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Computing the controllability radius for higher order systems using semidefinite programming

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Abstract— **We propose here a new approach for computing the controllability radius for higher order systems. The original problem is restated as an eigenvalue minimization problem and further transformed into a semidefinite programming problem. We then relax our problem by imposing a sum-of-squares constraint. Experimental results show that our algorithm can obtain good results in a much smaller computation time than the existing algorithm.**

I. Introduction

Controllability is a generic property of linear systems in the sense that the set of uncontrollable systems has measure zero. Hence, it is meaningful to analyze the robustness of this property. A way to quantify robustness is the distance to the nearest uncontrollable system, which was introduced for first order systems in [9]. Later, in [4], the same problem was proven to be equivalent to a singular value minimization, which led to algorithms like those in [3]. A powerful method of computing the controllability radius is given in [5].

The case of higher order systems has recently been discussed in [8]. Although the algorithm from [8] can provide a good interval in which the solution lies, the method is very slow when increasing the accuracy.

The algorithm proposed here is based on the minimization problem equivalent to the original problem of the controllability radius. We recast the problem using positive polynomials. The problem thus obtained is then substituted with its relaxed version by changing the positivity requirement to a sum-of-squares constraint. We prove that our method can obtain good results in lesser computation time than the algorithm from [8].

The remainder of this paper is organized as follows. We state the problem of computing the controllability radius in Section II and describe the computation of the distance to uncontrollability in Section III. The results are presented in Section IV. We conclude in Section V.

Notation: Multivariate entities (vectors, matrices) are denoted by bold characters. $\mathbf{Tr}\,\boldsymbol{M}$ is the trace of the matrix $\boldsymbol{M}$. We denote a Hermitian positive semidefinite matrix $\boldsymbol{M}$ by $\boldsymbol{M} \succeq \mathbf{0}$. $\lambda_{\min}(\boldsymbol{M})$ and $\sigma_{\min}(\boldsymbol{M})$ are the minimum eigenvalue and minimum singular value of the matrix $\boldsymbol{M}$, respectively. The superscript ^{H} denotes conjugate transposition. We denote by $\|\cdot\|$ the 2-norm. $\boldsymbol{I}_n$ is the identity matrix of size $n\times n$; when used without the subscript, the size is clear from the context. $\otimes$ stands for the Kronecker product.

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II. The distance to uncontrollability

Let us consider a k-th order continuous-time system

$$\boldsymbol{K}_k\boldsymbol{x}^{(k)}(t)+\ldots+\boldsymbol{K}_1\dot{\boldsymbol{x}}(t)+\boldsymbol{K}_0\boldsymbol{x}(t) \quad = \quad \boldsymbol{B}\boldsymbol{u}(t), \tag{1}$$

where $\boldsymbol{K}_i \in \mathbb{C}^{n\times n}, \forall i=0:k$, $\boldsymbol{B}\in\mathbb{C}^{n\times m}$, $\boldsymbol{x}$ is the state vector of the system, $\boldsymbol{u}$ is the vector of the inputs and $t\in\mathbb{R}$. We are interested "how far" the system (1) is from an uncontrollable one. The distance to uncontrollability for the system (1) was defined in [8] as

$$\begin{array}{rcl}\rho(\mathcal{K},\boldsymbol{\alpha}) & = & \min\limits_{\mathcal{K}} \quad \|\left[\begin{array}{cccc}\boldsymbol{\Delta}_{\boldsymbol{K}_k} & \ldots & \boldsymbol{\Delta}_{\boldsymbol{K}_0} & \boldsymbol{\Delta}_{\boldsymbol{B}}\end{array}\right]\| \\ & & \text{s.t.} \quad \mathcal{K}+\boldsymbol{\Delta}_{\mathcal{K}} \text{ is uncontrollable}\end{array} \tag{2}$$

where $\mathcal{K}=(\boldsymbol{K}_k,\ldots,\boldsymbol{K}_0,\boldsymbol{B})$ denotes the system from (1), $\boldsymbol{\Delta}_{\mathcal{K}}=(\alpha_k\boldsymbol{\Delta}_{\boldsymbol{K}_k},\ldots,\alpha_0\boldsymbol{\Delta}_{\boldsymbol{K}_0},\boldsymbol{\Delta}_{\boldsymbol{B}})$ represents the perturbations that affect the system (1) and $\boldsymbol{\alpha}=\left[\begin{array}{ccc}\alpha_k & \ldots & \alpha_0\end{array}\right]$ is a vector of given nonnegative real numbers used to modulate the perturbations. Note that the definition (2) would be the same for a discrete-time higher order system.

