Polynomial structure of 3×3 reciprocal inner matrices

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Abstract— **The objective of our work is the derivation of efficient algorithms for the synthesis of microwave multiplexers. In our opinion, an efficient frequency design process calls for the understanding of the structure of $n \times n$ inner (or lossless) reciprocal rational functions for $n > 2$. Whereas the case $n = 2$ is completely understood and a keystone of filter synthesis very little seems to be known about the polynomial structure of such matrices when they involve more than 2 ports.**

We therefore start with the analysis of the 3×3 case typically of practical use in the manufacturing of diplexers. Based on recent results obtained on minimal degree reciprocal Darlington synthesis [6], we derive a polynomial model for 3×3 reciprocal inner rational matrices with given McMillan degree.

I. INTRODUCTION

A. Frequency design of reciprocal electrical networks

A reciprocal network is by definition a circuit built with resistors, capacitors and inductors but no gyrators. These circuits are still currently actively studied [1] as they appear to be adapted for the modelling of microwave devices, such as resonating cavity filters, multiplexers and antennas. Although the functioning of such devices relies primary on Maxwell equations, classical modal analysis techniques yield equivalent reciprocal electrical circuits [2] that exhibit good approximation properties in a restricted frequency range. These models present an interesting trade off between flexibility and accuracy that justifies their central position among the techniques used for the design of microwave devices. In particular, the design of microwave filters relies heavily on the polynomial structure of 2×2 reciprocal lossless squattering matrices. This structure allows to cast the design, under modulus constraints, of the response of a filter to a quasi-convex optimisation Zolatariov problem [3]. More recently the polynomial structure of 3×3 reciprocal lossless matrices was derived under the assumption that the underlying circuit is composed of two filters connected at a common port [4]. We tackle here the problem of deriving the general polynomial structure of 3×3 reciprocal lossless matrices under no particular assumption on the underlying circuit. Divisibility conditions are derived involving the numerators of the transmission elements $S_{1,2}$ and $S_{1,3}$ and a scalar spectral factor of the term $1 - S_{1,1}S_{1,1}^*$. Using the latter a constructive method is derived to count and compute all possible 3×3 reciprocal extensions of a Shur function $S_{1,1} = \frac{p}{q}$ while keeping the MacMillan degree unchanged.

B. Preliminaries and notations

As usual, we denote by Π^+ and Π^- the right and left open half-planes and by $i\mathbb{R}$ the imaginary axis; similarly, $\mathbb{D}$ denotes the unit disk and $\mathbb{T}$ its boundary. For a complex matrix M, M^T stands for its transpose and M^* for the transposed conjugate. In System Theory, a rational function whose poles lie in Π^- (resp. $\Pi^- \cup i\mathbb{R}$) is called *asymptotically stable* (resp. *stable*), and a rational function which is finite (resp. vanishing) at infinity is called *proper* (resp. *strictly proper*). We say that a $p \times p$ matrix valued function S is a *Schur function* if it is contractive in Π^+ :

$$S(s)S(s)^* \leq I, \quad s \in \Pi^+.$$

Note that a rational matrix which is contractive in Π^+ is automatically analytic there. In system theory, a Schur function is called *bounded real*. A Schur function S is said to be *lossless* or *inner* if, in addition,

$$S(s)S(s)^* = I, \qquad s \in i\mathbb{R}. \tag{1}$$

Notice that if $\omega \in \Pi^+$, the Blaschke factor $b_\omega = \frac{s-\omega}{s+\bar{\omega}}$ is an inner function of McMillan degree one. Every scalar rational inner function can be obtained as a product of Blaschke factors $\epsilon\, b_{\omega_1}(s) b_{\omega_2}(s) \ldots b_{\omega_n}(s)$, $\epsilon \in \mathbb{T}$, $w_i \in \Pi^+$ [8].

C. Darlington synthesis

Let S be a $p \times p$ symmetric Schur function. We shall presently study the $(p+m) \times (p+m)$ lossless extensions $\mathcal{S}$ which are symmetric as well:

$$\mathcal{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S \end{pmatrix}, \text{ with } \begin{cases} S_{11} &= S_{11}^T \\ S_{21} &= S_{12}^T. \end{cases} \tag{2}$$

The symmetric Darlington synthesis problem has been little studied and the existing solutions require an important increase of the size of the extension [5]. In [7], it was proved that a $p \times p$ symmetric Schur matrix S do possess a $2p \times 2p$ symmetric lossless extension if and only if its zeros have even multiplicity. If this condition is not satisfied, given a $p \times p$ symmetric Schur matrix S of McMillan degree n, two dual extension problems can be formulated:

- either we fix the size of the extension to $2p \times 2p$ and we look for a minimal degree extension,
- or we fix the degree n of the extension and we look for a minimum size extension.

The solutions obtained in [7], [6] for each of these problems improve significantly the results in the literature.

The left lower block S_{21} of the completion (2) is a spectral factor of $I - SS^*$,

$$I - SS^* = S_{21}S_{21}^*.$$

Here, S^* denotes $S^*(s) = S(-\bar{s})^*$. There are many ways to achieve such a factorization. We assume that S is strictly contractive at least at some point of $i\mathbb{R}$.

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From [7], we get the inner extension $\mathcal{S}$ of S.

Theorem 1: Let S be a $p \times p$ Schur function such that $I + S$ is invertible in H^∞. Let S_{21} be a spectral factor of $I - SS^*$. Every inner completion $\mathcal{S}$ of $[S_{21}\ S]$ can be written as:

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix} \tag{3}$$

with

$$\begin{aligned} S_{12} &= -MS_{21}^*(I+S^*)^{-1}(I+S) \\ S_{11} &= M - MS_{21}^*(I+S^*)^{-1}S_{21} \end{aligned}$$

where M is a left DSS inner factor of $(I+S)^{-1}S_{21}$, i.e.

$$MS_{21}^*(I+S^*)^{-1}$$

is stable. The extension $\mathcal{S}$ has same degree as $[S_{21}\ S]$ if and only if M has minimal degree.

Proof: See [7] ■

We now consider $\mathcal{S}$ to be of minimal degree, $\deg \mathcal{S} = n$. Now we don't have a symmetric extension. To symmetrize $\mathcal{S}$ we will, as in proposition 1 of [7], multiply it to the right with a matrix using $Q = S_{21}^{-1}S_{12}^T$ and obtain Σ symmetric and unitary, i.e. verify (1).

$$\Sigma = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}$$

The extension Σ has a McMillan degree of $2n - n_0$ where n_0 is the number of zeros of S_{21} on the imaginary axe. Clearly, Σ is stable (resp. asymptotically stable) if and only if the poles of $\det Q$ have a non positive (resp. negative) real part.

D. Minimal size extension

To reduce the McMillan degree of Σ, we can use the following so-called Potapov factorization defined in the following lemma extracted from [7]

Lemma 1 (Symmetric Potapov factorization): Let T be a $p \times p$ symmetric inner function of McMillan degree N. The following assertions are equivalent:

1) T has a zero ω of *multiplicity* greater than 1
2) there exists a unit vector which satisfies the conditions

$$\begin{aligned} T(\omega)u &= 0 \\ u^T T'(\omega)u &= 0 \end{aligned}$$

3) the matrix $T(s)$ can be factored as

$$T = B_{\omega,u}^T\, R\, B_{\omega,u}$$

for some rational inner matrix R of degree $N-2$.

