On a nonlinear two-directionally continuous repetitive process

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***Abstract*— In the paper we consider a nonlinear, continuous version of the well-known discrete and differential repetitive process. A two-directionally linear continuous version of repetitive process has been introduced in paper [9]. The aim of this paper is to introduce a nonlinear version of the system considered in [9] and to prove the fundamental results for such systems: the existence, uniqueness and the continuous dependence of solutions on functional parameters (controls), as it has been obtained in [9] for the linear system.**

I. INTRODUCTION

In the paper we consider a control system

$$\begin{cases} \frac{\partial^2 z}{\partial t \partial x}(t,x) = f^1(t,x,z(t,x), \frac{\partial z}{\partial t}(t,x), \frac{\partial z}{\partial x}(t,x), \\ \qquad\qquad w(t,x), k(t,x)) \\ \frac{\partial w}{\partial t}(t,x) = f^2(t,x,z(t,x), \frac{\partial z}{\partial t}(t,x), \\ \qquad\qquad w(t,x), l(t,x)) \\ \qquad\qquad \text{for a.e. } (t,x) \in P := [0,1] \times [0,1] \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} z(t,0) = 0 & \text{for } t \in [0,1] \\ w(0,x) = 0 & \text{for } x \in [0,1] \end{cases} \tag{2}$$

The parameters k and l are referred to as controls. We assume that k and l are summable functions ($k \in L^1(P, \mathbb{R}^{m_1})$, $l \in L^1(P, \mathbb{R}^{m_2})$) with values in the given compact sets M_1, M_2 i.e. $k(t,x) \in M_1, l(t,x) \in M_2$ for a.e. $(t,x) \in P$.

System (1) is a nonlinear, continuous version of the well-known discrete and differential repetitive process. Repetitive processes are important from the practical point of view; their can be applied, among others, in modeling of long-wall coal cutting, metal rolling operations [3], [4] and in the theory of ILC schemes [5].

II. PRELIMINARIES AND BASIC ASSUMPTIONS

For system (1) we define the spaces $AC_0^{t,x}(P, \mathbb{R}^{n_1})$ and $AC_0^t(P, \mathbb{R}^{n_2})$ of trajectories z and w respectively, where

$$AC_0^{t,x}(P, \mathbb{R}^{n_1}) := \{z : P \to \mathbb{R}^{n_1} : \\ \text{there exists } \zeta \in L^1(P, \mathbb{R}^{n_1}) \text{ such that} \\ z(t,x) = \int_0^t \int_0^x \zeta(t_1, x_1)\, dt_1 dx_1 \text{ for } (t,x) \in P\}$$

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and

$$AC_0^t(P, \mathbb{R}^{n_1}) := \{w : P \to \mathbb{R}^{n_2} : \\ \text{there exists } \omega \in L^1(P, \mathbb{R}^{n_2}) \text{ such that} \\ w(t,x) = \int_0^t \omega(t_1, x)\, dt_1 \text{ for } t \in [0,1] \text{ and a.e. } x \in [0,1]\}.$$

For a function $z \in AC_0^{t,x}(P, \mathbb{R}^{n_1})$ we have that

$$\frac{\partial z}{\partial t}(t,x) = \int_0^x \zeta(t, x_1)\, dx_1, \\ \frac{\partial z}{\partial x}(t,x) = \int_0^t \zeta(t_1, x)\, dt_1, \\ \frac{\partial^2 z}{\partial t \partial x}(t,x) = \int_0^t \int_0^x \zeta(t_1, x_1)\, dt_1 dx_1$$

for a.e. $(t,x) \in P$ and

$$z(0,x) = z(t,0) = 0 \text{ for } t, x \in [0,1]$$

(see [6] for the details).