In the following sections we extend the work proposed in [3] to the case of higher order systems—we formulate a new solution for computing the controllability radius of higher order systems with the aid of sum-of-squares polynomials.

III. Transformation to sum-of-squares

It was proved first in [8] and then in [6] that the problem (2) reduces to the singular value minimization problem

$$\rho(\mathcal{K},\boldsymbol{\alpha}) \quad = \quad \min_{\lambda\in\mathbb{C}}\sigma_{\min}\left(\boldsymbol{H}(\lambda)\right), \tag{3}$$

where

$$\boldsymbol{H}(\lambda) \quad = \quad \left[\begin{array}{cc}\dfrac{\boldsymbol{P}(\lambda)}{\sqrt{s_{\boldsymbol{\alpha}}(|\lambda|)}} & \boldsymbol{B}\end{array}\right], \tag{4}$$

$$\boldsymbol{P}(\lambda) \quad = \quad \sum_{i=0}^{k}\lambda^i\boldsymbol{K}_i \tag{5}$$

and

$$s_{\boldsymbol{\alpha}}(|\lambda|) \quad = \quad \sum_{i=0}^{k}\alpha_i^2|\lambda|^{2i}. \tag{6}$$

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We recast the problem (3) as an eigenvalue minimization problem using the fact that $\sigma_{\min}(\boldsymbol{M}) = \sqrt{\lambda_{\min}(\boldsymbol{M}\boldsymbol{M}^H)}$, for an arbitrary matrix $\boldsymbol{M}$. Thus, we obtain the problem

$$\begin{aligned} \tau_o &= \max_{\tau\geq 0} \quad \tau \\ &\quad \text{s.t.} \quad \boldsymbol{H}(\lambda)\cdot\boldsymbol{H}(\lambda)^H \succeq \tau\cdot\boldsymbol{I}, \quad \forall\lambda\in\mathbb{C} \end{aligned} \tag{7}$$

The controllability radius is $\rho(\boldsymbol{\mathcal{K}},\boldsymbol{\alpha}) = \sqrt{\tau_o}$. Moreover, the problem (7) can be written as

$$\begin{aligned} \tau_o &= \max_{\tau\geq 0} \quad \tau \\ &\quad \text{s.t.} \quad \boldsymbol{R}(\lambda) \succeq \boldsymbol{0}, \quad \forall\lambda\in\mathbb{C} \end{aligned} \tag{8}$$

where

$$\boldsymbol{R}(\lambda) = \boldsymbol{P}(\lambda)\boldsymbol{P}(\lambda)^H + s_{\boldsymbol{\alpha}}(|\lambda|)\boldsymbol{B}\boldsymbol{B}^H - s_{\boldsymbol{\alpha}}(|\lambda|)\tau\boldsymbol{I}. \tag{9}$$

We aim now to transform the polynomial $\boldsymbol{R}(\lambda)$ into a bivariate matrix polynomial by splitting the complex variable λ into its real and imaginary part: $\lambda = x + jy$. Hence, the polynomial $\boldsymbol{R}(\lambda)$ becomes

$$\boldsymbol{R}(x,y) = \boldsymbol{P}(x,y)\boldsymbol{P}(x,y)^H + s_{\boldsymbol{\alpha}}(x,y)\boldsymbol{B}\boldsymbol{B}^H - s_{\boldsymbol{\alpha}}(x,y)\tau\boldsymbol{I}. \tag{10}$$

Finding a maximum τ in (8) using (10) under a positive polynomial constraint is a hard task. In order to avoid the positivity constraint we "relax" our problem by requiring the polynomial $\boldsymbol{R}(x,y)$ to be sum-of-squares. Using this sufficient condition we replace the problem (8) by

$$\begin{aligned} \tau_1 &= \max_{\tau\geq 0} \quad \tau \\ &\quad \text{s.t.} \quad \boldsymbol{R}(x,y) \text{ is sum-of-squares} \end{aligned} \tag{11}$$

The problem (11) is more conservative than problem (8) so we obtain $\tau_1 \leq \tau_o$. Next, we transform the problem (11) into a semidefinite programming (SDP) problem [2].

