A polynomial-algebraic approach to Lyapunov stability analysis of higher-order 2-D systems

Paolo Rapisarda, Kiyotsugu Takaba and Chiaki Kojima

Abstract— **We introduce a four-variable polynomial matrix equation which plays an essential role in the stability analysis of discrete 2-D systems and in the computation of Lyapunov functions for such systems; we call this the 2-D polynomial Lyapunov equation (2-D PLE). We also give necessary and sufficient conditions for the stability of "square" 2-D systems based on solutions of the 2-D PLE satisfying additional properties.**

I. PROBLEM STATEMENT

The central object of interest in this paper is the following four-variable polynomial matrix equation:

$$\begin{aligned}&(1-\zeta_1\eta_1)\Psi_1(\zeta_1,\zeta_2,\eta_1,\eta_2)\\&+(1-\zeta_2\eta_2)\Psi_2(\zeta_1,\zeta_2,\eta_1,\eta_2)\\&=-\Delta(\zeta_1,\zeta_2,\eta_1,\eta_2)+Y(\eta_1,\eta_2,\zeta_1,\zeta_2)^\top R(\eta_1,\eta_2)\\&\ +R(\zeta_1,\zeta_2)^\top Y(\zeta_1,\zeta_2,\eta_1,\eta_2)\ ,\end{aligned} \tag{1}$$

where Δ and R are given square polynomial matrices respectively in the four indeterminates $\zeta_1,\zeta_2,\eta_1,\eta_2$ and in the two variables ξ_1,ξ_2; and Ψ_i, $i=1,2$ and Y are unknown square polynomial matrices in the four indeterminates $\zeta_1,\zeta_2,\eta_1,\eta_2$. For reasons which are made apparent later on in the paper, we call (1) the (discrete) *2-D polynomial Lyapunov equation*, often abbreviated as *2-D PLE* in the following. The purpose of this paper is to show how the 2-D PLE arises in the context of stability analysis of discrete 2-D systems, and to discuss its role in the computation of Lyapunov functions for discrete 2-D systems. The setting for our investigation is the behavioral approach to 2-D systems pioneered in [14] and successively studied by several other authors; moreover, we use the notion of stability for discrete 2-D systems introduced in [15]. In order to make the paper as self-contained as possible, we will summarize the essential background concepts and definitions in section II. In section III we state the main result of this paper, a characterization of 2-D stability in terms of solutions Ψ_1,Ψ_2 to the equation (1) satisfying some additional properties. The paper ends with some concluding remarks, contained in section IV.

Notation: We denote with $\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1,\xi_2]$ (respectively, $\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$) the set of all $\mathtt{r}\times\mathtt{w}$ matrices with entries in the ring $\mathbb{R}[\xi_1,\xi_2]$ of polynomials in 2 indeterminates, with real coefficients (respectively in the ring $\mathbb{R}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$ of Laurent polynomials in 2 indeterminates with real coefficients). Given a nonzero Laurent polynomial $p(\xi_1,\xi_2)=\sum_{\ell,m}p_{\ell,m}\xi_1^\ell\xi_2^m\in\mathbb{R}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$, the *Laurent variety* of p is defined as

$$\mathcal{V}_L(p):=\{(\alpha,\beta)\in\mathbb{C}\times\mathbb{C}\mid\alpha\beta\neq 0,p(\alpha,\beta)=0\}$$

This definition extends to sets $\mathcal{I}$ of Laurent polynomials, with $\mathcal{V}(\mathcal{I})$ being the intersection of the Laurent varieties of all polynomials in the set. Let $R\in\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$ have full column rank (as a rational matrix); then its *characteristic ideal* is the ideal of $\mathbb{R}[\xi_1,\xi_2]$ generated by the determinants of all $\mathtt{w}\times\mathtt{w}$ minors of R, and its *characteristic variety* is the set of roots common to all polynomials in the ideal. Further properties and definitions, such as the concept of *right factor-prime* two-variable polynomial matrix used in the following, can be found in [2].

A set $\mathcal{K}\subset\mathbb{R}\times\mathbb{R}$ is called a *cone* if $\alpha\mathcal{K}\subset\mathcal{K}$ for all $\alpha\geq 0$. A cone is *solid* if it contains an open ball in $\mathbb{R}\times\mathbb{R}$, and *pointed* if $\mathcal{K}\cap-\mathcal{K}=\{(0,0)\}$. A cone is *proper* if it is closed, pointed, solid, and convex. It is easy to see that a proper cone is uniquely identified as the set of nonnegative linear combinations of two linearly independent vectors $v_1,v_2\in\mathbb{R}^2$, called the *generating vectors* of the cone. In the following we will often consider the intersection of a cone $\mathcal{K}$ with $\mathbb{Z}\times\mathbb{Z}$; whenever it will be clear from the context, we will be denoting this set with $\mathcal{K}$ instead of with $\mathcal{K}\cap\mathbb{Z}\times\mathbb{Z}$.