Similarly, for a function $w \in AC_0^t(P, \mathbb{R}^{n_1})$ ave have that

$$\frac{\partial w}{\partial t}(t,x) = \omega(t,x)$$

for a.e. $(t,x) \in P$ and

$$w(0,x) = 0 \text{ for a.e. } x \in [0,1]$$

(see [10] for the details). Another important fact is that $AC_0^{t,x}(P, \mathbb{R}^{n_1})$ and $AC_0^t(P, \mathbb{R}^{n_1})$ are Banach spaces with the norms

$$\|z\|_{AC_0^{t,x}(P,\mathbb{R}^m)} = \iint_P \left| \frac{\partial^2 z}{\partial t \partial x}(t,x) \right| dxdt, \\ \|w\|_{AC_0^t(P,\mathbb{R}^m)} = \iint_P \left| \frac{\partial w}{\partial x}(t,x) \right| dxdt,$$

respectively (cf. [1], [2]).

We impose the following assumptions:

(A1) There exists a constant $L > 0$ such that

$$\left| f^1(t,x,z,z_1,z_2,w,k) - f^1(t,x,\bar{z},\bar{z}_1,\bar{z}_2,\bar{w},k) \right| \\ \leq L(|z - \bar{z}| + |z_1 - \bar{z}_1| + |z_2 - \bar{z}_2| + |w - \bar{w}|)$$

and

$$\left| f^2(t,x,z,z_1,w,l) - f^2(t,x,\bar{z},\bar{z}_1,\bar{w},l) \right| \\ \leq L(|z - \bar{z}| + |z_1 - \bar{z}_1| + |w - \bar{w}|)$$

for a.e. $(t,x) \in P$, and any $z, z_1, z_2, \bar{z}, \bar{z}_1, \bar{z}_2 \in \mathbb{R}^{n_1}, w, \bar{w} \in \mathbb{R}^{n_2}, k \in \mathbb{R}^{m_1}, l \in \mathbb{R}^{m_2}$.

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(A2) The functions $f^1(\cdot,\cdot,z,z_1,z_2,w,k)$ and $f^2(\cdot,\cdot,z,z_1,w,l)$ are measurable on P for any $z,z_1,z_2\in\mathbb{R}^{n_1}, w\in\mathbb{R}^{n_2}, k\in\mathbb{R}^{m_1}, l\in\mathbb{R}^{m_2}$.

(A3) The functions $f^1(t,x,z,z_1,z_2,w,\cdot)$ and $f^2(t,x,z,z_1,w,\cdot)$ are continuous on $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$ resp. for a.e $(t,x)\in P, z,z_1,z_2\in\mathbb{R}^{n_1}, w\in\mathbb{R}^{n_2}$.

(A4) There exists a constant $b>0$ such that

$$\left|f^1(t,x,0,0,0,0,k)\right|+\left|f^2(t,x,0,0,0,l)\right|\le b$$

for a.e $(t,x)\in P, k\in M_1, l\in M_2$.

III. MAIN RESULTS

We have the following theorem on the existence of solutions to (1)-(2).

Theorem 1: For any controls k and l there exists an unique solution $(z,w)_{k,l}\in AC_0^{t,x}(P,\mathbb{R}^{n_1})\times AC_0^{t}(P,\mathbb{R}^{n_2})$ to (1)-(2) corresponding to the controls.

Proof: Fix $k\in L^1(P,\mathbb{R}^{m_1})$ and $l\in L^1(P,\mathbb{R}^{m_2})$. Define the operator

$$\Lambda: L^1(P,\mathbb{R}^{n_1})\times L^1(P,\mathbb{R}^{n_2}) \to L^1(P,\mathbb{R}^{n_1})\times L^1(P,\mathbb{R}^{n_2})$$

$$\Lambda(\zeta,\omega)(t,x)=(\Lambda_1(\zeta,\omega)(t,x),\Lambda_2(\zeta,\omega)(t,x)), \quad (3)$$

where

$$\Lambda_1(\zeta,\omega)(t,x) := f^1\Big(t,x,\int_0^t\int_0^x\zeta(t_1,x_1)\,dt_1dx_1,\int_0^x\zeta(t,x_1)\,dx_1, \int_0^t\zeta(t_1,x)\,dt_1,\int_0^t\omega(t_1,x)\,dt_1,k(t,x)\Big) \quad (4)$$

and

$$\Lambda_2(\zeta,\omega)(t,x) := f^2\Big(t,x,\int_0^t\int_0^x\zeta(t_1,x_1)\,dt_1dx_1, \int_0^x\zeta(t,x_1)\,dx_1,\int_0^t\omega(t_1,x)\,dt_1,l(t,x)\Big). \quad (5)$$

