

# Real Radius of Controllability for Systems Described by Polynomial Matrices: SIMO case

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**Abstract**—In this paper we discuss the problem of computing the real radius of controllability of the Single Input Multi Output (SIMO) systems described by univariate polynomial matrices. The problem is equivalent to computing the nearest noncoprime polynomial matrix to the polynomial matrix describing the system in some prescribed norm. A particular case of this problem is to compute approximate GCD of univariate polynomials. Further this problem is shown to be equivalent to the Structured Low Rank Approximation (SLRA) of a linearly structured resultant matrix associated with the given polynomial matrix. The radius of controllability is then computed by finding the nearest SLRA of this resultant matrix.

**Index Terms**—real radius of controllability, univariate polynomial matrices, Singular Value Decomposition (SVD), Structured Low Rank Approximation (SLRA)

## I. INTRODUCTION

Controllability is a central idea in systems theory. However it is not always enough to know whether the system is controllable. A small perturbation in system parameters may render the system uncontrollable. Thus checking for controllability is not a numerically stable problem. In order to overcome this difficulty a continuous metric instead of yes/no kind of controllability check was introduced in [1], [2], [3] for the systems represented in state space form. An algorithm to compute the distance between the given system in state space representation and the nearest uncontrollable system was discussed in [4], [5]. Recently an algorithm to compute the nearest uncontrollable system to the given SISO system represented using polynomial matrix was discussed in [6]. However this problem is equivalent to the problem of computing approximate GCD of two univariate polynomials. Real radius of controllability for the systems modeled as state space systems was introduced in [7]. This radius quantifies the maximum perturbation so that the perturbed system is still controllable. Consider a system represented in the state space form as in the following equation.

$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= cx\end{aligned}$$

where  $A \in \mathbb{R}^{g \times g}$ ,  $b \in \mathbb{R}^{g \times 1}$  and  $c \in \mathbb{R}^{1 \times g}$ . Then the real radius of controllability, denoted as  $r_c$ , is defined as

$$r_c = \min_{\Delta A, \Delta b} \{ \|\Delta A \ \Delta b\|_F \mid \text{the pair } (A + \Delta A, b + \Delta b) \text{ is uncontrollable} \}. \quad (1)$$

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The word *real* in the definition of real radius of controllability indicates the perturbations that are allowed in the system matrices are real matrices. When the complex perturbations are allowed, the term is defined as complex radius of controllability.

In this paper we compute the real radius of controllability for SIMO systems described by univariate polynomial matrices. We show that this problem is equivalent to computing the nearest noncoprime polynomial matrix. Further we construct a sequence of structured resultant matrices and show that the nearest noncoprime matrix can be obtained by computing the nearest Structured Low Rank Approximation (SLRA) of a certain resultant matrix from this sequence.

The paper is organized as follows. The remainder of this section is devoted to preliminaries. In Section II we formulate the problem formally. In Section III we prove the main results including the equivalence of the problem with the SLRA of linearly structured resultant matrix associated with the polynomial matrix. Further in Section IV we formally define the SLRA problem and discuss a numerical algorithm to compute the nearest SLRA to a given matrix. In Section V we show some numerical results and comparisons with existing results in the literature for the SISO case. Finally we conclude in Section VI.

### A. Preliminaries: Polynomial Matrices

Let  $\mathbb{R}[s]$  denote the ring of polynomials in a single variable  $s$  with coefficients from the real field. Let  $R(s) \in \mathbb{R}^{g \times w}[s]$  be a polynomial matrix of size  $g \times w$  with entries from the ring of polynomials  $\mathbb{R}[s]$ . Another useful way to represent a polynomial matrix is in the matrix polynomial form as follows:

$$R(s) = R_0 + R_1s + R_2s^2 + \dots + R_ns^n \quad (2)$$

where  $R_i \in \mathbb{R}^{g \times w}$ . Further  $n$  is called the degree of the polynomial matrix. The matrix polynomial representation of a polynomial matrix plays a key role in the discussion that follows. The nullspace of  $R(s)$ , denoted as  $\mathcal{N}$ , is defined as

$$\mathcal{N} = \{v \in \mathbb{R}^{w \times 1}[s] \mid R(s)v = 0\}.$$

We now define minimal polynomial basis for  $\mathcal{N}$ .