Denoting a basis $\boldsymbol{\Psi}(v) = [\ \boldsymbol{I}\ v\boldsymbol{I}\ \ldots\ v^k\boldsymbol{I}\ ]$, $v\in\mathbb{R}$, the polynomial $\boldsymbol{R}(x,y)$ is sum-of-squares if and only if ($\exists$) a matrix $\boldsymbol{Q}\succeq\boldsymbol{0}$, $\boldsymbol{Q}\in\mathbb{C}^{N\times N}$, such that

$$\boldsymbol{R}(x,y) = \boldsymbol{\Psi}(x,y)^T\cdot\boldsymbol{Q}\cdot\boldsymbol{\Psi}(x,y) \tag{12}$$

where $\boldsymbol{\Psi}(x,y) = \boldsymbol{\Psi}(y)\otimes\boldsymbol{\Psi}(x)$ and $N = (k+1)^2 n$. Thus, we have the SDP problem

$$\begin{aligned} \tau_1 &= \max_{\tau\geq 0,\boldsymbol{Q}} \quad \tau \\ &\quad \text{s.t.} \quad (12), \quad \boldsymbol{Q}\succeq\boldsymbol{0} \end{aligned} \tag{13}$$

The polynomial $\boldsymbol{R}(x,y)$ is of form

$$\boldsymbol{R}(x,y) = \sum_{k_1=0}^{2k}\sum_{k_2=0}^{2k}\boldsymbol{R}_{k_1,k_2}x^{k_1}y^{k_2}, \tag{14}$$

$\boldsymbol{R}_{k_1,k_2}\in\mathbb{C}^{n\times n}$. Considering (14) and (10), the problem (13) can be written [2], for each element of the matrix coefficients of the polynomial $\boldsymbol{R}(x,y)$, as

$$\begin{aligned} \tau_1 &= \max_{\tau\geq 0,\boldsymbol{Q}} \quad \tau \\ &\quad \text{s.t.} \quad (\boldsymbol{R}_{k_1,k_2})_{ij} = \mathbf{Tr}[(\boldsymbol{\Upsilon}_{k_2}\otimes\boldsymbol{\Upsilon}_{k_1}\otimes\boldsymbol{E}_{ji})\cdot\boldsymbol{Q}] \\ &\qquad\quad \boldsymbol{Q}\succeq\boldsymbol{0} \end{aligned} \tag{15}$$

where $k_1, k_2 = 0 : 2k$, $i = 0 : n$, $j = 0 : i$, $\boldsymbol{\Upsilon}_{k_p}$, $p = 1 : 2$, is a Hankel matrix with ones on the k_p-th antidiagonal and zeros elsewhere and E_{ji} is a matrix which has one on the (j,i) position and zeros elsewhere. (The matrix $\boldsymbol{Q}$ is called Gram matrix.)

The complexity of solving the problem (15) using SDP methods as the ones in the SeDuMi [10] library is $\mathcal{O}(N^6)$.

IV. Experimental results

We present in this section the results obtained for the controllability radius for several systems from the literature. We compare our results with the ones obtained by the algorithm from [8]. To test the algorithm from [8] we have used the `poly_dist_uncont` Matlab routine from http://home.ku.edu.tr/∼emengi/software/robuststability.html; setting an input accuracy ε, the function computes the margins of an interval $(\rho_\ell, \rho_u]$ in which lies the controllability radius, with $\rho_u - \rho_\ell \leq \varepsilon$. The problem (15) was solved using the SeDuMi [10] library. All the tests have been made on a 2 GHz PC with 2 GB RAM.