Proof: See [7], $B_{\omega,u} = I + (b_\omega - 1)\, uu^*$ is called a Potapov factor. ■

We can thus remove even multiplicity zeros of $\det Q$. Yet, we won't generally achieve an extension of degree n. To obtain an extension of degree n, the idea is to increase the size of the extension such that we double the zeros of odd multiplicities. This idea comes also from [7] and will allow us to exactly achieve the degree n extension after a proper use of Potapov factorization with an increase of only one dimension.

$$\hat{\Sigma} = \begin{bmatrix} S_{11}Q & 0 & S_{12} \\ 0 & \det Q & 0 \\ S_{12}^T & 0 & S \end{bmatrix}. \tag{4}$$

Thus, for S a $p \times p$ function, $\hat{\Sigma}$ is $(2p+1) \times (2p+1)$.

E. The 2×2 case

We will now focus on a scalar $S = \frac{p}{q}$ with p and q monic polynomials of degree n such that $\frac{p}{q}$ is irreducible and q is stable. Let r be a stable polynomial of degree $n_0 < n$ such that $qq^* - pp^* = rr^*$. If r is an auto reciprocal polynomial, there exists a symmetric extension of degree n. Otherwise, we can consider the following extension of degree at most $n + n_0$ of $\frac{p}{q}$

$$\begin{bmatrix} \frac{(-1)^n p^* r^*}{qr} & \frac{r^*}{q} \\ \frac{r^*}{q} & \frac{p}{q} \end{bmatrix}$$

which has I_2 as a limit when s goes to the infinity. To reduce the McMillan degree, one can use the symmetric Potapov factorization for zeros of r of even multiplicities. Generally, we can't reach this way an extension of size 2×2 of degree n. However, from the preceding section, we know that there are extensions of size 3×3 of McMillan degree n.

F. Objectives

In the following sections, we will use the minimal size extension approach to build and count all the extensions 3×3 of a Schur rational function $\frac{p}{q}$. Through divisibility relations we will also see that any extension 3×3 given by (3) can be obtain from a specific extension (4). Using symmetric Potapov factorization properties, we will then build all the possible extensions of $\frac{p}{q}$. We will define an equivalence class of extensions using orthogonal matrices transformations. Then, we will give the number of finite class of extensions. The Potapov factorization approach will be different for single roots of r and multiple roots. The single roots case will be first studied. More specifically, on one hand, the factorization of a single root gives us two choices. On the other hand, if the multiplicity m of a root is greater than 1, there are less choices : $m+1$ possible choices to eliminate all the factors corresponding to this root.

II. Minimal size extension

Extensions of $\frac{p}{q}$ of size 3×3 can be obtained like in section I-D. That is

$$\Sigma = \begin{bmatrix} (-1)^n \frac{p^* r^*}{qr} & 0 & \frac{r^*}{q} \\ 0 & \frac{r^*}{r} & 0 \\ \frac{r^*}{q} & 0 & \frac{p}{q} \end{bmatrix} \tag{5}$$

We thus have doubled the zeros of r and can apply an efficient Potapov symmetric factorization we will write in the form

$$\Sigma = B^T \Sigma_0 B.$$

where B is inner and Σ_0 is an extension of degree n of $\frac{p}{q}$.

Note that any imaginary root of r is directly simplified in the 2×2 upper left block of Σ and won't thus be factorized, effectively decreasing the McMillan degree of Σ. Those imaginary roots will appear in factor of both following polynomials p_1 and p_2. In the next sections, we won't pay attention to those roots that will only reappear in the final result.

Let's consider p_1 and p_2 polynomials such that $p_1p_1^* + p_2p_2^* = rr^*$. B must then be of the form

$$B = \frac{1}{r}\begin{bmatrix} p_1^* & (-1)^{n+1}p_2 & 0 \\ p_2^* & (-1)^n p_1 & 0 \\ 0 & 0 & r \end{bmatrix}. \quad (6)$$

Using $B^{-1} = B^*$, we obtain, after some computation, the matrix Σ_0 as

$$\Sigma_0 = \frac{1}{q}\begin{bmatrix} \frac{(-1)^n p^* p_1^2 + q(p_2^*)^2}{rr^*} & \frac{(-1)^{n+1}p^*p_1p_2 + qp_1^*p_2^*}{rr^*} & p_1 \\ \frac{(-1)^{n+1}p^*p_1p_2 + qp_1^*p_2^*}{rr^*} & \frac{(-1)^n p^* p_2^2 + q(p_1^*)^2}{rr^*} & p_2 \\ p_1 & p_2 & p \end{bmatrix}.$$

where every element inside the matrix block is a polynomial. Thus, rr^* divides

$$\begin{cases} (-1)^n p^* p_1^2 + q(p_2^*)^2 \\ (-1)^{n+1}p^*p_1p_2 + qp_1^*p_2^* \\ (-1)^n p^* p_2^2 + q(p_1^*)^2 \end{cases} \quad (7)$$

We will come back later to what those divisibility relations imply and have first a look at general 3×3 extensions of degree n.

III. General 3×3 extensions of degree n

As seen in (3), any inner symmetric extension 3×3 of $\frac{p}{q}$ can be written

$$\Sigma_0 = \begin{bmatrix} M - MS_{21}^*(1+S^*)^{-1}S_{21} & S_{12} \\ S_{21} & S \end{bmatrix}$$

with

$$S = \frac{p}{q}, S_{21} = S_{12}^T = [\frac{p_1}{q}\frac{p_2}{q}]$$

$$M = \frac{1}{l}\begin{bmatrix} m_1^* & -\eta m_2 \\ m_2^* & \eta m_1 \end{bmatrix}$$

with $\eta \in \mathbb{T}$ and $ll^* = m_1m_1^* + m_2m_2^*$.

Consider $H = MS_{21}^*(1+S^*)^{-1}$. M is inner, thus

$$M^*H = S_{21}^*(1+S^*)^{-1},$$

i.e.

$$M^*H = \frac{1}{p^*+q^*}\begin{bmatrix} p_1^* \\ p_2^* \end{bmatrix}.$$

Since zeros of q are in Π^- and $\frac{p}{q}$ is Schur, $p+q$ has all zeros in Π^- and thus p^*+q^* has all the zeros is Π^+. H being stable by definition of M, all the zeros of p^*+q^* are poles of M^*. This means, M being of minimal degree, that $l = \mu(p+q)$, $\mu \in \mathbb{C} - \{0\}$. Since μ ends up playing no real role later, we will skip it in the following.

$$S_{12} = -MS_{21}^*(1+\frac{p^*}{q^*})^{-1}(1+\frac{p}{q}).$$

That is,

$$\frac{1}{q}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \frac{1}{p+q}\frac{1}{q^*}\frac{q^*}{p^*+q^*}\frac{p+q}{q}\begin{bmatrix} m_1^*p_1^* - \eta m_2 p_2^* \\ m_2^*p_1^* + \eta m_1 p_2^* \end{bmatrix} = 0$$

that gives, when q does not vanish, the following equations

$$\begin{cases} m_1^*p_1^* - \eta m_2p_2^* + (p^*+q^*)p_1 &= 0 \\ m_2^*p_1^* + \eta m_1p_2^* + (p^*+q^*)p_2 &= 0. \end{cases}$$