**Definition 1.1:** Let  $B = \{m_1(s), m_2(s), \dots, m_k(s)\} \subset \mathbb{R}^w[s]$  be a generating set for  $\mathcal{N}$  with degrees  $\delta_1 \leq \delta_2 \leq \dots \leq \delta_k$ . This generating set is called minimal basis (see [8]) if for any other basis of  $\mathcal{N}$  with degrees  $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_k$ , it turns out that  $\delta_i \leq \gamma_i$  for  $i = 1, 2, \dots, k$ .

The degree of the nullspace  $\mathcal{N}$  is defined as the  $\max(\delta_1, \dots, \delta_k)$  where  $\delta_i$ 's are degrees of basis vectors in

a minimal nullspace basis. A set of vectors  $\{v_1, v_2, \dots, v_n\} \subset \mathbb{R}^{w \times 1}[s]$  is called polynomially independent set if  $a_1(s)v_1 + a_2(s)v_2 + \dots + a_n(s)v_n = 0$  implies that  $a_i(s) = 0$  for all  $i = 1, 2, \dots, n$  identically. If a set of polynomial vectors is not polynomially independent, then it is called polynomially dependent. Rank or normal rank of a polynomial matrix is defined as the number  $\max_{\lambda \in \mathbb{R}} \text{rank } R(\lambda)$ . It can be shown that this number is same as the number of polynomially independent rows in  $R(s)$ . A square polynomial matrix is called unimodular if its determinant is a nonzero real number. Thus inverse of a unimodular matrix is also a polynomial matrix. A polynomial matrix  $R(s)$  is called left coprime if it has full row rank everywhere in the complex plane, that is  $R(z)$  as a matrix of complex numbers is full row rank for all  $z \in \mathbb{C}$ . Similarly we can define when  $R(s)$  is right coprime. Note that in case of  $g = 1$ , the polynomial matrix  $R(s)$  being left coprime is equivalent to the definition of coprimeness of a set of  $w$  univariate polynomials.

### B. Preliminaries: Systems Theory

Use of polynomial matrices to represent systems can be found in [9], [10]. We define some basic notions about systems represented using polynomial matrices. A behavior  $\mathfrak{B}$  is said to be a linear differential behavior if it is governed by a system of linear differential equations.  $\mathfrak{B}$  is said to have a kernel representation if

$$\mathfrak{B} = \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \mid R\left(\frac{d}{dt}\right)w = 0\} \quad (3)$$

where  $R(s) \in \mathbb{R}^{g \times w}[s]$ . A kernel representation (3) is said to be minimal if  $R(s)$  is full row rank. Here rank is normal rank of a polynomial matrix, that is all  $g$  rows of  $R(s)$  are polynomially independent. A behavior is said to be controllable if we can patch any two trajectories in the behavior in finite time. An algebraic test to check controllability is to check rank of matrix  $R(\lambda)$ ,  $\forall \lambda \in \mathbb{C}$ . The behavior is controllable iff this rank is constant for all  $\lambda \in \mathbb{C}$ . Thus when a behavior is represented in a minimal kernel representation, controllability is equivalent to the the polynomial matrix  $R(s)$  being left coprime. A behavior is said to have an image representation if one can write the behavior as

$$\mathfrak{B} = \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \mid w = M\left(\frac{d}{dt}\right)\ell \text{ where } \ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^1)\} \quad (4)$$

where  $M(s) \in \mathbb{R}^{w \times 1}[s]$ . An important characterization of controllability of a behavior is the existence of an image representation. Further  $M(s)$  in the image representation (4) is such that  $R(s)M(s) = 0$ . Another important notion in behavioral theory is of McMillan degree which we define now.