Example 1: We analyze first a descriptor system example considered in [8], with

$$\boldsymbol{K}_1 = \boldsymbol{I}_4, \quad \boldsymbol{K}_0 = \begin{bmatrix} 1 & 3 & 0 & 0 \\ -2 & 1 & 3 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & -2 & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}. \tag{16}$$

We consider $\varepsilon \in \{10^{-4}, 10^{-5}, 10^{-6}\}$ and apply perturbations in two cases: only to $\boldsymbol{K}_0$ ($\boldsymbol{\alpha} = [\ 0\ 1\ ]$) and both $\boldsymbol{K}_0$ and $\boldsymbol{K}_1$ ($\boldsymbol{\alpha} = [\ 1\ 1\ ]$). (Perturbations on the $\boldsymbol{B}$ matrix are always considered.)

We present in Table I the distances to uncontrollability obtained using the two different kinds of perturbations and setting several values of accuracy. One can observe that our algorithm offers good values for the controllability radius in less than two seconds, while the algorithm from [8] needs a much bigger computation time. Also note that the controllability radius is smaller when perturbations are applied also to $\boldsymbol{K}_1$.

Example 2: We take now a model for a tubular ammonia reactor (see e.g. [7]) with $\boldsymbol{K}_1 = \boldsymbol{I}_9$,

$$\boldsymbol{K}_{0_{(:,1:4)}} = \begin{bmatrix} -4.019 & 5.12 & 0 & 0 \\ -0.346 & 0.986 & 0 & 0 \\ -7.909 & 15.407 & -4.069 & 0 \\ -21.816 & 35.606 & -0.339 & -3.87 \\ -60.196 & 98.188 & -7.907 & 0.34 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{17}$$

TABLE I

CONTROLLABILITY RADII FOR THE SYSTEM CONSIDERED IN EXAMPLE 1.

α	$\sqrt{\tau_1}$	Algorithm from [8] $\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-6}$
[0 1]	0.4769411	(0.47688, 0.47696]	(0.476935, 0.476942]	(0.4769402, 0.4769411]
[1 1]	0.1425212	(0.14243, 0.14252]	(0.142515, 0.142523]	(0.1425205, 0.1425212]
		Time (in seconds)		
[0 1]	0.8	88	770	6809
[1 1]	1.1	64	708	10134

$$\boldsymbol{K}_{0_{(:,5:8)}} = \begin{bmatrix} -2.082 & 0 & 0 & 0 \\ -2.34 & 0 & 0 & 0 \\ -6.45 & 0 & 0 & 0 \\ -17.8 & 0 & 0 & 0 \\ -53.008 & 0 & 0 & 0 \\ 94 & -147.2 & 0 & 53.2 \\ 0 & 94 & -147.2 & 0 \\ 0 & 12.8 & 0 & -31.6 \\ 12.8 & 0 & 0 & 18.8 \end{bmatrix}, \tag{18}$$

$$\boldsymbol{K}_{0_{(:,9)}} = \begin{bmatrix} 0.87 & 0.97 & 2.68 & 7.39 & 20.4 & \mathbf{0} & -31.6 \end{bmatrix}^T \tag{19}$$

and

$$\boldsymbol{B}_{(1:4,:)} = \begin{bmatrix} 0.010 & 0.003 & 0.009 & 0.024 \\ -0.011 & -0.021 & -0.059 & -0.162 \\ -0.151 & 0 & 0 & 0 \end{bmatrix}^T, \tag{20}$$

$$\boldsymbol{B}_{(5:9,:)} = \begin{bmatrix} 0.068 & 0 & 0 & 0 & 0 \\ -0.445 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T. \tag{21}$$

Table II shows the results obtained for the controllability radius for different settings. One can see that the proposed algorithm offers good computation times comparing with the algorithm from [8], but fails to offer a good result in the case when we apply perturbations only to $\boldsymbol{K}_0$.