From both above equations, by eliminating successively m_1 then m_2, one obtains the following equations :

$$m_1rr^* + (p+q)(p_1^*)^2 + \bar{\eta}(p^*+q^*)(p_2)^2 = 0 \quad (8)$$

and

$$m_2rr^* + (p+q)p_1^*p_2^* - \bar{\eta}(p^*+q^*)p_1p_2 = 0$$

From (8) we get the following

$$\bar{\eta}m_1^*rr^* + (p+q)(p_2^*)^2 + \bar{\eta}(p^*+q^*)(p_1)^2 = 0 \quad (9)$$

Let's multiply the three equations by q and use $qq^* = pp^* + rr^*$ to obtain :

$$\begin{cases} (qm_1 + \bar{\eta}p_2^2)rr^* + (p+q)(q(p_1^*)^2 + \bar{\eta}p^*(p_2)^2) &= 0 \\ (qm_2 - \bar{\eta}p_1p_2)rr^* + (p+q)(qp_1^*p_2^* - \bar{\eta}p^*p_1p_2) &= 0 \\ \bar{\eta}(qm_1^* + p_1^2)rr^* + (p+q)(q(p_2^*)^2 + \bar{\eta}p^*(p_1)^2) &= 0 \end{cases} \quad (10)$$

Consider ω a common zero of rr^* and $p+q$. Then, $(pp^*)(\omega) = (qq^*)(\omega)$ which can be written, using $p(\omega) = -q(\omega)$,

$$q(\omega)(p^*+q^*)(\omega) = 0$$

p and q being coprime, $q(\omega) \neq 0$, thus ω is a zero of p^*+q^*. All zeros of $p+q$ are in Π^- and all zeros of p^*+q^* are in Π^+ contradicting that $p+q$ and p^*+q^* having a common zero. Thus, we have proved that rr^* and $p+q$ are coprime. So, from (10) we get that rr^* divides

$$\begin{cases} q(p_1^*)^2 + \bar{\eta}p^*(p_2)^2 \\ qp_1^*p_2^* - \bar{\eta}p^*p_1p_2 \\ q(p_2^*)^2 + \bar{\eta}p^*(p_1)^2. \end{cases} \quad (11)$$

Since $\mathcal{S}(\infty) = I_3$, we get, after some computation, $\eta = (-1)^n$ and preceding relations are the same as (7). We will thus study more closely those divisibility relations in following section.

IV. Divisibility relations

Last two sections lead us to the following theorem.

Theorem 2: Let Σ_0 be a symmetric inner extension 3×3 of McMillan degree n of $\frac{p}{q}$. Then, there exists an inner matrix B such that

$$\Sigma = B^T\Sigma_0 B.$$

Proof: Straightforward from last section and computation. ■

Let $\omega_1, .., \omega_{n_0}$ be the zeros of r. Then, the divisibility conditions (7) at point where those relations have a sense are reduced to a set of interpolation conditions

$$\begin{cases} \left(\frac{p_2^*}{p_1}(\omega_j)\right)^2 = (-1)^{n+1}\frac{p^*}{q}(\omega_j) \\ \left(\frac{p_1^*}{p_2}(\omega_j)\right)^2 = (-1)^{n+1}\frac{p^*}{q}(\omega_j) \end{cases} \quad j = 1 \ldots n_0. \quad (12)$$

We can also consider the conditions at zeros of r^*. (12) gives us some insight on how we will be able to build p_1 and p_2 from the knowledge of p and q. But those are not interpolation conditions between polynomials but rational fractions and we need to be careful. Note, however, that the Schur assumption gives us a good regularity of the conditions in Π^+. We will see later that the values of $\frac{p^*}{q}$ at well chosen points will be determining to the construction of p_1 and p_2. The multiplicities of the zeros of r will play an important role in the following construction based on the factorization of matrix B defined by (6). But when the roots are single, the interpolation condition being quadratic with respect to p_1 and p_2, we have two choices of square root value of the right part of the conditions (12) for a total of 2^k extensions of degree n if k is the degree of r.

V. Factorization and classes of equivalence

In this section, we will define the equivalence classes of the potential extensions 3×3 of $\frac{p}{q}$. We will also introduce a proposition that will be used later for root elimination. From the Potapov symmetric factorization, all rational inner matrices are generated by products of elementary Blaschke factors of the form $UB_\omega(s)U^*$ where U is a unitary matrix and

$$B_\omega(s) = \begin{bmatrix} b_\omega & 0 \\ 0 & I_2 \end{bmatrix}.$$

Consider $u \in \mathbb{C}^3$ the unit vector first column of U, then

$$UB_\omega(s)U^* = I_3 + \left(\frac{s-\omega}{s+\bar{\omega}} - 1\right)uu^*. \tag{13}$$

We thus obtain that elementary Blaschke factors all converge to I_3 at infinity. Note that this factor preserves $\frac{p}{q}$ which implies that with the choices of U that there is a diagonal block decomposition with 1 at the lower part. That means that

$$B^*(s) = V \times \prod_{i=1}^{n_0} \frac{1}{s+\bar{\omega_i}} \begin{bmatrix} l_1^i(s) & -l_2^{i*}(s) & 0 \\ l_2^i(s) & l_1^{i*}(s) & 0 \\ 0 & 0 & s+\bar{\omega_i} \end{bmatrix}$$

with $\omega_1, ..., \omega_{n_0}$ the zeros of r and V a unitary matrix (we include the $(-1)^n$ in factor of the second line of B^* in V). We call B_j^* the following

$$B_j^*(s) = \prod_{i=1}^{j} \frac{1}{s+\bar{\omega_i}} \begin{bmatrix} l_1^i(s) & -l_2^{i*}(s) & 0 \\ l_2^i(s) & l_1^{i*}(s) & 0 \\ 0 & 0 & s+\bar{\omega_i} \end{bmatrix}$$

which will also be written

$$B_j^*(s) = \frac{1}{\prod_{i=1}^{j}(s+\bar{\omega_i})} \begin{bmatrix} p_1^j(s) & p_2^j(s) & 0 \\ -p_2^{j*}(s) & p_1^{j*}(s) & 0 \\ 0 & 0 & \prod_{i=1}^{j}(s+\bar{\omega_i}) \end{bmatrix}.$$

One can then express p_1^{j+1} and p_2^{j+1} in terms of p_1^j, p_2^j, l_1^{j+1} and l_2^{j+1} :

$$p_1^{j+1} = p_1^j l_1^{j+1} + p_2^j l_2^{j+1} \tag{14}$$

$$p_2^{j+1} = p_2^j l_1^{j+1*} - p_1^j l_2^{j+1*} \tag{15}$$

By convention, we can take $p_1^0 = 1$ and $p_2^0 = 0$ and have $B_0^* = I_3$ multiply the B_j^* on the left. The p_1^j and p_2^j will be the seeds of p_1 and p_2 but with the operation of V. Now let's introduce a lemma that will be useful for the construction of p_1 and p_2. In this lemma, we use an analytic function λ of the variable s. The following notations will ease up a bit the statement of the lemma

$$\begin{aligned} \alpha_j^\lambda &\triangleq p_2^{j*} + \lambda p_1^j \\ \beta_j^\lambda &\triangleq p_1^{j*} - \lambda p_2^j. \end{aligned}$$