*Definition 1.2:* Let  $\mathfrak{B}$  be represented by a minimal kernel representation as in equation (3). Consider all  $\binom{w}{g}$  determinants of order  $g \times g$  of  $R(s)$ . The highest degree among all the degrees of these determinants is called McMillan degree of  $\mathfrak{B}$ .

Note that the McMillan degree of a behavior is an invariant. In a similar manner we can define the McMillan degree of an image representation. The McMillan degree of a the

kernel representation of a controllable behavior is equal to the McMillan degree of the observable image representation of the behavior.

An input/output partition of  $w$  can be done as follows: let  $\mathfrak{B}$  be represented in a minimal kernel representation with  $R(s) \in \mathbb{R}^{g \times w}[s]$  as in equation (3). Then we chose the  $g \times g$  minor  $R_1$  such that the degree of the determinant polynomial of  $R_1$  is equal to the McMillan degree of  $\mathfrak{B}$ . WLOG we assume that this minor  $R_1$  is such that we can partition  $R$  as  $R = [R_1 \ R_2]$  where  $R_2 \in \mathbb{R}^{g \times (w-g)}[s]$ . Then for  $w = [w_1 \ w_2]^T$  where  $w_1$  and  $w_2$  are compatible with  $R_1$  and  $R_2$ , we write

$$\begin{aligned} R_1 w_1 + R_2 w_2 &= 0. \\ \Rightarrow w_1 &= R_1^{-1} R_2 w_2. \end{aligned}$$

Thus variables  $w_1$  can be thought of as outputs and variables  $w_2$  as inputs. For the case when  $w = g + 1$ , we get SIMO systems. In this paper we consider the case when  $w = g + 1$ .

## II. PROBLEM FORMULATION

In this section we define the problem formally. Let  $\mathfrak{B}$  be a controllable behavior represented in minimal kernel representation as in equation (3). Then the system is controllable if and only if the polynomial matrix  $R(s)$  is left coprime. Thus if we find the nearest noncoprime matrix to  $R(s)$  in some norm, then the behavior corresponding to this matrix is uncontrollable. We start by defining the norm on the space  $\mathbb{R}^{g \times (g+1)}[s]$ . Let  $R(s) \in \mathbb{R}^{g \times (g+1)}[s]$ . In order to define a norm  $\|\cdot\|_*$  on  $\mathbb{R}^{g \times w}[s]$ , we use the matrix polynomial representation as introduced in equation (2). Let  $n$  be the degree of  $R(s)$ .

Construct the matrix  $X_R \in \mathbb{R}^{(n+1)g \times w}$  as  $X_R = \begin{bmatrix} R_0 \\ R_1 \\ \vdots \\ R_n \end{bmatrix}$ . The

norm  $\|\cdot\|_*$  is defined as  $\|R(s)\|_* = \|X_R\|_F$ , where  $\|\cdot\|_F$  is the Frobenius norm on  $\mathbb{R}^{(n+1)g \times w}$ . It is easy to verify that  $\|\cdot\|_*$  satisfies all the conditions of the norm function.

In order to compute the nearest noncoprime polynomial matrix to the given polynomial matrix  $R(s)$ , we need to perturb the polynomial matrix  $R(s)$  to  $R(s) + \Delta R(s)$  so that the perturbed matrix is not left coprime. The perturbation in the sense of norm defined above should be optimal. In this paper we restrict the class of perturbations as to the polynomial matrices in the set  $\mathbb{R}^{g \times (g+1)}[s]$  with the degree less than or equal to that of  $R(s)$ .

*Problem Statement 2.1:* Let  $\mathcal{P} \subset \mathbb{R}^{g \times (g+1)}[s]$  be the set of all polynomial matrices which are not left coprime. Let  $R(s) \in \mathbb{R}^{g \times (g+1)}[s]$  with degree  $n$  be a minimal kernel representation of  $\mathfrak{B}$  as in equation (3). Then we want to compute  $Q(s) \in \mathcal{P}$  with degree at most  $n$  such that

$$r_c = \min_{Q \in \mathcal{P}} \|R(s) - Q(s)\|_* \quad (5)$$

where  $r_c$  is called the real radius of controllability. Here the system corresponding to  $Q(s)$  is uncontrollable as  $Q(s)$  is not left coprime.