We denote with $\overline{\mathcal{P}_1}$ the closed unit polydisk:

$$\overline{\mathcal{P}_1}:=\{(\alpha,\beta)\in\mathbb{C}\times\mathbb{C}\mid|\alpha|\leq 1,|\beta|\leq 1\}$$

Given a set $\mathcal{S}\subset\mathbb{Z}\times\mathbb{Z}$, its *(discrete) convex hull* is the intersection of the convex hull of $\mathcal{S}$ (seen as a subset of $\mathbb{R}\times\mathbb{R}$) and of $\mathbb{Z}\times\mathbb{Z}$. In the following we will also refer to the (discrete) convex hull associated with an element $p\in\mathbb{R}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$, meaning the (discrete) convex hull of the *support of* p, i.e. the set

$$\begin{aligned}\text{supp}(p)\ :=\ &\{(h,k)\in\mathbb{Z}\times\mathbb{Z}\mid\ \text{the coefficient of } \xi_1^h\xi_2^k\\&\ \text{in } p(\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1})\ \text{is }\neq 0\}\end{aligned}$$

We denote with $\mathbb{W}^{\mathbb{T}}$ the set consisting of all trajectories from $\mathbb{T}$ to $\mathbb{W}$. We denote with σ_1, σ_2 the *shift operators* defined as

$$\begin{aligned}&\sigma_i:(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}}\to(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}}\ \ i=1,2\\&(\sigma_1 w)(x_1,x_2):=w(x_1-1,x_2)\\&(\sigma_2 w)(x_1,x_2):=w(x_1,x_2-1)\end{aligned}$$

P. Rapisarda is with the Information: Signals, Images, Systems group, School of Electronics and Computer Science, University of Southampton, UK pr3@ecs.soton.ac.uk

K. Takaba is with the Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Japan takaba@amp.i.kyoto-u.ac.jp

C. Kojima is with the Department of Information Physics and Computing, Graduate School of Information Science and Technology, The University of Tokyo, Japan chiaki_kojima@ipc.i.u-tokyo.ac.jp

ISBN 978-963-311-370-7

II. BACKGROUND MATERIAL

A. 2-D behaviors

We call $\mathfrak{B}$ a *linear discrete-time complete* 2-D *behavior* if it is the solution set of a system of linear, constant-coefficient difference equations with two independent variables; more precisely, if $\mathfrak{B}$ is the subset of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}}$ consisting of all solutions to

$$R(\sigma_1,\sigma_2)w = 0 \tag{2}$$

where $R \in \mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$. We call (2) a *kernel representation* of $\mathfrak{B}$. We denote the set consisting of all linear discrete-time complete 2-D behaviors with $\mathtt{w}$ external variables with $\mathcal{L}_2^{\mathtt{w}}$.

$\mathfrak{B} \in \mathcal{L}_2^{\mathtt{w}}$ is *autonomous* if there exists a proper cone $\mathcal{K} \subset \mathbb{R}\times\mathbb{R}$ such that

$$\left[w_1, w_2 \in \mathfrak{B} \text{ and } w_{1|\mathcal{K}\cap\mathbb{Z}\times\mathbb{Z}} = w_{2|\mathcal{K}\cap\mathbb{Z}\times\mathbb{Z}}\right] \Longrightarrow [w_1 = w_2]$$

Such a cone $\mathcal{K}\cap\mathbb{Z}\times\mathbb{Z}$ will be called a proper *characteristic* cone for $\mathfrak{B}$. Intuitively, we can look at the characteristic cone $\mathcal{K}$ as the "past"; then a behavior is autonomous if any two trajectories whose values in the past coincide, are equal. Note that this implies that the behavior has no "inputs", see [14].

Proper characteristic cones play an important role in the definition of stability of a 2-D system according to Valcher, and we now proceed to characterize them algebraically, following closely the original source [15]. The following result holds.

Theorem 1: Let $\mathfrak{B} \in \mathcal{L}_2^{\mathtt{w}}$ be autonomous, and let $\mathfrak{B} = \ker R(\sigma_1,\sigma_2)$ for some $R \in \mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$. Assume there exist $H \in \mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$ right factor prime, and $S \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$ nonsingular, such that $R = H\cdot S$.

Moreover, denote $\delta := \det(S) \in \mathbb{R}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$. The following statements are equivalent:

1) The proper cone $\mathcal{K}$ is characteristic for $\mathfrak{B}$;
2) The proper cone $\mathcal{K}$ is characteristic for $\ker\ S(\sigma_1,\sigma_2)$;
3) The proper cone $\mathcal{K}$ is characteristic for $\ker\ \delta(\sigma_1,\sigma_2)$;
4) The discrete convex hull $\mathcal{H}_\delta$ of δ satisfies the following two conditions:
   - $4a.$ $-\mathcal{H}_\delta \subset \mathcal{K}$;
   - $4.b.$ $-\mathcal{H}_\delta \subset \mathcal{K}$ intersects the generating lines of $\mathcal{K}$ only in $(0,0)$.

If $\mathfrak{B}$ is autonomous, and $\mathfrak{B} = \ker\ R(\sigma_1,\sigma_2)$ for some square nonsingular Laurent matrix R, then $\mathfrak{B}$ is called a *square autonomous behavior*; this is the class of behaviors we will be considering in this paper.

Intuitively, stability of an autonomous behavior corresponds to the trajectories dying out in some "future cone", given arbitrary initial conditions in the "past characteristic cone"; note also that given a characteristic cone $\mathcal{K}$, it is natural to consider as "future cone" the cone $-\mathcal{K}$. However, in the square autonomous case the set of points in which a trajectory can be freely assigned is infinite, and consequently it may happen that particular choices of the "initial conditions" correspond to trajectories of the behavior which do not die out within a proper characteristic cone $\mathcal{K}$: a sharper definition is in order. To state it, we need to introduce the following notation: given a proper cone $\mathcal{K}$, we denote with $\delta(-\mathcal{K})$ the *boundary* of $-\mathcal{K}$, i.e. the generating lines of $-\mathcal{K}$. Moreover, we denote with $(\delta(-\mathcal{K}))^n$ the set consisting of the points of $\mathbb{Z}\times\mathbb{Z}$ whose distance from $\delta(-\mathcal{K})$ is less than n:

$$\begin{aligned}&(\delta(-\mathcal{K}))^n := \{(i,j)\in\mathbb{Z}\times\mathbb{Z}\mid\\ &\min\{|i-h|+|j-k| \text{ with } (h,k)\in\delta(-\mathcal{K})\}\le n\}\end{aligned}$$

The definition of $\mathcal{K}$-stable square autonomous behavior is as follows.