Lemma 2: Let $j < n_0$, $j \in \mathbb{N}$. For all λ analytic, on the domain of analyticity of λ,

$$\begin{aligned} \alpha_{j+1}^\lambda &= l_1^{j+1}\alpha_j^\lambda - l_2^{j+1}\beta_j^\lambda \\ \beta_{j+1}^\lambda &= l_2^{j+1*}\alpha_j^\lambda + l_1^{j+1*}\beta_j^\lambda \end{aligned}$$

Proof: The proof is straightforward with some computation using (14) and (15).

$$p_1^{j+1*} = p_1^{j*} l_1^{j+1*} + p_2^{j*} l_2^{j+1*} \tag{16}$$

$$p_2^{j+1*} = p_2^{j*} l_1^{j+1} - p_1^{j*} l_2^{j+1} \tag{17}$$

Let λ be a complex analytic function. Now with computations of (17) plus λ times (14) and (16) minus λ times (15) and factorizations by the (l_i^{j+1}) and (l_i^{j+1*}), we obtain directly the result. ■

From this lemma, we can prove by iteration the following proposition

Proposition 1: Let $j < n_0$, $j \in \mathbb{N}$ and $k \leq j$, $k \in \mathbb{N}$. For all λ analytic complex, on the domain of analyticity of λ,

$$\begin{aligned} \alpha_{j+1}^\lambda &= L_{1,j}^k \alpha_{j-k}^\lambda - L_{2,j}^k \beta_{j-k}^\lambda \\ \beta_{j+1}^\lambda &= L_{2,j}^{k*\alpha_{j-k}^\lambda} + L_{1,j}^{k*}\beta_{j-k}^\lambda \end{aligned}$$

where $L_{1,j}^0 = l_1^{j+1}$, $L_{2,j}^0 = l_2^{j+1}$ and for $j \geq 1$ and $k \geq 1$

$$\begin{aligned} L_{1,j}^k &= l_1^{j+1} L_{1,j-1}^{k-1} - l_2^{j+1} L_{2,j-1}^{k-1*} \\ L_{2,j}^k &= l_2^{j+1} L_{1,j-1}^{k-1*} + l_1^{j+1} L_{2,j-1}^{k-1} \end{aligned}$$

with $\deg(L_{1,j}^k) = k+1$ and $\deg(L_{2,j}^k) \leq k$.

Let's have a quick look at the action of V. First of all, V preserve the third column and third lines, so we can write

$$V = \begin{bmatrix} v_{11} & v_{12} & 0 \\ v_{21} & v_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where the (v_{ij}) are constants. Then, $B = V \times B_{n_0}^*$, so in particular,

$$\begin{aligned} p_1 &= v_{11}p_1^{n_0} - v_{12}p_2^{n_0*} \\ p_2 &= v_{11}p_2^{n_0} + v_{12}p_1^{n_0*}. \end{aligned}$$

V being a unitary matrix, we have $VV^* = I_3$, thus $|v_{11}|^2 + |v_{12}|^2 = 1$. We can now remark that $p_1p_1^* + p_2p_2^* = p_1^{n_0}p_1^{n_0*} + p_2^{n_0}p_2^{n_0*}$. Considering values at infinity of S and Σ, we get $V^TV = I_3$, this means that V has real coefficient, thus V is an orthogonal matrix. We now consider the orthogonal matrix group action over S and thus define an equivalence class of solutions. In the following, if not mentioned otherwise, it is considered that the representation of the equivalence class of (p_1, p_2) will be the couple (p_1, p_2) such that the degree of

p_2 is strictly lesser than the degree of p_1. Using Proposition 1 with $j = n_0 - 1$, $k = j$ and $\lambda = 0$, we have :

$$\begin{array}{rcl} p_2^{n_0*} & = & L_{1,n_0-1}^{n_0-1} p_2^{0*} - L_{2,n_0-1}^{n_0-1} p_1^{0*} \\ p_1^{n_0*} & = & L_{2,n_0-1}^{n_0-1*} p_2^{0*} + L_{1,n_0-1}^{n_0-1*} p_1^{0*} \end{array}$$

That is, considering that $p_1^0 = 1$ and $p_2^0 = 0$,

$$\begin{array}{rcl} p_2^{n_0} & = & -L_{2,n_0-1}^{n_0-1*} \\ p_1^{n_0} & = & L_{1,n_0-1}^{n_0-1}. \end{array}$$

This means that the degree of $p_2^{n_0}$ is strictly less than the degree of $p_1^{n_0}$.

We can thus consider V as the matrix

$$V = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 1 \end{array}\right].$$

having $p_1 = p_1^{n_0}$ and $p_2 = p_2^{n_0}$ to represent the class of equivalence. We can also remark that in this case, $V^T \Sigma V = \Sigma$ allowing us to temporarily forget about this matrix V.

VI. Roots elimination

We recall that $\omega_1, ..., \omega_{n_0}$ are the roots of r and potentially, there may be i and j different from i such that $\omega_i = \omega_j$. We will now develop how the relations of divisibility expressed at the end of section IV will be used to compute the polynomials l_1^i and l_2^i. Since q is stable, we can consider r as stable when sharing the roots of rr^* between r and r^* and have $q(-\bar{\omega}_i) \neq 0$ for all i integer in $\{1, ..., n_0\}$.

We will consider the factorization of B^* with the roots of r taken in with following order : $i \leq j \implies (Re(\omega_i) < Re(\omega_j)) \vee (Re(\omega_i) = Re(\omega_j) \wedge Im(\omega_i) \leq Im(\omega_j))$. That means that any multiple root of r will have indexes in a subset of $\{1, ..., n_0\}$ of consecutive integers. Note that this choice of order of the roots of r won't have any impact of the generality of the choice of B^* since for a given B^* there exists a Potapov factorization using this order.