The Frobenius norm used in the definition of  $\|\cdot\|_*$  measures the total perturbations in the coefficient matrices, that

would make the system uncontrollable or equivalently, make the polynomial matrix noncoprime. We propose an algorithm to compute the radius of controllability. The approach towards the solution involves constructing certain special structured matrices. Further left coprimeness is shown to be equivalent to the full column rank condition of these structured matrices. Thus the problem of computing the radius of controllability is equivalent to computing when these structured matrices would lose full column rank property.

*Remark 2.2:* The special case of  $g = 1$  is equivalent to finding the smallest perturbations of coefficients of two coprime polynomials that would render them noncoprime. The problem is then equivalent to a well studied problem of finding approximate GCD of two polynomials (See [11]). Thus our solution also gives a way of finding approximate GCD of polynomials.

*Remark 2.3:* The matrix  $R(s) \in \mathbb{R}^{g \times (g+1)}[s]$  is left coprime if and only if all the  $g + 1$  determinant polynomials of  $g \times g$  minors are coprime. Thus for a given polynomial matrix when we compute the nearest noncoprime polynomial matrix, the  $g + 1$  determinant polynomials become noncoprime. However note that the problem of computing the approximate GCD of  $g + 1$  polynomials is a more difficult problem than the problem discussed in this paper.

### III. MAIN RESULTS

In this section we prove the main results of this paper. We now construct a sequence of structured matrices from the given polynomial matrix  $R(s) \in \mathbb{R}^{g \times (g+1)}[s]$  with degree  $n$  and discuss the relation of the nullspace  $\mathcal{N}$  with the nullspaces of these structured matrices.

Let  $X_0 = \begin{bmatrix} R_0 \\ R_1 \\ \vdots \\ R_n \end{bmatrix} \in \mathbb{R}^{(n+1)g \times (g+1)}$ . We now construct the sequence of structured matrices  $X_1, X_2, \dots$  as follows:

$$X_1 = \left[ \begin{array}{c|c} X_0 & 0 \\ \hline 0 & X_0 \end{array} \right], \quad X_2 = \left[ \begin{array}{c|cc} X_0 & 0 & 0 \\ \hline 0 & & X_1 \\ 0 & & \end{array} \right], \dots \quad (6)$$

where 0's in the above equation are zero matrices of size  $g \times (g + 1)$ . For any  $i \in \mathbb{N}$ ,  $X_i \in \mathbb{R}^{(i+n+1)g \times (i+1)(g+1)}$ . Let  $\mathcal{X}_i$  be the nullspace of matrix  $X_i$  and let  $d_i = \dim(\mathcal{X}_i)$ . The nullspace  $\mathcal{N}$  of the polynomial matrix is related to the nullspaces  $\mathcal{X}_i$  of structured matrices in the following way: for any  $i \in \mathbb{N} \cup \{0\}$ , let  $y \in \mathcal{X}_i$ . Then partition  $y \in \mathbb{R}^{(i+1)(g+1)}$

as  $y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_i \end{bmatrix}$  where  $y_j \in \mathbb{R}^{(g+1)}$  for  $j = 0, 1, \dots, i$ . Let  $y(s) = \sum_{j=0}^i y_j s^j \in \mathbb{R}^{(g+1) \times 1}[s]$ . It is easy to verify that  $y(s) \in \mathcal{N}$ .

Note that at the  $i^{\text{th}}$  stage of this sequence, if  $\mathcal{X}_i \neq \{0\}$ , then we get the an element of  $\mathcal{N}$  of degree  $i$ . We prove some important properties of the sequence  $\{d_i\}_{i=0,1,\dots}$  in the following theorem.