Definition 2: Let $\mathcal{K}$ be a proper characteristic cone such that $-\mathcal{K}$ is characteristic for a square autonomous behavior $\mathfrak{B} \in \mathcal{L}_2^{\mathtt{w}}$. $\mathfrak{B}$ is $\mathcal{K}$-stable if there exists some positive integer n such that

$$\begin{aligned}&[w\in\mathfrak{B}, w \text{ bounded in } (\delta(-\mathcal{K}))^n]\\ &\Longrightarrow \left[\lim_{\substack{(i,j)\in\mathcal{K}\\ |i|+|j|\to+\infty}} \|w(i,j)\| = 0\right]\end{aligned}$$

The following is an algebraic characterization of $\mathcal{K}$-stability ; in order to avoid cumbersome details, in the following we often emulate [15], and only consider proper cones generated by unimodular integer matrices, which are then isomorphic to the first orthant of $\mathbb{Z}\times\mathbb{Z}$, in the sense that there exists a nonsingular square matrix $T:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}$ such that $T(\mathcal{K})$ is the first orthant.

Theorem 3: Let $\mathfrak{B} = \ker\ S(\sigma_1,\sigma_2)$ be a square autonomous behavior, and let $\mathcal{K}$ be a proper characteristic cone for $\mathfrak{B}$ which is T-isomorphic to the first orthant. Denote $\delta := \det(S)$, and assume w.l.o.g. that $\mathcal{H}_\delta \subset \mathcal{K}$ and that $\mathcal{H}_\delta \cap \delta\mathcal{K} = \{(0,0)\}$. Denote with $(t_1(\ell,m), t_2(\ell,m))$ the image of $(\ell,m)\in\mathbb{Z}\times\mathbb{Z}$ under T. Define

$$S_T(\xi_1,\xi_2) := \sum_{\ell,m} S_{\ell,m}\xi_1^{t_1(\ell,m)}\xi_2^{t_2(\ell,m)}$$

Then the following two statements are equivalent:

1) $\mathfrak{B}$ is $\mathcal{K}$-stable;
2) The Laurent variety of $\det\ S_T$ does not intersect the closed unit polydisk $\overline{\mathcal{P}_1}$.

Proof: See Theorem 3.6 of [15]. ■

B. Bilinear- and quadratic difference forms for 2-D systems

In the pioneering paper [16], it has been shown that bilinear- and quadratic functionals of 1-D continuous-time system variables and their derivatives can be efficiently represented by two-variable polynomial matrices; this has been extended to the 1-D discrete-time case in [7]. In order to represent bilinear- and quadratic functionals of the variables of continuous-time 2-D-systems, 4-variable polynomial matrices are used, see [13]. We now examine the extension of quadratic difference forms to the 2-D discrete setting; some preliminary results in this sense have been obtained in [8].

In order to simplify the notation, define the multi-indices $\mathbf{k} := (k_1,k_2)$, $\mathbf{l} := (l_1,l_2)$, and the notation $\zeta := (\zeta_1,\zeta_2)$ and $\eta := (\eta_1,\eta_2)$, and define $\zeta^{\mathbf{k}}\eta^{\mathbf{l}} := \zeta_1^{k_1}\zeta_2^{k_2}\eta_1^{l_1}\eta_2^{l_2}$.

Let $\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta,\eta]$ denote the set of real $\mathtt{w}_1\times\mathtt{w}_2$ polynomial matrices in the four indeterminates ζ_i and η_i, $i=1,2$; that is, an element of $\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta,\eta]$ is of the form

$$\Phi(\zeta,\eta)=\sum_{\mathbf{k},\mathbf{l}}\Phi_{\mathbf{k},\mathbf{l}}\zeta^{\mathbf{k}}\eta^{\mathbf{l}}$$

where $\Phi_{\mathbf{k},\mathbf{l}}\in\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}$; the sum ranges over the nonnegative multi-indices $\mathbf{k}$ and $\mathbf{l}$, and is assumed to be finite. Such matrix induces a *bilinear difference form* (*BDF* in the following) L_Φ

$$\begin{aligned}L_\Phi &: (\mathbb{R}^{\mathtt{w}_1})^{\mathbb{Z}\times\mathbb{Z}}\times(\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}\times\mathbb{Z}}\longrightarrow(\mathbb{R})^{\mathbb{Z}\times\mathbb{Z}}\\ L_\Phi(v,w)&:=\sum_{\mathbf{k},\mathbf{l}}(\sigma^{\mathbf{k}}v)^\top\Phi_{\mathbf{k},\mathbf{l}}(\sigma^{\mathbf{l}}w)\end{aligned}$$

where the $\mathbf{k}$-th shift operator $\sigma^{\mathbf{k}}$ is defined as $\sigma^{\mathbf{k}}:=\sigma_1^{k_1}\sigma_2^{k_2}$, and analogously for $\sigma^{\mathbf{l}}$.