Example 3: We take now a model binary distillation column with condenser, reboiler and nine plates (see e.g. [7]) with $\boldsymbol{K}_1 = \boldsymbol{I}_8$,

$$\boldsymbol{K}_{0_{(:,1:4)}} = \begin{bmatrix} -0.991 & 0.529 & 0 & 0 \\ 0.522 & -1.051 & 0.596 & 0 \\ 0 & 0.522 & -1.118 & 0.596 \\ 0 & 0 & 0.522 & -1.548 \\ 0 & 0 & 0 & 0.922 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{22}$$

$$\boldsymbol{K}_{0_{(:,5:8)}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.718 & 0 & 0 & 0 \\ -1.64 & 0.799 & 0 & 0 \\ 0.922 & -1.721 & 0.901 & 0 \\ 0 & 0.922 & -1.823 & 1.021 \\ 0 & 0 & 0.922 & -1.943 \end{bmatrix} \tag{23}$$

and

$$\boldsymbol{B}_{(1:4,:)} = 10^{-3} \times \begin{bmatrix} 3.84 & 4.00 & 37.60 & 3.08 \\ -2.88 & -3.04 & -2.80 & -2.32 \end{bmatrix}^T, \tag{24}$$

$$\boldsymbol{B}_{(5:8,:)} = 10^{-3} \times \begin{bmatrix} 2.36 & 2.88 & 3.08 & 3.00 \\ -3.32 & -3.82 & -4.12 & -3.96 \end{bmatrix}^T. \tag{25}$$

The results presented in Table III show that our method offers good results comparing with the other algorithm.

Example 4: We take now a model for a binary distillation column with pressure variation (see e.g. [7]) with $\boldsymbol{K}_1 = \boldsymbol{I}_{11}$,

$$\boldsymbol{K}_{0_{(:,1:4)}} = 10^{-2} \times \begin{bmatrix} -1.40 & 0.43 & 0 & 0 \\ 0.95 & -1.38 & 0.46 & 0 \\ 0 & 0.95 & -1.41 & 0.63 \\ 0 & 0 & 0.95 & -1.58 \\ 0 & 0 & 0 & 0.95 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2.55 & 0 & 0 & 0 \end{bmatrix}, \tag{26}$$

$$\boldsymbol{K}_{0_{(:,5:8)}} = 10^{-2} \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.10 & 0 & 0 & 0 \\ -3.12 & 1.50 & 2.20 & 0 \\ 2.02 & -3.52 & 2.20 & 0 \\ 0 & 2.02 & -4.22 & 2.80 \\ 0 & 0 & 2.02 & -4.82 \\ 0 & 0 & 0 & 2.02 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

TABLE II
CONTROLLABILITY RADII FOR THE SYSTEM CONSIDERED IN EXAMPLE 2.

$\boldsymbol{\alpha}$	$\sqrt{\tau_1}$	Algorithm from [8] $\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$
[0 1]	0.08671	(0.046, 0.054]	(0.0538, 0.0545]	(0.05439, 0.05448]
[1 1]	0.04069	(0.033, 0.041]	(0.0402, 0.0409]	(0.04062, 0.04071]
		Time (in seconds)		
[0 1]	5.8	4.6	72	416
[1 1]	6.8	4.9	73	363

TABLE III
CONTROLLABILITY RADII FOR THE SYSTEM CONSIDERED IN EXAMPLE 3.

$\boldsymbol{\alpha}$	$\sqrt{\tau_1}$	Algorithm from [8] $\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-6}$
[0 1]	0.0013169	(0.00125, 0.00133]	(0.001312, 0.001319]	(0.0013160, 0.0013168]
[1 1]	0.0013141	(0.00123, 0.00132]	(0.001306, 0.001314]	(0.0013136, 0.0013143]
		Time (in seconds)		
[0 1]	4.6	298	3691	28169
[1 1]	3.9	154	3017	46786

$$\boldsymbol{K}_{0_{(:,9:11)}} = 10^{-2} \times \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.05 \\ 0 & 0 & 0.02 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3.70 & 0 & 0.02 \\ -5.72 & 4.20 & 0.05 \\ 2.02 & -4.83 & 0.05 \\ 0 & 2.55 & -1.85 \end{bmatrix} \quad (28)$$

and

$$\boldsymbol{B} = 10^{-4} \times \begin{bmatrix} 0 & 0 & 0 \\ 0.05 & -0.4 & 25 \\ 0.02 & -0.2 & 50 \\ 0.01 & -0.1 & 50 \\ 0 & 0 & 50 \\ 0 & 0 & 50 \\ -0.05 & 0.1 & 50 \\ -0.1 & 0.3 & 50 \\ -0.4 & 0.05 & 25 \\ -0.2 & 0.02 & 25 \\ 4.60 & 4.60 & 0 \end{bmatrix}. \quad (29)$$

As Table IV shows the time needed for our algorithm to compute the controllability radius is less than two seconds while the execution of the algorithm from [8] takes more than twelve hours in all cases.