Using the Banach contraction principle we show that Λ has a unique fixed point $(\zeta^*,\omega^*)\in L^1(P,\mathbb{R}^{n_1})\times L^1(P,\mathbb{R}^{n_2})$ which implies immediately that the pair $(z^*,w^*)\in AC_0^{t,x}(P,\mathbb{R}^{n_1})\times AC_0^{t}(P,\mathbb{R}^{n_2})$ defined by the formula

$$z^*(t,x)=\int_0^t\int_0^x\zeta^*(t_1,x_1)\,dt_1dx_1, \quad w^*(t,x)=\int_0^t\omega^*(t_1,x)\,dt_1 \quad \text{for a.e. } (t,x)\in P \quad (6)$$

is the unique solution to (1)-(2). To apply the Banach contraction principle we show that Λ is contractive with respect to the following norm defined by Bielecki (see [8])

$$\|(\zeta,\omega)\|_\beta := \iint_P e^{-\beta(t+x)}|\zeta(t,x)|\,dtdx + \iint_P e^{-\beta(t+x)}|\omega(t,x)|\,dtdx, \quad (7)$$

where $\beta>0$ is a given constant. The above norm is equivalent to the classical norm in L^1.

By direct calculation (integrating by parts) we get

$$\begin{aligned}
&\left\|\Lambda(\zeta,\omega)-\Lambda(\bar\zeta,\bar\omega)\right\|_\beta \\
&=\left\|\left(\Lambda_1(\zeta,\omega)-\Lambda_1(\bar\zeta,\bar\omega),\Lambda_2(\zeta,\omega)-\Lambda_2(\bar\zeta,\bar\omega)\right)\right\|_\beta \\
&=\iint_P e^{-\beta(t+x)}\left|\Lambda_1(\zeta,\omega)(t,x)-\Lambda_1(\bar\zeta,\bar\omega)(t,x)\right|dtdx \\
&+\iint_P e^{-\beta(t+x)}\left|\Lambda_2(\zeta,\omega)(t,x)-\Lambda_2(\bar\zeta,\bar\omega)(t,x)\right|dtdx \\
&\le\left(\frac{2L}{\beta^2}+\frac{3L}{\beta}\right)\iint_P e^{-\beta(t+x)}\left|\zeta(t,x)-\bar\zeta(t,x)\right|dtdx \\
&\quad+\frac{2L}{\beta}\iint_P e^{-\beta(t+x)}|\omega(t,x)-\bar\omega(t,x)|\,dtdx \\
&\le\left(\frac{2L}{\beta^2}+\frac{3L}{\beta}\right)\left\|(\zeta,\omega)-(\bar\zeta,\bar\omega)\right\|_\beta
\end{aligned}$$

Choosing β such that $\frac{2L}{\beta^2}+\frac{3L}{\beta}<1$ we get that the operator Λ is contracting, thus it possesses exactly one fixed point $(\zeta^*,\omega^*)\in L^1(P,\mathbb{R}^{n_1})\times L^1(P,\mathbb{R}^{n_2})$. Consequently, applying (6) we get the unique solution to (1)-(2). ■

Further by $(z,w)_{k,l}\in AC_0^{t,x}(P,\mathbb{R}^{n_1})\times AC_0^{t}(P,\mathbb{R}^{n_2})$ we shall denote the unique solution to (1)-(2) corresponding to the controls $k\in L^1(P,\mathbb{R}^{m_1}), l\in L^1(P,\mathbb{R}^{m_2})$.

We have that the solution $(z,w)_{k,l}$ depends continuously on k and l.

Theorem 2: Let $\{(k_s,l_s)\}_{s\in\mathbb{N}}\subset L^1(P,\mathbb{R}^{m_1})\times L^1(P,\mathbb{R}^{m_2})$ be a sequence of controls such that $(k_s,l_s)\to(k_0,l_0)$ in the space $L^1(P,\mathbb{R}^{m_1})\times L^1(P,\mathbb{R}^{m_2})$ as $s\to\infty$. Then $(z_s,w_s)\to(z_0,w_0)$ in the space $AC_0^{t,x}(P,\mathbb{R}^{n_1})\times AC_0^{t}(P,\mathbb{R}^{n_2})$ as $s\to\infty$, where (z_s,w_s) denotes the solution to (1)-(2) corresponding to the control (k_s,l_s) for $s=0,1,2,\ldots$.