A 4-variable polynomial matrix $\Phi(\zeta_1,\zeta_2,\eta_1,\eta_2)\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is called *symmetric* if $\Phi(\zeta_1,\zeta_2,\eta_1,\eta_2)=\Phi(\eta_1,\eta_2,\zeta_1,\zeta_2)^\top$, concisely written as $\Phi(\zeta,\eta)=\Phi(\eta,\zeta)^\top$. In this case, Φ induces also a quadratic functional

$$\begin{aligned}Q_\Phi &: (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}}\longrightarrow(\mathbb{R})^{\mathbb{Z}\times\mathbb{Z}}\\ Q_\Phi(w)&:=L_\Phi(w,w)\end{aligned}$$

We will call Q_Φ the *quadratic difference form* (in the following abbreviated with QDF) associated with the four-variable polynomial matrix Φ.

In this paper we also consider "vectors" of 4-variable polynomial matrices $\Psi\in(\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta,\eta])^2$, i.e.

$$\Psi(\zeta,\eta)=\begin{bmatrix}\Psi_1(\zeta,\eta)\\ \Psi_2(\zeta,\eta)\end{bmatrix}=:\mathrm{col}(\Psi_i(\zeta,\eta))_{i=1,2}$$

with $\Psi_i\in\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta,\eta]$. Ψ induces a *vector bilinear difference form* (abbreviated *VBDF*), defined as

$$\begin{aligned}L_\Psi &: (\mathbb{R}^{\mathtt{w}_1})^{\mathbb{Z}\times\mathbb{Z}}\times(\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}\times\mathbb{Z}}\longrightarrow(\mathbb{R}^2)^{\mathbb{Z}\times\mathbb{Z}}\\ L_\Psi(v,w)&:=\begin{bmatrix}L_{\Psi_1}(v,w)\\ L_{\Psi_2}(v,w)\end{bmatrix}=\mathrm{col}(L_{\Psi_i}(v,w))_{i=1,2}\ .\end{aligned}$$

Finally, we introduce the notion of (discrete) divergence of a VBDF. Given a VBDF $L_\Psi=\mathrm{col}(L_{\Psi_1},L_{\Psi_2})$, we define its *divergence* as the BDF defined by

$$\begin{aligned}(\nabla L_\Psi)(w_1,w_2):=&(L_{\Psi_1}(w_1,w_2)-\sigma_1(L_{\Psi_1}(w_1,w_2)))\\ &+(L_{\Psi_2}(w_1,w_2)-\sigma_2(L_{\Psi_2}(w_1,w_2)))\quad(3)\end{aligned}$$

for all w_1,w_2. If L_Φ is the divergence of $L_\Psi=\mathrm{col}(L_{\Psi_1},L_{\Psi_2})$, it is straightforward to verify that in terms of the 4-variable polynomial matrices associated with the BDF's, their relationship is

$$\begin{aligned}\Phi(\zeta_1,\zeta_2,\eta_1,\eta_2)=&(1-\zeta_1\eta_1)\Psi_1(\zeta_1,\zeta_2,\eta_1,\eta_2)\\ &+(1-\zeta_2\eta_2)\Psi_2(\zeta_1,\zeta_2,\eta_1,\eta_2)\ ,\end{aligned}$$

written concisely as $\Phi=\mathrm{div}\ \mathrm{col}(\Psi_1,\Psi_2)$.

The definition and properties described above can be adapted to a vector quadratic difference form (VQDF) in a obvious manner.

We now introduce the notion of positivity of a QDF. We define a QDF Q_Δ induced by a four-variable polynomial matrix $\Delta\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1,\zeta_2,\eta_1,\eta_2]$ to be *nonnegative* if $Q_\Delta(w(x_1,x_2))\geq 0$ for all $(x_1,x_2)\in\mathbb{Z}\times\mathbb{Z}$ and for all $w\in(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}}$. This will be denoted with $Q_\Delta\geq 0$ or $\Delta(\zeta,\eta)\geq 0$. We call Q_Δ positive, denoted $Q_\Delta>0$ or $\Delta(\zeta,\eta)>0$, if $Q_\Delta\geq 0$ and $Q_\Delta(w(x_1,x_2))=0$ for all (x_1,x_2) implies $w=0$. Often in the following we will also consider QDFs induced by matrices of the form $\Delta(e^{-i\omega},\zeta_2,e^{i\omega},\eta_2)$, i.e. matrices in the indeterminates ζ_2,η_2 with coefficients being polynomials in $e^{i\omega}$ for some $\omega\in\mathbb{R}$. The definition of nonnegativity and positivity in this case is readily adapted from the above definition.

Finally, we define the equivalence of QDFs along a behavior. Let $\mathfrak{B}\in\mathcal{L}_2^{\mathtt{w}}$ and $\Phi_i\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1,\zeta_2,\eta_1,\eta_2]$, $i=1,2$. Then Q_{Φ_1} is *equivalent modulo* $\mathfrak{B}$ *to* Q_{Φ_2}, denoted $Q_{\Phi_1}\stackrel{\mathfrak{B}}{=}Q_{\Phi_2}$, if $Q_{\Phi_1}(w)=Q_{\Phi_2}(w)$ for all $w\in\mathfrak{B}$. Now let $\mathfrak{B}=\ker\ R(\sigma_1,\sigma_2)$; then it can be shown that $Q_{\Phi_1}\stackrel{\mathfrak{B}}{=}Q_{\Phi_2}$ if and only if there exists $X\in\mathbb{R}^{\bullet\times\mathtt{w}}[\zeta_1,\zeta_2,\eta_1,\eta_2]$ such that

$$\begin{aligned}\Phi_1(\zeta_1,\zeta_2,\eta_1,\eta_2)=&\Phi_2(\zeta_1,\zeta_2,\eta_1,\eta_2)\\ &+R^\top(\zeta_1,\zeta_2)X(\zeta_1,\zeta_2,\eta_1,\eta_2)\\ &+X^\top(\eta_1,\eta_2,\zeta_1,\zeta_2)R(\eta_1,\eta_2)\end{aligned}$$

(see Proposition 10 in [8]). In this case we also write

$$\Phi_1(\zeta_1,\zeta_2,\eta_1,\eta_2)=\Phi_2(\zeta_1,\zeta_2,\eta_1,\eta_2)\ \mathrm{mod}\ R\ ,$$

or $\Phi_1(\zeta_1,\zeta_2,\eta_1,\eta_2)-\Phi_2(\zeta_1,\zeta_2,\eta_1,\eta_2)=0$ mod R.