*Theorem 3.1:* Let  $R(s) \in \mathbb{R}^{g \times (g+1)}[s]$  be a polynomial matrix of degree  $n$  and rank  $g$ . Let  $\{X_i\}_{i=0,1,\dots}$  be the sequence of structured matrices constructed from  $R(s)$  as in the equation (6). Let  $\mathcal{X}_i = \ker(X_i)$  and  $d_i = \dim(\mathcal{X}_i)$ . Then the following statements hold:

- (a) The sequence  $\{d_i\}_{i=0,1,2,\dots}$  is a nondecreasing sequence of nonnegative integers.
- (b) There exists  $n_0 \in \mathbb{N}$  such that  $d_{k+1} = d_k + 1$  for all  $k \geq n_0$ .

*Proof:* (a) Let  $n_0 \in \mathbb{N}$  be the smallest positive integer such that  $d_{n_0} > 0$ . Let  $y \in \mathbb{R}^{(n_0+1)(g+1)}$  be such that  $y \in \mathcal{X}_{n_0}$ . Then from the structure of matrices  $X_i$  it is clear that for  $0 \in \mathbb{R}^{g+1}$ ,  $\begin{bmatrix} y \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \in \mathcal{X}_{n_0+1}$ . Thus  $d_{n_0+1} \geq 2d_{n_0}$ . In particular  $d_{n_0+1} > d_{n_0}$ . Let  $d_{n_0+1} = 2d_{n_0} + \alpha_1$ . Then using similar argument we can show that  $d_{n_0+2} = 3d_{n_0} + 2\alpha_1 + \alpha_2 = d_{n_0+1} + d_{n_0} + \alpha_1 + \alpha_2 > d_{n_0+1}$ . Generalizing this argument we can show that

$$\begin{aligned} d_{n_0+j} &= (j+1)d_{n_0} + \sum_{k=1}^j (j-k+1)\alpha_k \\ &= d_{n_0+j} + \left( d_{n_0} + \sum_{k=1}^j \alpha_k \right) \\ &> d_{n_0+j} \end{aligned}$$

for  $j = 0, 1, 2, \dots$ . The first  $n_0 - 1$  terms of the sequence are 0. This proves that  $\{d_i\}_{i=0,1,2,\dots}$  is a nondecreasing sequence of nonnegative integers.

(b) It is clear that once we find a polynomial vector  $y(s) \in \mathcal{N}$  such that  $\deg y(s)$  is same as the degree of  $\mathcal{N}$ , all polynomial vectors in  $\mathcal{N}$  can be obtained from linear span of polynomial vectors  $y(s), sy(s), s^2y(s), \dots$ . Further every vector in  $\mathcal{X}_i$  corresponds to a polynomial vector in  $\mathcal{N}$  with a degree less than or equal to  $i$ . This proves that  $\alpha_j = 0$  for  $j = 1, 2$  in part (a). Then it follows that  $d_{k+1} = d_k + 1$  for  $k \geq n_0$ . ■

*Corollary 3.2:* Let  $R(s) \in \mathbb{R}^{g \times (g+1)}[s]$  be a given polynomial matrix with degree  $n$  and normal rank  $g$ . Construct the sequence  $\{d_i\}_{i=0,1,\dots}$  from  $R(s)$  as discussed above. Then the degree of  $\mathcal{N}$  denoted by  $n_0$  is the positive integer such that  $d_{n_0} = 1$ .

In the following theorem we show the relation of the nullspace of polynomial matrix and the left coprimeness.

*Theorem 3.3:* Let  $\mathfrak{B}$  be a behavior represented in minimal kernel representation form as in equation (3). Let  $R(s) \in \mathbb{R}^{g \times (g+1)}[s]$ . Let  $M(s) \in \mathbb{R}^{(g+1)}[s]$  be such that  $R(s)M(s) = 0$  and  $\deg M(s) = \deg \mathcal{N}$ . Then the McMillan degree of controllable part of  $\mathfrak{B}$  is equal to  $\deg M(s)$ .