Example 5: We present now an example of scalable size which describes the control of the heat flow in a thin rod (see e.g. [7]). The descriptor system is characterized by $\boldsymbol{K}_1 = \boldsymbol{I}_n$,

$$\boldsymbol{K}_0 = \begin{bmatrix} -1/h & 1/h & 0 & \cdots & 0 \\ 1/h & -2/h & 1/h & \ddots & \vdots \\ 0 & 1/h & -2/h & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1/h \\ 0 & \cdots & 0 & 1/h & -2/h \end{bmatrix} \quad (30)$$

and

$$\boldsymbol{B} = \begin{bmatrix} 0 & \cdots & 0 & 1/h \end{bmatrix}^T, \quad (31)$$

with $h = 1/(n+1)$. The values presented in Table V for the controllability radius were obtained considering $n = 25$—the maximum value for which we were able to run our algorithm due to memory limitations. The proposed algorithm competes well in terms of accuracy with the method from [8]. Considering the computation time, the other method can obtain a smaller execution time only for $\varepsilon = 10^{-3}$. For $\varepsilon \in \{10^{-4}, 10^{-5}\}$ our method offers the results considerably faster.

Example 6: Let us take now a second order drum brake system [8] determined by

$$\boldsymbol{K}_0 = g \begin{bmatrix} (\mathrm{s}(\gamma) + \mu\mathrm{c}(\gamma))\mathrm{s}(\gamma) & -\mu - (\mathrm{s}(\gamma) + \mu\mathrm{c}(\gamma))\mathrm{c}(\gamma) \\ (\mu\mathrm{s}(\gamma) - \mathrm{c}(\gamma))\mathrm{s}(\gamma) & 1 + (-\mu\mathrm{s}(\gamma) + \mathrm{c}(\gamma))\mathrm{c}(\gamma) \end{bmatrix}, \quad (32)$$

and

$$\boldsymbol{K}_2 = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad K_1 = \boldsymbol{0}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (33)$$

with $\mathrm{s}(\gamma) = \sin\gamma$ and $\mathrm{c}(\gamma) = \cos\gamma$. We take $m = 5$, $g = 1$, $\gamma = \pi/100$ and $\mu \in \{0.05, 0.15, 0.50, 10, 100, 1000\}$.

We discuss here two cases: the case of an absolute radius and the case of a relative radius. In the first case we consider $\boldsymbol{\alpha} = [\ 1\ 0\ 1\ ]$ and $\boldsymbol{\alpha} = [\ 1\ 1\ 1\ ]$. The results are presented in Table VI, which shows that our algorithm offers

TABLE IV
CONTROLLABILITY RADII FOR THE SYSTEM CONSIDERED IN EXAMPLE 4.

$\boldsymbol{\alpha}$	$\sqrt{\tau_1}$	Algorithm from [8] $\varepsilon = 10^{-5}$	$\varepsilon = 10^{-6}$
[0 1]	0.0000675	(0.000061, 0.000068]	(0.0000668, 0.0000677]
[1 1]	0.0000674	(0.000061, 0.000069]	(0.0000668, 0.0000675]
Time (in seconds)			
[0 1]	1.4	6570	55844
[1 1]	1.4	6154	100811

TABLE V
CONTROLLABILITY RADII FOR THE SYSTEM CONSIDERED IN EXAMPLE 5.