III. THE 2-D POLYNOMIAL LYAPUNOV EQUATION

Having introduced the definition of 2-D stability in section II-A and 2-D bilinear- and quadratic difference forms in section II-B, in this section we show how the 2-D Lyapunov equation allows to give a necessary and sufficient condition for an autonomous square behavior $\mathfrak{B}\in\mathcal{L}_2^{\mathtt{w}}$ to be asymptotically stable. In this section we use the result of Theorem 3, and we deal only with stability with respect to the proper cone consisting of the first orthant of $\mathbb{Z}\times\mathbb{Z}$, denoted with $\mathcal{K}_0$ in the following.

We begin this section with a straightforward but important refinement of Proposition 3.5 of [15].

Proposition 4: Let $\mathfrak{B}\in\mathcal{L}_2^{\mathtt{w}}$ be square and autonomous, and let $\mathfrak{B}=\ker\ S(\sigma_1,\sigma_2)$ with $S\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1,\xi_2]$ nonsingular. Assume that $\delta:=\det\ S$ is such that $\mathcal{H}_\delta$ is a subset of $\mathcal{K}_0$, the first orthant of $\mathbb{Z}\times\mathbb{Z}$, that intersects the coordinate axes only in the origin. Then the following statements are equivalent:

1) $\mathfrak{B}$ is $\mathcal{K}_0$-stable;
2) For all $\omega\in\mathbb{R}$, the polynomial $\delta(e^{j\omega},\xi_2)$ has all its roots outside of the closed unit disk $\{z_2\in\mathbb{C}\mid |z_2|\geq 1\}$, and the polynomial $\delta(\xi_1,e^{j\omega})$ has all its roots outside of the closed unit disk $\{z_1\in\mathbb{C}\mid |z_1|\geq 1\}$.

Proof: The proof follows from Theorem 3 and from the equivalence of statements $i)$ and $iv)$ in Proposition 3.1 of [6]. ■

Proposition 4 shows that the stability of a square autonomous behavior can be checked by ascertaining the stability of two families of complex polynomials depending on the parameter $\omega\in\mathbb{R}$. In the scalar case, Geronimo and

Woerdeman in [6] use an ω-dependent complex Hermitian polynomial analogous to the Bézoutian used in the case of univariate polynomials (see Chapter 8 of [3]) in order to do this. We now generalize their result to the multivariable case, and state an equivalent condition in terms of a pair of quadratic difference forms satisfying the 2-D PLE.

In order to do this, we need to introduce yet some more notation; in the following we denote with $\mathrm{Per}_2 \subset (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}}$ the set consisting of all trajectories $v \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}}$ such that the restriction of v to the lines $\{(i,j) \mid j \in \mathbb{Z}\}$ is periodic for all $i \in \mathbb{Z}$, i.e.

$$\mathrm{Per}_2 := \{v \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}} \mid v(i,\cdot) \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{R}} \text{ is periodic for all } i \in \mathbb{Z}\}$$

and analogously we define

$$\mathrm{Per}_1 := \{v \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}} \mid v(\cdot,j) \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{R}} \text{ is periodic for all } j \in \mathbb{Z}\} \ .$$

The following is the main result of this paper, and shows how the 2-D PLE arises naturally in the study of the stability of 2-D square autonomous behaviors.

Theorem 5: Let $\mathcal{B}$ be a 2-D square autonomous linear behavior, and let $\mathcal{B} = \ker\ R(\sigma_1, \sigma_2)$. Then the following statements are equivalent:

1) $\mathcal{B}$ is asymptotically stable.
2) There exists a VQDF $Q_\Phi = \mathrm{col}(Q_{\Phi_1}, Q_{\Phi_2})$ and a QDF Q_Δ such that
   - $(2a)$ $\nabla Q_\Phi \overset{\mathfrak{B}}{=} -Q_\Delta$;
   - $(2b)$ $Q_{\Phi_1}(w), Q_\Delta(w) > 0$ for all $w \in \mathfrak{B} \cap \mathrm{Per}_2$, and $Q_{\Phi_2}(w), Q_\Delta(w) > 0$ for all $w \in \mathfrak{B} \cap \mathrm{Per}_1$.
3) There exist $\Phi = \mathrm{col}(\Phi_1, \Phi_2)$ and Δ, with $\Phi_1, \Phi_2, Y \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$, $\Delta \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ such that
   - $(3a)$ $$\begin{aligned}&(1-\zeta_1\eta_1)\Phi_1(\zeta_1,\zeta_2,\eta_1,\eta_2)\\ &+(1-\zeta_2\eta_2)\Phi_2(\zeta_1,\zeta_2,\eta_1,\eta_2)\\ &= -\Delta(\zeta_1,\zeta_2,\eta_1,\eta_2)\\ &+R(\zeta_1,\zeta_2)^\top Y(\zeta_1,\zeta_2,\eta_1,\eta_2)\\ &+Y(\eta_1,\eta_2,\zeta_1,\zeta_2)^\top R(\eta_1,\eta_2);\end{aligned}$$
   - $(3b)$ $$\begin{aligned}&\Phi_1(\zeta_1,\zeta_2,\eta_1,\eta_2) \overset{\mathfrak{B}\cap\mathrm{Per}_2}{>} 0,\\ &\Phi_2(\zeta_1,\zeta_2,\eta_1,\eta_2) \overset{\mathfrak{B}\cap\mathrm{Per}_1}{>} 0,\\ &\Delta(\zeta_1,\zeta_2,\eta_1,\eta_2) \overset{\mathfrak{B}\cap\mathrm{Per}_i}{>} 0,\ i=1,2.\end{aligned}$$

Proof: The equivalence of statements (2) and (3) follows by standard arguments of the calculus of QDFs; consequently we only prove the equivalence of (3) and (1) in the following.