Consider a family of complex $(\xi_i)_{i\in\{1,...,n_0\}}$ defined by

$$\xi_i^2 = (-1)^{n+1} \frac{p^*}{q}(-\bar{\omega}_i).$$

Of course, there are multiple choices of such families but we will span all the cases later. Let's now introduce the following matrices, product of Σ and the B_j^*. For j in $\{1, ..., n_0\}$

$$T_j = B_j^{*T} \Sigma B_j^*$$

Of course, $T_{n_0} = \Sigma_0$ and by convention, $T_0 = \Sigma$. Let $(r_j)_{j\in\{1,...,n_0\}}$ be the family of polynomials defined by

$$\forall j \in \{1, ..., n_0\}, r = r_j \prod_{i=1}^{j} (s - \omega_i)$$

Thus, $\prod_{i=1}^{j} (s + \bar{\omega}_i) = \frac{r^*}{r_j^*}$. We can compute the $(T_j)_{j\in\{1,...,n_0\}}$ using the explicit forms of Σ and the B_j^*. For all j in $\{1, ..., n_0\}$

$$T_j = \frac{1}{q} \left[\begin{array}{ccc} \frac{(r_j^*)^2}{rr^*}\varphi_j & \frac{(r_j^*)^2}{rr^*}\psi_j & r_j^* p_1^j \\ \frac{(r_j^*)^2}{rr^*}\psi_j & \frac{(r_j^*)^2}{rr^*}\theta_j & r_j^* p_2^j \\ r_j^* p_1^j & r_j^* p_2^j & p \end{array}\right].$$

with φ_j, ψ_j and χ_j (introduced only for writing purpose) defined by

$$\begin{array}{rcl} \varphi_j & = & p^*(p_1^j)^2 + (-1)^n q (p_2^{j*})^2 \\ \psi_j & = & p^* p_1^j p_2^j - (-1)^n q p_1^{j*} p_2^{j*} \\ \theta_j & = & p^*(p_2^j)^2 + (-1)^n q (p_1^{j*})^2. \end{array}$$

We also define for writing purpose only

$$\begin{array}{rcl} \Phi_j & = & (p_2^{j*})^2 - \xi_{j+1}^2 (p_1^j)^2 \\ \Psi_j & = & p_1^{j*} p_2^{j*} + \xi_{j+1}^2 p_1^j p_2^j \\ \Theta_j & = & (p_1^{j*})^2 - \xi_{j+1}^2 (p_2^j)^2. \end{array}$$

Since for all i greater than j, $r_j(\omega_i) = 0$, it's easy to see that

$$\forall i > j, T_j(-\bar{\omega}_i) = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{p}{q}(-\bar{\omega}_i) \end{array}\right].$$

Let's take a look at how to compute T_{j+1} from T_j for j in $\{1, ..., n_0\}$. Consider such a j. We can notice that $r_j = r_{j+1}(s - \omega_{j+1})$. We notice also

$$\begin{array}{rcl} \alpha_j^{\xi_{j+1}} \alpha_j^{-\xi_{j+1}} & = & \Phi_j \\ \beta_j^{\xi_{j+1}} \beta_j^{-\xi_{j+1}} & = & \Psi_j \\ \alpha_j^{\xi_{j+1}} \beta_j^{-\xi_{j+1}} & = & \Theta_j + \xi_{j+1} \frac{rr^*}{r_j r_j^*}. \end{array}$$

From above, we have

$$T_j(-\bar{\omega}_{j+1}) = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{p}{q}(-\bar{\omega}_{j+1}) \end{array}\right]$$

and by denoting $(.)'$ the derivative of $(.)$ with respect to the variable s, we also have, knowing that $r_j(-\bar{\omega}_{j+1}) = 0$:

$$T_j'(-\bar{\omega}_{j+1}) = \left[\begin{array}{ccc} -\frac{r_j^* r_{j+1}^*}{rr^*}\Phi_j & \frac{r_j^* r_{j+1}^*}{rr^*}\Theta_j & r_{j+1}^* \frac{p_1^j}{q} \\ \frac{r_j^* r_{j+1}^*}{rr^*}\Theta_j & -\frac{r_j^* r_{j+1}^*}{rr^*}\Psi_j & r_{j+1}^* \frac{p_2^j}{q} \\ r_{j+1}^* \frac{p_1^j}{q} & r_{j+1}^* \frac{p_2^j}{q} & (\frac{p}{q})' \end{array}\right](-\bar{\omega}_{j+1}).$$

Let's first have a look at the following factor

$$\frac{r_j^* r_{j+1}^*}{rr^*} = \frac{r_{j+1}^*}{r \prod_{i=1}^{j} (s + \bar{\omega}_i)}$$

Now, one can see that if $r_{j+1}^*(-\bar{\omega}_{j+1}) = 0$,

$$T_j'(-\bar{\omega}_{j+1}) = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\frac{p}{q})'(-\bar{\omega}_{j+1}) \end{array}\right]$$

This case, since $r_{j+1}^* = \prod_{i=j+2}^{n_0} (s + \bar{\omega}_i)$, corresponds to $\omega_{j+2} = \omega_{j+1}$ that is a root of r of multiplicity greater than one.

From lemma 1, we need to find a unitary vector $u_j = (u_{1j}, u_{2j}, 0)^T$ such that

$$\begin{array}{rcl} T_j(-\bar{\omega}_{j+1})u_j & = & 0 \\ u_j^T T_j'(-\bar{\omega}_{j+1})u_j & = & 0. \end{array} \quad (18)$$

A. Single multiplicity

Let's now consider $\omega_{j+1} \neq \omega_j$. In the case where $\omega_{j+2} = \omega_{j+1}$, it is clear that any choice of u_{1j} and u_{2j} will be suitable. Otherwise, let

$$\begin{array}{l} \nu_j^+ = p_1^{j*}(-\bar{\omega}_{j+1}) - \xi_{j+1}p_2^j(-\bar{\omega}_{j+1}) \\ \mu_j^+ = p_2^{j*}(-\bar{\omega}_{j+1}) + \xi_{j+1}p_1^j(-\bar{\omega}_{j+1}) \\ \nu_j^- = p_1^{j*}(-\bar{\omega}_{j+1}) + \xi_{j+1}p_2^j(-\bar{\omega}_{j+1}) \\ \mu_j^- = p_2^{j*}(-\bar{\omega}_{j+1}) - \xi_{j+1}p_1^j(-\bar{\omega}_{j+1}) \end{array}$$

There are two possible choices :

$$\begin{array}{l} u_{1j}^+ = \frac{\nu_j^+}{(|\mu_j^+|^2+|\nu_j^+|^2)^{\frac{1}{2}}} \\ u_{2j}^+ = \frac{\mu_j^+}{(|\mu_j^+|^2+|\nu_j^+|^2)^{\frac{1}{2}}} \end{array}$$

and

$$\begin{array}{l} u_{1j}^- = \frac{\nu_j^-}{(|\mu_j^-|^2+|\nu_j^-|^2)^{\frac{1}{2}}} \\ u_{2j}^- = \frac{\mu_j^-}{(|\mu_j^-|^2+|\nu_j^-|^2)^{\frac{1}{2}}} \end{array} \quad (19)$$

when $(\mu^+, \nu^+) \neq (0,0)$ and $(\mu^-, \nu^-) \neq (0,0)$. Combining μ_j^+ and ν_j^- and inversely won't give a proper solution.

