

$$\begin{aligned} \hat{d} &= -[\mathcal{Q}_b - \mathcal{D}_b^*\mathcal{Q}\mathcal{D}(\mathcal{P} + \mathcal{D}^*\mathcal{Q}\mathcal{D})^{-1} \qquad (30) \\ &\quad \times \mathcal{D}^*\mathcal{Q}\mathcal{D}_b]^{-1}\left(\mathcal{D}_b^*\mathcal{Q}\mathcal{D}(\mathcal{P} + \mathcal{D}^*\mathcal{Q}\mathcal{D})^{-1}\mathcal{D}^*\mathcal{Q}\varphi_0\right). \end{aligned}$$

Proof: Inserting (20) and (27) into (19) then after a short calculation we obtain

$$\begin{aligned} 2J(d, \hat{\mathcal{W}}(d)) &= \left\langle d, [\mathcal{Q}_b - \mathcal{D}_b^*\mathcal{Q}\mathcal{D}(\mathcal{P} + \mathcal{D}^*\mathcal{Q}\mathcal{D})^{-1}\mathcal{D}^*\mathcal{Q}\mathcal{D}_b]d \right\rangle \\ &\quad - 2\left\langle d, \mathcal{D}_b^*\mathcal{Q}\mathcal{D}(\mathcal{P} + \mathcal{D}^*\mathcal{Q}\mathcal{D})^{-1}\mathcal{D}^*\mathcal{Q}\varphi_0 \right\rangle + J_1, \end{aligned}$$

where the terms collected in J_1 are not depending on d. The conclusions of the theorem then can be deduced directly as before from this representation. ■

VI. Conclusions and Future Work

This work describes a study on the boundary control for repetitive processes with smoothing, since many applications find interpass smoothness a more suitable model. We concentrated on pass-controllability and observability since point controllability follows the classical approach. We also investigated conditions for the existence and uniqueness of the solution of a disturbed linear quadratic optimal problem with the same 2–D dynamics.

In the future we would like to study the same problem using an operator approach, as well as make a study of the stability for boundary control.

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