In order to show that (3) $\Longrightarrow$ (1), we proceed as follows. First, note that in the following we consider behaviors $\mathfrak{B}$ whose trajectories take values in $\mathbb{C}^{\mathtt{w}}$, obtained e.g. by complexification of real behaviors $\mathfrak{B}'$:

$$[w \in \mathfrak{B}] \iff [\text{the real and the imaginary part of } w \text{ belong to } \mathfrak{B}'] \ .$$

Now let $(\lambda, \mu) \in \mathbb{C}^2$ be in the characteristic variety of R, which we denote with $\mathcal{C}(R)$ in the following. Since $\mathcal{C}(R) = \mathcal{C}(R')$ for any polynomial matrix R' inducing a kernel representation of $\mathfrak{B}$, in the following we will also speak without confusion about the characteristic variety of the behavior $\mathfrak{B}$, denoted with $\mathcal{C}(\mathfrak{B})$. Since $(\lambda, \mu) \in \mathcal{C}(\mathfrak{B})$, there exists a vector $v \in \mathbb{C}^{\mathtt{w}}$ (which depends on λ and μ) such that the trajectory w defined by $w(x_1, x_2) := v\ \lambda^{x_1}\ \mu^{x_2}$ belongs to $\mathfrak{B}$. It is easy to see that v is such that $R(\lambda, \mu)v = 0$, i.e. $v \in \ker\ R(\lambda, \mu)$.

We now prove that if μ lies on the unit circle, i.e. $\mu = e^{i\omega}$ for some $\omega \in \mathbb{R}$, then $\mid \lambda \mid > 1$. Once this will have been established, statement (1) follows from Proposition 4.

Let $\zeta_1 = \overline{\lambda}$, $\eta_1 = \lambda$, $\zeta_2 = \overline{\mu} = e^{-i\omega}$, $\eta_2 = \mu = e^{i\omega}$ in $(3a)$, and multiply the resulting expression on the left by $v^\top$ and on the right by v. It follows from the fact that $v \in \ker\ R(\lambda, \mu)$ that

$$(1-\overline{\lambda}\lambda)\ v^\top \Phi_1(\overline{\lambda}, e^{-i\omega}, \lambda, e^{i\omega})v = -v^\top \Delta(\overline{\lambda}, e^{-i\omega}, \lambda, e^{i\omega})v$$

The right-hand side of this equation is strictly negative; on the left-hand side it holds that $v^\top \Phi_1(\overline{\lambda}, e^{-i\omega}, \lambda, e^{i\omega})v > 0$, and consequently it follows that $1 - \overline{\lambda}\lambda < 0$. An analogous argument is used when $w(t_1, t_2) = v\ e^{i\omega t_1}\mu^{t_2}$. This proves the claim.

The proof of (1) $\Longrightarrow$ (3) will be achieved by showing the existence of matrices $\Phi_i \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$, $i = 1, 2$, and $\Delta \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ such that $(3a) - (3b)$ hold. In this way we will also produce a solution of the 2-D PLE, equation $(3a)$.

The two-variable polynomial matrix R can be seen as a polynomial matrix in one of the two variables, with coefficients being polynomial matrices in the other one; that is, $R(\xi_1, \xi_2) = \sum_{i=0}^{L_1} R_i(\xi_2)\xi_1^{L_1} = \sum_{i=0}^{L_2} R_i'(\xi_1)\xi_2^{L_2}$, where L_i is the highest power of ξ_i in R, $i = 1, 2$. Now define the four-variable polynomial matrix

$$\begin{aligned}\Gamma(\zeta_1,\zeta_2,\eta_1,\eta_2) &:= R(\zeta_1,\zeta_2)^\top R(\eta_1,\eta_2) \\ &- \zeta_1^{L_1}\zeta_2^{L_2}\eta_1^{L_1}\eta_2^{L_2} R(\eta_1^{-1},\eta_2^{-1})^\top R(\zeta_1^{-1},\zeta_2^{-1}) \ ,\end{aligned} \tag{4}$$

the two-variable polynomial matrix $Y(\xi_1, \xi_2) := \frac{1}{2}R(\xi_1, \xi_2)$ and the four-variable one $\Delta(\zeta_1, \zeta_2, \eta_1, \eta_2) := \zeta_1^{L_1}\eta_1^{L_1}\zeta_2^{L_2}\eta_2^{L_2}R(\eta_1^{-1}, \eta_2^{-1})^\top R(\zeta_1^{-1}, \zeta_2^{-1})$. It is a matter of straightforward verification to see that

$$\begin{aligned}\Gamma(\zeta_1,\zeta_2,\eta_1,\eta_2) &= -\Delta(\zeta_1,\zeta_2,\eta_1,\eta_2) \\ &+ Y(\zeta_1,\zeta_2)^\top R(\eta_1,\eta_2) + R(\zeta_1,\zeta_2)^\top Y(\eta_1,\eta_2) \ .\end{aligned}$$

From (4) it follows that $\partial\Gamma = 0$, where the "del" operator ∂ is defined as

$$\begin{aligned}&\partial : \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta_1,\zeta_2,\eta_1,\eta_2] \longrightarrow \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}] \\ &\partial\Phi(\xi_1,\xi_2) := \Phi(\xi_1^{-1},\xi_2^{-1},\xi_1,\xi_2) \ .\end{aligned}$$

We now prove the following Lemma, which allows us to conclude that Γ is the divergence of some VQDF.