We know that $T_{j+1}(-\bar{\omega}_{j+1})$ is well defined, thus

$$\left\{ \begin{array}{rcl} (p^*(p_1^{j+1})^2 + (-1)^n q(p_2^{j+1*})^2)(-\bar{\omega}_{j+1}) & = & 0 \\ (p^*(p_2^{j+1})^2 + (-1)^n q(p_1^{j+1*})^2)(-\bar{\omega}_{j+1}) & = & 0 \end{array} \right.$$

This means, using lemma 2 that we can choose either

$$\left\{ \begin{array}{rcl} l_1^{j+1}(-\bar{\omega}_{j+1})\mu_j^+ - l_2^{j+1}(-\bar{\omega}_{j+1})\nu_j^+ & = & 0 \\ l_2^{j+1*}(-\bar{\omega}_{j+1})\mu_j^+ + l_1^{j+1*}(-\bar{\omega}_{j+1})\nu_j^+ & = & 0 \end{array} \right.$$

or

$$\left\{ \begin{array}{rcl} l_1^{j+1}(-\bar{\omega}_{j+1})\mu_j^- - l_2^{j+1}(-\bar{\omega}_{j+1})\nu_j^- & = & 0 \\ l_2^{j+1*}(-\bar{\omega}_{j+1})\mu_j^- + l_1^{j+1*}(-\bar{\omega}_{j+1})\nu_j^- & = & 0 \end{array} \right.$$

to define l_1^{j+1} and l_2^{j+1}. Consider $(\mu_j, \nu_j) \in \{(\mu_j^+, \nu_j^+), (\mu_j^-, \nu_j^-)\}$ which will reflect one of both choices. Consider also $l_1^{j+1} = s + a_{j+1}$ and $l_2^{j+1} = b_{j+1}$. Then

$$\left\{ \begin{array}{rcl} (a_{j+1} - \bar{\omega}_{j+1})\mu_j - b_{j+1}\nu_j & = & 0 \\ \bar{b}_{j+1}\mu_j + (\bar{a}_{j+1} + \bar{\omega}_{j+1})\nu_j & = & 0 \end{array} \right.$$

which gives the system

$$\left\{ \begin{array}{rcl} a_{j+1}(|\mu_j|^2 + |\nu_j|^2) & = & \bar{\omega}_{j+1}|\mu_j|^2 - \omega_{j+1}|\nu_j|^2 \\ b_{j+1}(|\mu_j|^2 + |\nu_j|^2) & = & -\bar{\nu_j}\mu_j(\bar{\omega}_{j+1} + \omega_{j+1}) \end{array} \right. \quad (20)$$

This system is singular if and only if $(\mu_j, \nu_j) = (0,0)$. Since

$$p_1^j(-\bar{\omega}_{j+1})\mu_j + p_2^j(-\bar{\omega}_{j+1})\nu_j = (p_1^j p_1^{j*} + p_2^j p_2^{j*})(-\bar{\omega}_{j+1})$$

and knowing that

$$p_1^j p_1^{j*} + p_2^j p_2^{j*} = \prod_{i=1}^{j}(\omega_i - s)(\bar{\omega}_i + s)$$

this means that $\omega_{j+1} = \omega_j$. Conversely, if $\omega_{j+1} = \omega_j$, then $\xi_j^2 = \xi_{j+1}^2$ and we get $(\mu_j, \nu_j) = (0,0)$.

Of course, both (19) with (13) or (20) will give two choices of couples l_1^{j+1} and l_2^{j+1} when $\omega_{j+1} \neq \omega_j$

$$\left\{ \begin{array}{rcl} l_1^{j+1} & = & s + \frac{\bar{\omega}_{j+1}|\mu_j|^2 - \omega_{j+1}|\nu_j|^2}{|\mu_j|^2+|\nu_j|^2} \\ l_2^{j+1} & = & -\frac{\bar{\nu_j}\mu_j(\bar{\omega}_{j+1}+\omega_{j+1})}{|\mu_j|^2+|\nu_j|^2} \end{array} \right. \quad (21)$$

This means that in the case where r has only roots of multiplicity one, we can build all the possible matrices Σ_0 from Σ and that there are 2^{n_0} possibilities. More generally, if m is the number of single roots of r, there are 2^m factors to eliminate those roots.

B. Multiplicity two

First we just consider a multiplicity of two. We will generalize later to higher multiplicities. Now let's have a look at the case where $\omega_{j+1} = \omega_j$ for some j. As seen in the preceding section, any vector $u_j = (u_{1j}, u_{2j}, 0)^T$ will verify (18) for the first of the double roots. This is equivalent to taking any suitable couple $(\mu_{j-1}, \nu_{j-1}) \neq (0,0)$.

On the other hand, for the second root, writing relations between order $j+1$ and order j gives tautological relations. Let $\lambda = ((-1)^{n+1}\frac{p^*}{q})^{\frac{1}{2}}$. That is, λ is a function that associate s to a complex value such that the square of this value is $(-1)^{n+1}\frac{p^*}{q}(s)$.

Lemma 3: The function $\lambda : s \mapsto ((-1)^{n+1}\frac{p^*}{q})^{\frac{1}{2}}(s)$ is analytic around every root of r^*.

Proof: λ is analytic around every point where $\frac{p^*}{q}$ is well defined and not null since such neighborhood can be chosen simply connected. Since r and q are both stable, q does not vanish at a root of r^* so $\frac{p^*}{q}$ is well defined at these points. Since $rr^* + pp^* = qq^*$, if, at a root of r^*, p^* vanishes, then qq^* also. Since p and q are coprime, the same is true for p^* and q^* and we can conclude. ■

In the following, we study our functions at the roots of r^* so we have always λ analytic in a well chosen neighborhood.

Since $\alpha_{j+1}^{\lambda} \times \alpha_{j+1}^{-\lambda} = p_2^{j+1*} + (-1)^n\frac{p^*}{q}(p_1^{j+1})^2$ and $T_{j+1}(-\bar{\omega}_j)$ is well defined, we get that $(s+\bar{\omega}_j)^2$ divides $\alpha_{j+1}^{\lambda} \times \alpha_{j+1}^{-\lambda}$. There are thus two possibilities :

- $\alpha_{j+1}^{\lambda}(-\bar{\omega}_j) = \alpha_{j+1}^{-\lambda}(-\bar{\omega}_j) = 0$
- $(\alpha_{j+1}^{\lambda}(-\bar{\omega}_j) = (\alpha_{j+1}^{\lambda})'(-\bar{\omega}_j) = 0) \vee (\alpha_{j+1}^{-\lambda}(-\bar{\omega}_j) = (\alpha_{j+1}^{-\lambda})'(-\bar{\omega}_j) = 0)$.

1) $\alpha_{j+1}^{\lambda}(-\bar{\omega}_j) = \alpha_{j+1}^{-\lambda}(-\bar{\omega}_j) = 0$: In this case, also the sum and difference of both functions is null at the point $-\bar{\omega}_j$. Thus, $(s+\bar{\omega}_j)$ divides p_1^{j+1} and p_2^{j+1*}. Also, since

$$\begin{array}{rl} \alpha_{j+1}^{\lambda} \times \beta_{j+1}^{-\lambda} = & p_1^{j+1*}p_2^{j+1*} + (-1)^{n+1}\frac{p^*}{q}p_1^{j+1}p_2^{j+1} \\ & +\lambda(p_1^{j+1}p_1^{j+1*} + p_2^{j+1}p_2^{j+1*}) \end{array}$$

and knowing that $(s+\bar{\omega}_j)^2$ divides both

$$p_1^{j+1*}p_2^{j+1*} + (-1)^{n+1}\frac{p^*}{q}p_1^{j+1}p_2^{j+1}$$

$(T_{j+1}(\bar{\omega}_j)$ being well defined) and

$$p_1^{j+1}p_1^{j+1*} + p_2^{j+1}p_2^{j+1*} = (\omega_j - s)(\bar{\omega}_j + s)\prod_{i=1}^{j}(\omega_i - s)(\bar{\omega}_i + s),$$