Proof: Let Λ_s denotes the operator defined by (3)-(5) with the functions k_s and l_s, and let (ζ_s^*,ω_s^*) be the unique fixed point of Λ_s for $s=0,1,2,\ldots$. We have that

$$z_s^*(t,x)=\int_0^t\int_0^x\zeta_s^*(t_1,x_1)\,dt_1dx_1, \quad w_s^*(t,x)=\int_0^t\omega_s^*(t_1,x)\,dt_1 \quad \text{for a.e. } (t,x)\in P,\ s=0,1,2,\ldots.$$

From the proof of theorem 1 it follows that the operator Λ_s is contractive with the constant $\lambda:=\frac{2L}{\beta^2}+\frac{3L}{\beta}<1$ and the norm $\|\cdot\|_\beta$. Thus, we have

$$\begin{aligned}
&\|(\zeta_s^*,\omega_s^*)-(\zeta_0^*,\omega_0^*)\|_\beta \\
&=\|\Lambda_s(\zeta_s^*,\omega_s^*)-\Lambda_0(\zeta_0^*,\omega_0^*)\|_\beta \\
&\le\|\Lambda_s(\zeta_s^*,\omega_s^*)-\Lambda_s(\zeta_0^*,\omega_0^*)\|_\beta \\
&+\|\Lambda_s(\zeta_0^*,\omega_0^*)-\Lambda_0(\zeta_0^*,\omega_0^*)\|_\beta \\
&\le\lambda\|(\zeta_s^*,\omega_s^*)-(\zeta_0^*,\omega_0^*)\|_\beta \\
&\quad+\|\Lambda_s(\zeta_0^*,\omega_0^*)-\Lambda_0(\zeta_0^*,\omega_0^*)\|_\beta\,.
\end{aligned}$$

Consequently

$$\|(\zeta_s^*,\omega_s^*)-(\zeta_0^*,\omega_0^*)\|_\beta \le \frac{1}{1-\lambda}\|\Lambda_s(\zeta_0^*,\omega_0^*)-\Lambda_0(\zeta_0^*,\omega_0^*)\|_\beta . \quad (8)$$

Since for $(\zeta,\omega)\in L^1(P,\mathbb{R}^{n_1})\times L^1(P,\mathbb{R}^{n_2})$

$$e^{-2\beta}\|(\zeta,\omega)\|\le\|(\zeta,\omega)\|_\beta\le\|(\zeta,\omega)\|,$$

where $\|\cdot\|$ denotes the standard norm in $L^1(P,\mathbb{R}^{n_1})\times L^1(P,\mathbb{R}^{n_2})$, then by (8)

$$\begin{aligned}\|z_s^*-z_0^*\|_{AC_0^{t,x}}+\|w_s^*-w_0^*\|_{AC_0^t} &= \iint_P|\zeta_s^*(t,x)-\zeta_0^*(t,x)|\,dtdx \\ &+\iint_P|\omega_s^*(t,x)-\omega_0^*(t,x)|\,dtdx \\ &\le e^{2\beta}\iint_P e^{-\beta(t+x)}|\zeta_s^*(t,x)-\zeta_0^*(t,x)|\,dtdx \\ &+e^{2\beta}\iint_P e^{-\beta(t+x)}|\omega_s^*(t,x)-\omega_0^*(t,x)|\,dtdx \\ &= e^{2\beta}\|(\zeta_s^*,\omega_s^*)-(\zeta_0^*,\omega_0^*)\|_\beta \\ &\le \frac{e^{2\beta}}{1-\lambda}\|\Lambda_s(\zeta_0^*,\omega_0^*)-\Lambda_0(\zeta_0^*,\omega_0^*)\|_\beta \\ &\le \frac{e^{2\beta}}{1-\lambda}\|\Lambda_s(\zeta_0^*,\omega_0^*)-\Lambda_0(\zeta_0^*,\omega_0^*)\|.\end{aligned}$$