Lemma 6: A BDF L_Φ is the divergence of some VBDF L_Ψ if and only if $\partial\Phi(\xi_1, \xi_2) = 0$.

Proof: That the condition $\partial\Phi(\xi_1, \xi_2) = 0$ is necessary follows immediately from the definition of discrete divergence, and its expression in terms of four-variable polynomial matrices. We now prove sufficiency. Observe first that the polynomials $1-\zeta_1\eta_1$ and $1-\zeta_2\eta_2$ form a Gröbner basis for the ideal generated by them (see [1] for a thorough introduction to Gröbner bases). Now let $p \in \mathbb{R}[\zeta_1,\zeta_2,\eta_1,\eta_2]$, and consider that the normal form of p modulo $1-\zeta_1\eta_1$ and $1-\zeta_2\eta_2$ only involves linear combinations of the terms ζ_k, η_k, $k=1,2$, and $\zeta_i\eta_k$, for $i,k=1,2$ with $i \neq k$. Observe that the image under ∂ of this normal form is zero if and only if the coefficients of the linear combination are all zero. Conclude that if $p \in \mathbb{R}[\zeta_1,\zeta_2,\eta_1,\eta_2]$ is such that $\partial p = 0$, then necessarily its normal form modulo $1-\zeta_1\eta_1$ and $1-\zeta_2\eta_2$ is zero, i.e. there exist polynomials $\varphi_i \in \mathbb{R}[\zeta_1,\zeta_2,\eta_1,\eta_2]$, $i=1,2$, such that $p(\zeta_1,\zeta_2,\eta_1,\eta_2) = (1-\zeta_1\eta_1)\varphi_1(\zeta_1,\zeta_2,\eta_1,\eta_2)+(1-\zeta_2\eta_2)\varphi_2(\zeta_1,\zeta_2,\eta_1,\eta_2)$. This argument can be extended entrywise to polynomial matrices. This concludes the proof. ■

We resume the proof of the implication (1) $\Longrightarrow$ (3) of Theorem 5. Conclude from Lemma 6 that there exists $\Phi = \mathrm{col}(\Phi_1,\Phi_2) \in \mathbb{R}^{2\mathtt{w}\times\mathtt{w}}[\zeta_1,\zeta_2,\eta_1,\eta_2]$ such that div $\Phi(\zeta_1,\zeta_2,\eta_1,\eta_2) = \Gamma(\zeta_1,\zeta_2,\eta_1,\eta_2)$. This proves $(3a)$.

In order to prove $(3b)$ we proceed as follows. First, note that

$$\begin{aligned}(1-\zeta_1\eta_1)\Phi_1(\zeta_1,e^{-i\omega},\eta_1,e^{i\omega}) &= -\Delta(\zeta_1,e^{-i\omega},\eta_1,e^{i\omega})\\ &+ R(\zeta_1,e^{-i\omega})^\top Y(\eta_1,e^{i\omega}) + Y(\zeta_1,e^{-i\omega})R(\eta_1,e^{i\omega}). \quad (5)\end{aligned}$$

Following [9], [10] (see equation (4) of [10]) we call (5) a *ω-dependent 1-D two-variable polynomial Lyapunov equation*.

Now from Proposition 4 it follows that since $\mathfrak{B}$ is $\mathcal{K}_0$-stable, for all $\omega \in \mathbb{R}$ the polynomial det $R(\xi_1,e^{i\omega})$ is anti-Schur, i.e. all its roots have modulus greater than one. Consequently, from the fact that

$$\Delta(\zeta_1,e^{-i\omega},\eta_1,e^{i\omega}) = \zeta_1^{L_1}\eta_1^{L_1}R(\eta_1^{-1},e^{i\omega})^\top R(\zeta_1^{-1},e^{-i\omega})$$

is a "square" it follows that $\Delta(\zeta_1,e^{-i\omega},\eta_1,e^{i\omega}) \geq 0$ for all $\omega \in \mathbb{R}$. Use the fact that det $\xi_1^{L_1}R(\xi_1^{-1},e^{i\omega})$ is Schur in order to conclude that $\Delta(\zeta_1,e^{-i\omega},\eta_1,e^{i\omega}) \overset{\mathfrak{B}\cap\mathrm{Per}_2}{>} 0$. Now apply Theorem 1 of [9] to conclude that $\Phi_1(\zeta_1,e^{-i\omega},\eta_1,e^{i\omega}) \overset{\mathfrak{B}\cap\mathrm{Per}_2}{\geq} 0$. In order to prove that $\Phi_1(\zeta_1,e^{-i\omega},\eta_1,e^{i\omega}) \overset{\mathfrak{B}\cap\mathrm{Per}_2}{>} 0$, assume by contradiction that there exists a trajectory in $\mathfrak{B}\cap\mathrm{Per}_2$ along which the QDF induced by $\Phi_1(\zeta_1,e^{-i\omega},\eta_1,e^{i\omega})$ is zero; then from (5) it follows that also the QDF induced by $\Delta(\zeta_1,e^{-i\omega},\eta_1,e^{i\omega})$ is zero along the same trajectory, a contradiction with the result $\Delta(\zeta_1,e^{-i\omega},\eta_1,e^{i\omega}) \overset{\mathfrak{B}\cap\mathrm{Per}_2}{>} 0$ established previously. This proves half of the claim $(3b)$. The other half is proved following a similar argument. This concludes the proof of the claim.