we obtain that $\beta^{-\lambda}_{j+1}(-\bar{\omega}_j) = 0$. By computation of $\alpha^{-\xi_j}_{j+1} \times \beta^{\xi_j}_{j+1}$ and an analog argument, we obtain also that $\beta^{\lambda}_{j+1}(-\bar{\omega}_j) = 0$. Thus the same for the sum and difference and we get that $(s+\bar{\omega}_j)$ divides p_2^{j+1} and p_1^{j+1*}. Considering that

$$\frac{p_1^{j+1}p_1^{j+1*} + p_2^{j+1}p_2^{j+1*}}{p_1^{j-1}p_1^{j-1*} + p_2^{j-1}p_2^{j-1*}} = (\omega_j - s)^2(\bar{\omega}_j + s)^2,$$

we finally can express

$$\begin{array}{rcl} p_1^{j+1} & = & (s-\omega_j)(\bar{\omega}_j + s)p_1^{j-1} \\ p_2^{j+1} & = & (s-\omega_j)(\bar{\omega}_j + s)p_2^{j-1} \end{array} \qquad (22)$$

We now consider relations between orders $j+1$ and $j-1$. For that, we use Proposition 1 with $k = 1$. We thus have

$$\left\{ \begin{array}{rcl} \alpha^{\lambda}_{j+1} & = & L^1_{1,j}\alpha^{\lambda}_{j-1} - L^1_{2,j}\beta^{\lambda}_{j-1} \\ \beta^{\lambda}_{j+1} & = & L^{1*}_{2,j}\alpha^{\lambda}_{j-1} + L^{1*}_{1,j}\beta^{\lambda}_{j-1} \end{array} \right.$$

From Proposition 1 we get also

$$\left\{ \begin{array}{rcl} L^1_{1,j} & = & l_1^{j+1}l_1^j - l_2^{j+1}l_2^{j*} \\ L^1_{2,j} & = & l_2^{j+1}l_1^{j*} + l_1^{j+1}l_2^j \end{array} \right.$$

So, computing $\alpha^{\lambda}_{j+1} - \alpha^{-\lambda}_{j+1}$ we get

$$p_1^{j+1} = (l_1^{j+1}l_1^j - l_2^{j+1}l_2^{j*})p_1^{j-1} + (l_2^{j+1}l_1^{j*} + l_1^{j+1}l_2^j)p_2^{j-1}$$

which by identification with (22) gives

$$\left\{ \begin{array}{rcl} l_1^{j+1}l_1^j - l_2^{j+1}l_2^{j*} & = & (s-\omega_j)(\bar{\omega}_j + s) \\ l_2^{j+1}l_1^{j*} + l_1^{j+1}l_2^j & = & 0 \end{array} \right. \qquad (23)$$

Let a_{j+1} and b_{j+1} in $\mathbb{C}$ defined by $l_1^{j+1}(s) = s + a_{j+1}$ and $l_2^{j+1}(s) = b_{j+1}$. Let $(\mu, \nu) \neq (0,0)$ be the choice of (μ_{j-1}, ν_{j-1}) made at rank j, thus we have

$$\left\{ \begin{array}{rcl} a_j(|\mu|^2 + |\nu|^2) & = & \bar{\omega}_j|\nu|^2 - \omega_j|\mu|^2 \\ b_j(|\mu|^2 + |\nu|^2) & = & -\bar{\nu}\mu(\bar{\omega}_j + \omega_j) \end{array} \right.$$

Some computation shows that (23) is equivalent to

$$\left\{ \begin{array}{rcl} l_1^{j+1} & = & s + \frac{\bar{\omega}_j|\nu|^2 - \omega_j|\mu|^2}{|\mu|^2+|\nu|^2} \\ l_2^{j+1} & = & -\frac{\bar{\nu}\mu(\bar{\omega}_j+\omega_j)}{|\mu|^2+|\nu|^2} \end{array} \right. \qquad (24)$$

2) $(\alpha^{\lambda}_{j+1}(-\bar{\omega}_j) = (\alpha^{\lambda}_{j+1})'(-\bar{\omega}_j) = 0) \vee (\alpha^{-\lambda}_{j+1}(-\bar{\omega}_j) = (\alpha^{-\lambda}_{j+1})'(-\bar{\omega}_j) = 0)$*:* Let's consider that the relation holds with λ and not $-\lambda$. It will then clearly appear that the proof will be the same in case of the reverse. We have

$$(\alpha^{\lambda}_j)' = (p_2^{j*})' + \lambda(p_1^j)' + \lambda' p_1^j$$

Also, using proposition 1 with $k = 0$ we get

$$(\alpha^{\lambda}_{j+1})' = l_1^{j+1}(\alpha^{\lambda}_j)' - l_2^{j+1}(\beta^{\lambda}_j)' + \alpha^{\lambda}_j \qquad (25)$$

Since $(s+\bar{\omega}_j)^2$ divides $\alpha^{-\lambda}_{j+1} \times \beta^{\lambda}_{j+1}$ and $\alpha^{-\lambda}_{j+1}(-\bar{\omega}_j) \neq 0$, we obtain that $\beta^{\lambda}_{j+1}(-\bar{\omega}_j) = (\beta^{\lambda}_{j+1})'(-\bar{\omega}_j) = 0$. We have

$$(\beta^{\lambda}_j)' = (p_1^{j*})' - \lambda(p_2^j)' - \lambda' p_2^j$$

and with proposition 1,

$$(\beta^{\lambda}_{j+1})' = l_2^{j+1*}(\alpha^{\lambda}_j)' + l_1^{j+1*}(\beta^{\lambda}_j)' - \beta^{\lambda}_j \qquad (26)$$

Let's call $(\mu', \nu') \neq (0,0)$ the value of $((\alpha^{\lambda}_j)', \beta^{\lambda}_j)')$ at $\bar{\omega}_j$. We obtain from (25) and (26) at point $\bar{\omega}_j$ the following relations

$$\left\{ \begin{array}{rcl} 0 & = & (a_{j+1} - \bar{\omega}_j)\mu' - b_{j+1}\nu' \\ 0 & = & \bar{b}_{j+1}\mu' + (\bar{a}_{j+1} + \bar{\omega}_j)\nu' \end{array} \right.$$

with a_{j+1} and b_{j+1} in $\mathbb{C}$ defined by $l_1^{j+1}(s) = s + a_{j+1}$ and $l_2^{j+1}(s) = b_{j+1}$. This leads us to the values of l_1^{j+1} and l_2^{j+1}

$$\left\{ \begin{array}{rcl} l_1^{j+1} & = & s + \frac{\bar{\omega}_j|\mu'|^2 - \omega_j|\nu'|^2}{|\mu'|^2+|\nu'|^2} \\ l_2^{j+1} & = & -\frac{\bar{\nu}'\mu'(\bar{\omega}_j+\omega_j)}{|\mu'|^2+|\nu'|^2} \end{array} \right. \qquad (27)$$

Remark: We have seen at the beginning of the single case section that for multiplicities greater than 1, the first factor can be taken freely. Computation shows that this case corresponds to the case where $\alpha^{\lambda}_{j+1}(-\bar{\omega}_j) = \alpha^{-\lambda}_{j+1}(-\bar{\omega}_j) = 0$.