$\boldsymbol{\alpha}$	$\sqrt{\tau_1}$	Algorithm from [8] $\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
[0 1]	0.178176	(0.1774, 0.1783]	(0.17811, 0.17818]	(0.178169, 0.178179]
[1 1]	0.003954	(0.0033, 0.0039]	(0.00390, 0.00399]	(0.003950, 0.003957]
Time (in seconds)				
[0 1]	1466	935	11277	73467
[1 1]	1305	1087	10509	99948

good controllability radius in a faster execution than the algorithm from [8]. In the second case we consider the relative perturbations $\boldsymbol{\alpha} = [\ \eta_2\ 0\ \eta_0\ ]$ and $\boldsymbol{\alpha} = [\ \eta_2\ 1\ \eta_0\ ]$, where $\eta_i = \|\boldsymbol{K}_i\|$, $i \in \{0, 2\}$. In this case our algorithm fails to provide a good controllability radius when μ is high—$\mu = 1000$; for the rest of the values chosen for μ our method offers a good controllability radius, comparing it with the one computed by the algorithm from [8]. The results are presented in Table VII.

Example 7: We consider now a second order model [1] (describing a four-mass driving accelerance) determined by $\boldsymbol{K}_2 = \boldsymbol{I}_4$,

$$\boldsymbol{K}_1 = \begin{bmatrix} 0.0490 & -0.0882 & -0.2034 & 0.0236 \\ -0.0882 & 0.6629 & -0.0759 & 0.1068 \\ -0.2034 & -0.0759 & 1.3181 & 0.0127 \\ 0.0236 & 0.1068 & 0.0127 & 0.7344 \end{bmatrix}, \tag{34}$$

$$\boldsymbol{K}_0 = \begin{bmatrix} 0.1877 & -0.5873 & -0.4114 & 0.4359 \\ -0.5873 & 3.2409 & -0.0973 & -1.3774 \\ -0.4114 & -0.0973 & 2.5883 & -0.0377 \\ 0.4359 & -1.3774 & -0.0377 & 3.5517 \end{bmatrix} \tag{35}$$

and

$$\boldsymbol{B} = \begin{bmatrix} 0.3605 & 0 & 0 & 0 \end{bmatrix}^T. \tag{36}$$

The results for the controllability radii are presented in Table VIII. One can observe that our method offers good values in a smaller execution time compared with the algorithm from [8].

V. CONCLUSIONS

We have presented an algorithm for computing the controllability radius for higher order systems using SDP. The initial norm minimization problem was transformed into an eigenvalue minimization problem of a bivariate real matrix polynomial. The new problem was replaced with its relaxed version by changing the positive polynomial constraint with a sum-of-squares one, thus obtaining a more conservative problem.

The examples on which we have tested our algorithm show that the proposed method can obtain good results for the controllability radius. Even more, our algorithm has a much smaller execution time than the one from [8].

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TABLE VI
CONTROLLABILITY RADII FOR THE SYSTEM CONSIDERED IN EXAMPLE 6.

μ	$\boldsymbol{\alpha}$	$\sqrt{\tau_1}$	Algorithm from [8] $\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$
0.05	[1 0 1]	0.05872	(0.051, 0.059]	(0.0582, 0.0588]	(0.05863, 0.05872]
	[1 1 1]	0.05872	(0.051, 0.059]	(0.0582, 0.0588]	(0.05863, 0.05872]
0.15	[1 0 1]	0.14701	(0.140, 0.147]	(0.1465, 0.1471]	(0.14694, 0.14703]
	[1 1 1]	0.14699	(0.140, 0.147]	(0.1465, 0.1471]	(0.14694, 0.14703]
0.50	[1 0 1]	0.42271	(0.418, 0.426]	(0.4221, 0.4228]	(0.42263, 0.42272]
	[1 1 1]	0.42225	(0.414, 0.422]	(0.4216, 0.4223]	(0.42217, 0.42226]
10	[1 0 1]	0.99588	(0.989, 0.997]	(0.9952, 0.9959]	(0.99583, 0.99591]
	[1 1 1]	0.99583	(0.989, 0.997]	(0.9952, 0.9959]	(0.99576, 0.99585]
100	[1 0 1]	0.99982	(0.992, 1.000]	(0.9992, 0.9999]	(0.99974, 0.99983]
	[1 1 1]	0.99982	(0.992, 1.000]	(0.9992, 0.9999]	(0.99974, 0.99983]
1000	[1 0 1]	0.99976	(0.992, 1.0002]	(0.9992, 0.9999]	(0.99971, 0.99980]
	[1 1 1]	0.99975	(0.992, 1.0002]	(0.9992, 0.9999]	(0.99971, 0.99980]
			Time (in seconds)		
0.05	[1 0 1]	2.2	12	180	1263
	[1 1 1]	0.8	12	179	1268
0.15	[1 0 1]	1	10	152	1224
	[1 1 1]	1.2	10	154	1224
0.50	[1 0 1]	1.1	19	169	1297
	[1 1 1]	0.8	10	192	1029
10	[1 0 1]	0.9	13	103	1387
	[1 1 1]	0.9	13	104	1074
100	[1 0 1]	1	18	151	1284
	[1 1 1]	1.3	18	145	1378
1000	[1 0 1]	1.1	14	146	1340
	[1 1 1]	0.9	14	156	1282