Since $k_s(t,x)\in M_1, l_s(t,x)\in M_2$ for a.e. $(t,x)\in P$, $s=0,1,2,\ldots$, where M_1,M_2 are compact then by (A2)-(A3) we get applying Krasnosielski'a theorem that $\|\Lambda_s(\zeta_0^*,\omega_0^*)-\Lambda_0(\zeta_0^*,\omega_0^*)\|_\beta\to0$ as $s\to\infty$, which completes the proof. ■

Now, by $\mathcal{S}$ let us denote the family of all solutions to (1)-(2) parametrizied by (k,l). More precisely,

$$\mathcal{S}:=\{(z,w)_{k,l}\in AC_0^{t,x}(P,\mathbb{R}^{n_1})\times AC_0^t(P,\mathbb{R}^{n_2}): (k,l)\in L^1(P,\mathbb{R}^{m_1})\times L^1(P,\mathbb{R}^{m_2})\}.$$

We have the following

Lemma 1: The family $\mathcal{S}$ is equibounded on P, i.e. there exists a constant $c>0$ such that for any control (k,l) we have

$$|z_{k,l}(t,x)|\le c,\ \text{for}\ (t,x)\in P\ \text{and}\quad |w_{k,l}(t,x)|\le c,\ \text{for a.e.}\ (t,x)\in P$$

Moreover, for any control (k,l) we have that

$$\left|\frac{\partial^2 z_{k,l}(t,x)}{\partial t\partial x}\right|\le c\ \text{and}\ \left|\frac{\partial w_{k,l}(t,x)}{\partial t}\right|\le c \quad \text{for a.e.}\ (t,x)\in P,$$

(where c does not depend on (k,l)).

Proof: Fix $(k,l)\in L^1(P,\mathbb{R}^{m_1})\times L^1(P,\mathbb{R}^{m_2})$. As it was showed in the proof of theorem 1 the solution $(z,w)_{k,l}$ is defined by formula

$$z_{(k,l)}(t,x)=\int_0^t\int_0^x\zeta^*(t_1,x_1)\,dt_1dx_1,\quad w_{(k,l)}(t,x)=\int_0^t\omega^*(t_1,x)\,dt_1 \quad \text{for a.e. } (t,x)\in P$$

where (ζ^*,ω^*) is the unique fixed point of the operator Λ defined by (3)-(5). From the proof of the Banach contraction principle it follows that

$$(\zeta^*,\omega^*)=\lim_{s\to\infty}\Lambda^s(0,0)\quad(\text{in } L^1(P,\mathbb{R}^{m_1})\times L^1(P,\mathbb{R}^{m_2})),$$

where Λ^s denotes, the $s-th$ superposition of Λ. We have that for $(t,x)\in P$ and $s\ge2$

$$|\Lambda^s(0,0)(t,x)| \le \sum_{i=2}^{s}\left|\Lambda^s(0,0)(t,x)-\Lambda^{s-1}(0,0)(t,x)\right| + |\Lambda(0,0)|(t,x).$$

Moreover it can be proved by arduous calculation that

$$\left|\Lambda^s(0,0)(t,x)-\Lambda^{s-1}(0,0)(t,x)\right| \le 2c_1\sum_{i=2}^{s}\frac{b\cdot L^{s-2}\cdot4^{s-2}}{\left[\frac{s-2}{2}\right]!}$$

where $c_1>0$ does not depend on (t,x) nor (k,l). Consequently, we get that

$$|(\zeta^*,\omega^*)(t,x)|\le c,\ \text{for a.e.}\ (t,x)\in P.$$

Therefore we get immediately that

$$\left|\frac{\partial^2 z_{k,l}(t,x)}{\partial t\partial x}\right|=|\zeta^*(t,x)|\le c,$$

$$\left|\frac{\partial w_{k,l}(t,x)}{\partial t}\right|=|\omega^*(t,x)|\le c,$$

$$\left|z_{(k,l)}(t,x)\right|\le\int_0^t\int_0^x|\zeta^*(t_1,x_1)|\,dt_1dx_1 \le\int_0^1\int_0^1|\zeta^*(t_1,x_1)|\,dt_1dx_1\le c,$$

$$|w_{k,l}(t,x)|\le\int_0^t|\omega^*(t_1,x)|\,dt_1\le\int_0^1|\omega^*(t_1,x)|\,dt_1\le c,$$

for a.e. $(t,x)\in P$. ■

In the proof of the next theorem we use the following Arzeli-Ascola theorem for absolutely continuous functions of two variables (see [9]).