C. Higher multiplicities

Let k be the multiplicity of the root ω_j of r. As in the multiplicity two case, we know that $(s + \bar{\omega}_j)^k$ divides $\alpha^{\lambda}_{j+k-1} \times \alpha^{-\lambda}_{j+k-1}$. Thus, there exists an integer k' such that $(\alpha^{\lambda}_{j+k-1})^{(i)}(-\bar{\omega}_j) = 0$ for all $i \leq k'$ and $(\alpha^{\lambda}_{j+k-1})^{(k'+1)}(-\bar{\omega}_j) \neq 0$ where $^{(i)}$ is the i-th derivative along s. That is, the order of the Taylor series of α^{λ}_{j+k-1} around $-\bar{\omega}_j$ is k' and we will note it $o(\alpha^{\lambda}_{j+k-1})(-\bar{\omega}_j) = k'$. Let's consider m the minimum of the orders of the Taylor series of α^{λ}_{j+k-1} and $\alpha^{-\lambda}_{j+k-1}$ around $-\bar{\omega}_j$. We are in the same conditions as in section VI-B.1 and with exactly the same computations, we find that $(s+\bar{\omega}_j)^m$ divides p_1^{j+k-1}, p_1^{j+k-1*}, p_2^{j+k-1} and p_2^{j+k-1*}. Since the matrices B^* linked to those auto reciprocal parts of p_1 and p_2 are all of the form

$$\begin{pmatrix} \frac{s-\omega_j}{s+\bar{\omega}_j} & 0 & 0 \\ 0 & \frac{s-\omega_j}{s+\bar{\omega}_j} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (28)$$

these matrices commute with the other B^* and the order of elimination of the auto reciprocal part among all the other zeros is not important. We will thus consider that this part is handled at the end. Let's consider, for simplicity of notations, that the order of the Taylor series of $\alpha^{\lambda}_{j+k-2m-1}$ around $-\bar{\omega}_j$ is higher than the one of the Taylor series of $\alpha^{-\lambda}_{j+k-2m-1}$ around same point. We then have $\alpha^{\lambda}_{j+k-2m-1}(-\bar{\omega}_j) = (\alpha^{\lambda}_{j+k-2m-1})'(-\bar{\omega}_j) = ... = (\alpha^{\lambda}_{j+k-2m-1})^{(k-2m-1)}(-\bar{\omega}_j) = 0$. The same relations hold for $\alpha^{\lambda}_{j+k-2m-1}$ and its derivatives along s at the same point since $(s+\bar{\omega}_j)^{k-2m}$ divides $\alpha^{-\lambda}_{j+k-2m-1} \times \beta^{\lambda}_{j+k-2m-1}$ and $\alpha^{-\lambda}_{j+k-2m-1}(-\bar{\omega}_j) \neq 0$. Let i be an integer in $\{1, .., k-2m-1\}$.

Using Proposition 1 with $k = 0$ we get

$$\left\{ \begin{array}{rcl} (\alpha^{\lambda}_{j+i})^{(i)} & = & l_1^{j+i}(\alpha^{\lambda}_{j+i-1})^{(i)} - l_2^{j+i}(\beta^{\lambda}_{j+i-1})^{(i)} \\ & & +(\alpha^{\lambda}_{j+i-1})^{(i-1)} \\ (\beta^{\lambda}_{j+i})^{(i)} & = & l_2^{j+i*}(\alpha^{\lambda}_{j+i-1})^{(i)} + l_1^{j+i*}(\beta^{\lambda}_{j+i-1})^{(i)} \\ & & -(\beta^{\lambda}_{j+i-1})^{(i-1)} \end{array} \right. \qquad (29)$$

We have $(\alpha_{j+i}^{\lambda})^{(i)}(-\bar{\omega}_j) = (\beta_{j+i}^{\lambda})^{(i)}(-\bar{\omega}_j) = 0$ by construction of those functions. Let's call $(\mu_i, \nu_i) \neq (0,0)$ the value of $((\alpha_{j+i-1}^{\lambda})^{(i)}, \beta_{j+i-1}^{\lambda})^{(i)})$ at $\bar{\omega}_j$. By evaluating relations (29) at point $\bar{\omega}_j$ we get, as in section VI-B.2 the following

$$\left\{ \begin{array}{lcl} l_1^{j+i} & = & s + \frac{\bar{\omega}_j |\mu_{i-1}|^2 - \omega_j |\nu_{i-1}|^2}{|\mu_{i-1}|^2 + |\nu_{i-1}|^2} \\ l_2^{j+i} & = & -\frac{\bar{\nu}_{i-1}\mu_{i-1}(\bar{\omega}_j + \omega_j)}{|\mu_{i-1}|^2 + |\nu_{i-1}|^2} \end{array} \right. \quad (30)$$

which by iteration over i gives the solution.

Thus, a root of multiplicity k leads us to $k+1$ potential factors for the elimination of the k identical roots.

D. Construction of the extensions

To build an extension, one has to choose between λ and $-\lambda$ at each step. Depending on the multiplicities of zeros, the l_i^j will be given by the formulae (21), (24), (27) and (30). Note that in the case of (24), we would rather compute both factors corresponding to the root at the same time using (22) as done in the high multiplicities case.

E. Number of extensions

Theorem 3: Let $\frac{p}{q}$ be a Schur rational function with p and q monic polynomials of degree n such that $\frac{p}{q}$ is irreducible and q is stable. Let r be a stable polynomial of degree $n_0 < n$ such that $qq^* - pp^* = rr^*$. Let $k \leq n_0$ be the number, multiplicity included, of the non imaginary roots of r. Let $m \leq k$ the number of simple zeros of r and $\{m_1, ..., m_{k-m}\}$ potentially null (if $m = k$) the set of multiplicities of the multiple roots of r. Then, the number N of extensions 3×3 of degree n of $\frac{p}{q}$ is given by the formula

$$N = 2^m \prod_{i=1}^{k-m} (m_i + 1). \quad (31)$$

VII. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

We have determined and counted the different possible symmetric inner extensions of size 3×3 of minimal degree of a scalar rational Schur function. It is noteworthy that this number is finite and follows a non evident rule. The proof given is constructive, meaning that an algorithm based on the computations done in the proof will allow the explicit determination of all the extensions of size 3×3 using the iteration formulae as explained in section VI-D. As could be expected, over polynomials r ahving zeros with multiplicities the number of extension collapses, and such points can be seen as singularities of the real algebraic manifold defined by equations 7.

B. Future Works

From a physical point of view, it is more interesting to solve the inverse problem of, given polynomials p_1 and p_2, finding the p and q polynomials of degree n considered in this study. For this parametrization, we need to not only use the interpolation conditions (12) but also the asymptotic limits at infinity in the equation $1 - \frac{pp^*}{qq^*} = \frac{rr^*}{qq^*}$ for coefficients of p and q of degree higher than n_0.

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