TABLE VII

RELATIVE CONTROLLABILITY RADII FOR THE SYSTEM CONSIDERED IN EXAMPLE 6.

μ	$\boldsymbol{\alpha}$	$\sqrt{\tau_1}$	Algorithm from [8] $\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$
0.05	[η_2 0 η_0]	0.04642	(0.042, 0.049]	(0.0458, 0.0465]	(0.04633, 0.04642]
	[η_2 1 η_0]	0.04642	(0.042, 0.049]	(0.0458, 0.0465]	(0.04633, 0.04642]
0.15	[η_2 0 η_0]	0.11613	(0.110, 0.118]	(0.1155, 0.1162]	(0.11605, 0.11614]
	[η_2 1 η_0]	0.11612	(0.110, 0.118]	(0.1155, 0.1162]	(0.11605, 0.11614]
0.50	[η_2 0 η_0]	0.33135	(0.324, 0.332]	(0.3308, 0.3315]	(0.33129, 0.33138]
	[η_2 1 η_0]	0.33094	(0.324, 0.332]	(0.3302, 0.3309]	(0.33087, 0.33096]
10	[η_2 0 η_0]	0.80445	(0.800, 0.807]	(0.8039, 0.8046]	(0.80438, 0.80447]
	[η_2 1 η_0]	0.80358	(0.796, 0.804]	(0.8029, 0.8035]	(0.80350, 0.80359]
100	[η_2 0 η_0]	0.80939	(0.801, 0.809]	(0.8089, 0.8095]	(0.80934, 0.80943]
	[η_2 1 η_0]	0.80930	(0.801, 0.809]	(0.8089, 0.8095]	(0.80922, 0.80931]
1000	[η_2 0 η_0]	0.83830	(0.800, 0.807]	(0.8072, 0.8079]	(0.80764, 0.80773]
	[η_2 1 η_0]	0.83829	(0.800, 0.807]	(0.8072, 0.8079]	(0.80757, 0.80766]
			Time (in seconds)		
0.05	[η_2 0 η_0]	0.9	11	143	1144
	[η_2 1 η_0]	0.9	11	143	1147
0.15	[η_2 0 η_0]	1	14	148	1071
	[η_2 1 η_0]	1	14	147	1189
0.50	[η_2 0 η_0]	0.9	15	189	1410
	[η_2 1 η_0]	0.9	16	207	1415
10	[η_2 0 η_0]	0.7	16	156	1461
	[η_2 1 η_0]	0.7	10	149	1616
100	[η_2 0 η_0]	1.1	16	206	1607
	[η_2 1 η_0]	0.9	16	210	1334
1000	[η_2 0 η_0]	0.6	16	186	1349
	[η_2 1 η_0]	0.4	16	185	1193

TABLE VIII

CONTROLLABILITY RADII FOR THE SYSTEM CONSIDERED IN EXAMPLE 7.

$\boldsymbol{\alpha}$	$\sqrt{\tau_1}$	Algorithm from [8] $\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$
[0 0 1]	0.03123	(0.023, 0.032]	(0.0306, 0.0313]	(0.03115, 0.03124]
[0 1 1]	0.03023	(0.021, 0.030]	(0.0295, 0.0303]	(0.03017, 0.03024]
[1 1 1]	0.02847	(0.022, 0.030]	(0.0279, 0.0286]	(0.02839, 0.02848]
		Time (in seconds)		
[0 0 1]	4.1	11	143	1065
[0 1 1]	4.4	12	114	1134
[1 1 1]	4	17	192	1412