■

The VQDF $\Phi = \mathrm{col}(\Phi_1,\Phi_2)$ and the QDF Δ given in the proof of Theorem 5 can be considered as an ω-parametrized 2-D discrete-time version of the multivariable Bézoutian

$$\frac{R(\zeta)^\top R(\eta) - R(-\eta)^\top R(-\zeta)}{\zeta+\eta}$$

used in analyzing stability of 1-D continuous-time systems, see [4]. In the single-variable (i.e. $\mathtt{w}=1$) case, stability conditions based on the positivity of the coefficient matrix of an ω-dependent Bézoutian have been obtained by Geronimo and Woerdeman in [5], [6]; in order to see that the result of Theorem 5 is more than just a generalization of those results to the multivariable case, consider the following example.

Example 7: Consider the system described in kernel form by the polynomial

$$p(\zeta_1,\zeta_2,\eta_1,\eta_2) := 1 + \frac{1}{2}\xi_1 + \frac{1}{2}\xi_2 + \frac{1}{2}\xi_1\xi_2$$

The Bézoutian $B(\zeta_1,\zeta_2,\eta_1,\eta_2)$ can be shown to be the divergence of the VQDF induced by the two polynomials

$$\begin{aligned}\Phi_1(\zeta_1,\zeta_2,\eta_1,\eta_2) &:= \frac{1}{2}(\eta_2+\zeta_2+3\eta_2\zeta_2)\\ \Phi_2(\zeta_1,\zeta_2,\eta_1,\eta_2) &:= \frac{1}{4}(3+\eta_1+\zeta_1)\ .\end{aligned}$$

It is easy to see that

$$\begin{aligned}\Phi_1(\zeta_1,e^{-i\omega},\eta_1,e^{i\omega}) =& \frac{1}{2}(3+2\cos(\omega))\\ =& \Phi_2(e^{-i\omega},\zeta_2,e^{i\omega},\eta_2) > 0\end{aligned}$$

for all $\omega \in \mathbb{R}$: the system is stable.

We now compute another Lyapunov functional for ker $p(\sigma_1,\sigma_2)$. Define first the two-variable polynomial

$$\Delta'(\zeta_1,\zeta_2,\eta_1,\eta_2) := 1 + \frac{1}{4}(\zeta_1+\eta_1+\zeta_2+\eta_2+\zeta_1\eta_1+\zeta_2\eta_2)\ .$$

Since $\Delta'(\zeta_1,\zeta_2,\eta_1,\eta_2)$ can be rewritten as

$$\Delta'(\zeta_1,\zeta_2,\eta_1,\eta_2) = \frac{1}{2}+\frac{1}{4}(1+\zeta_1)(1+\eta_1)+\frac{1}{4}(1+\zeta_2)(1+\eta_2),$$

we have

$$\Delta'(\zeta_1,e^{-i\omega},\eta_1,e^{i\omega}) > 0 \text{ and } \Delta'(e^{-i\omega},\zeta_2,e^{i\omega},\eta_2) > 0$$

for all $\omega \in \mathbb{R}$. Now define

$$\begin{aligned}\Phi_1'(\zeta_1,\zeta_2,\eta_1,\eta_2) &:= \frac{1}{4}(1+\zeta_2)(1+\eta_2)+\frac{1}{4}\\ \Phi_2'(\zeta_1,\zeta_2,\eta_1,\eta_2) &:= \frac{1}{4}(1+\zeta_1)(1+\eta_1)+\frac{1}{4}\end{aligned}$$

and observe that

$$\begin{aligned}\Phi_1'(\zeta_1,e^{-i\omega},\eta_1,e^{i\omega}) =& \frac{1}{4} \mid 1+e^{i\omega} \mid^2 + \frac{1}{4}\\ =& \Phi_2'(e^{-i\omega},\zeta_2,e^{i\omega},\eta_2) > 0\end{aligned}$$

for all $\omega \in \mathbb{R}$. It is a matter of straightforward verification to check that with these positions,

$$\frac{1}{4}\begin{bmatrix}(1+\zeta_2)(1+\eta_2)+1\\(1+\zeta_1)(1+\eta_1)+1\end{bmatrix}$$

is a Lyapunov function for $\mathfrak{B} = \ker\, p(\sigma_1,\sigma_2)$ with divergence equal to $-\Delta'(\zeta_1,\zeta_2,\eta_1,\eta_2)$ along $\mathfrak{B}$.

The issue of how to efficiently solve the general 2-D PLE is a matter of ongoing research. In the 1-D case, an algorithm to solve the PLE has been presented in [11]; it is a matter of current investigation whether this procedure can inspire similar schemes for the solution of the 2-D PLE.

IV. CONCLUSIONS

The main result of this paper is Theorem 5, which states necessary and sufficient conditions for the asymptotic stability of a "square" 2-D behavior in the sense defined in [15]. In these stability conditions, an essential role is played by the 2-D polynomial Lyapunov equation (1). Current research efforts are directed at devising algorithms for solving the 2-D PLE in an efficient way.

ACKNOWLEDGMENTS

The first author gratefully acknowledges the financial support of *The Great Britain Sasakawa Foundation* for financially supporting the visit to Kyoto University during which the results presented in this paper were obtained.

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