Lemma 2: From any sequence $\{z_s\}_{s\in\mathbb{N}}\subset AC_0^{t,x}(P,\mathbb{R}^n)$ which is equibounded together with the sequence of the mixed derivatives $\left\{\frac{\partial^2 z_s}{\partial t\partial x}\right\}_{s\in\mathbb{N}}$ one can choose a subsequence $\{z_{s_i}\}_{i\in\mathbb{N}}$ which is uniformly convergent on P to some absolutely continuous function $z_0\in AC_0^{t,x}(P,\mathbb{R}^n)$.

Theorem 3: For any sequence $\{(z_s, w_s)\}_{s\in\mathbb{N}} \subset \mathcal{S}$ there exists a subsequence $\{(z_{s_i}, w_{s_i})\}_{i\in\mathbb{N}} \subset \mathcal{S}$ and a function $(z_0, w_0) \in AC_0^{t,x}(P, \mathbb{R}^{n_1}) \times AC_0^t(P, \mathbb{R}^{n_2})$ such that $z_{s_i} \to z_0$ uniformly and $w_{s_i} \rightharpoonup w_0$ weakly in $L^1(P, \mathbb{R}^{n_2})$ as $i \to \infty$.

Proof: Let $\{(z_s, w_s)\}_{s\in\mathbb{N}} \subset \mathcal{S}$ be any sequence. The fact that there is a subsequence $\{z_{s_i}\}_{i\in\mathbb{N}} \subset \mathcal{S}$ such that $z_{s_i} \rightrightarrows z_0 \in AC_0^{t,x}(P, \mathbb{R}^{n_1})$ as $i \to \infty$ follows immediately from lemmas 1 and 2. Let $\{\omega_s\}_{s\in\mathbb{N}} \subset L^1(P, \mathbb{R}^{n_2})$ be a sequence such that

$$w_s(t,x) = \int_0^t \omega_s(t_1, x)\, dt_1$$

for a.e $(t,x) \in P$ and $s \in \mathbb{N}$. By lemma 1 then the sequence $\left\{\frac{\partial w_s}{\partial t}\right\}_{s\in\mathbb{N}}$ is bounded in $L^2(P, \mathbb{R}^{n_2})$. From the reflexivity of $L^2(P, \mathbb{R}^{n_2})$ it follows, that there exists a function $\sigma \in L^2(P, \mathbb{R}^{n_2}) \subset L^1(P, \mathbb{R}^{n_2})$ and a subsequence $\{w_{s_i}\}_{i\in\mathbb{N}}$ such that $\frac{\partial w_{s_i}}{\partial t} \rightharpoonup \sigma$ weakly in $L^2(P, \mathbb{R}^{n_2})$ as $i \to \infty$. The weak convergence in $L^2(P, \mathbb{R}^{n_2})$ implies the weak conference in $L^1(P, \mathbb{R}^{n_2})$. Since the operator

$$\mathcal{T} : L^1(P, \mathbb{R}^{n_2}) \to L^1(P, \mathbb{R}^{n_2})$$

$$\mathcal{T}(g)(t,x) = \int_0^t g(t_1, x)\, dx$$

is linear and continuous then $w_{s_i} = \mathcal{T}\left(\frac{\partial w_{si}}{\partial t}\right) \rightharpoonup \mathcal{T}(\sigma)$ weakly in $L^1(P, \mathbb{R}^{n_2})$. Adopting

$$w_0(t,x) := \mathcal{T}(\sigma)(t,x) = \int_0^t \sigma(t_1, x)\, dt_1$$

we get that $w_{s_i} \rightharpoonup w_0 \in AC_0^t(P, \mathbb{R}^{n_2})$ weakly in $L^1(P, \mathbb{R}^{n_2})$ as $i \to \infty$. ■

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