*Proof:* Let  $R(s) \in \mathbb{R}^{g \times (g+1)}[s]$  be the polynomial matrix in the minimal kernel representation (3). Construct vector  $M(s) \in \mathbb{R}^{(g+1)}[s]$  as follows: let  $R_1(s), R_2(s), \dots, R_{g+1}(s)$  be the minors of  $R(s)$  such that  $R_i(s)$  is obtained from  $R(s)$  by removing  $i^{\text{th}}$  column. Then construct  $M(s)$  as

$$M(s) = \begin{bmatrix} \det R_1(s) \\ -\det R_2(s) \\ \vdots \\ (-1)^g \det R_{g+1}(s) \end{bmatrix}. \quad (7)$$

Then  $R(s)M(s) = 0$ , that is,  $M(s) \in \mathcal{N}$ . Further

$$\deg M(s) = \max_{i=1,2,\dots,g+1} \deg(\det R_i(s)). \quad (8)$$

We now consider two cases:

*Case 1:* Consider that  $\mathfrak{B}$  is controllable. This implies that all the entries in the vector  $M(s)$ , that is determinant polynomials of the minors, are coprime. Hence  $M(s)$  is such that  $\deg M(s) = \deg \mathcal{N}$ . This implies that  $M(s)$  forms minimal polynomial basis for the nullspace of  $R(s)$ . Hence from equation (8) it is clear that McMillan degree of  $\mathfrak{B}$  is equal to  $\deg \mathcal{N}$ .

*Case 2:* Consider that  $\mathfrak{B}$  is not controllable. Then the polynomial matrix  $R(s)$  in the minimal kernel representation (3) can be decomposed as  $R(s) = A(s)\tilde{R}(s)$  where  $A(s) \in \mathbb{R}^{g \times g}[s]$  and  $\tilde{R}(s) \in \mathbb{R}^{g \times (g+1)}[s]$  is a left coprime matrix. Roots of the determinant polynomial of  $A(s)$  are complex numbers where  $R(s)$  loses rank.  $\tilde{R}(s)$  corresponds to the controllable part of the behavior. Further note that the nullspace of  $\tilde{R}(s)$  is same as that of  $R(s) = A(s)\tilde{R}(s)$ . Then using similar arguments as in Case 1 above, we can show that  $\deg \mathcal{N}$  is equal to the McMillan degree of the behavior corresponding to  $\tilde{R}(s)$ , the controllable part of  $\mathfrak{B}$ . ■

We now explain how we compute the nearest uncontrollable system to the given system. Let  $R(s) \in \mathbb{R}^{g \times (g+1)}$  represent a minimal kernel representation of  $\mathfrak{B}$  as in equation (3). Let  $\mathfrak{B}$  be controllable. Then polynomial matrix  $R(s)$  is left coprime. Then the nearest noncoprime polynomial matrix to  $R(s)$  represents an uncontrollable behavior. In order to compute the nearest noncoprime polynomial matrix, we construct the sequence of structured matrices  $\{X_i\}_{i=0,1,\dots}$  as described in equation (6). At stage some  $n_0 \in \mathbb{N}$ , when  $d_{n_0} = 1$  from Corollary 3.2 and Theorem 3.3 we know that McMillan degree of  $\mathfrak{B}$  is  $n_0$ . We perturb the matrix  $X_{n_0-1}$  to  $\tilde{X}_{n_0-1}$  such that  $\tilde{X}_{n_0-1}$  is rank deficient and has the same structure as that of  $X_{n_0-1}$ . Then the behavior corresponding  $R(s) + \Delta R(s)$  obtained from  $\tilde{X}_{n_0-1}$  will be uncontrollable. If this perturbation is optimal in the sense of norm defined in Section II, then we compute the nearest uncontrollable system to the given controllable system. Note that, in order to compute the nearest uncontrollable system, we have used the equivalence of left coprimeness of minimal kernel representation with the controllability of  $\mathfrak{B}$ . In the next section, we formulate the SLRA problem formally and give a numerical algorithm to compute the nearest SLRA.

#### IV. SLRA: FORMULATION AND ALGORITHM

In this section we first state the problem of computing the nearest SLRA of a given linearly structured matrix. Then we formulate the problem of computing the nearest noncoprime polynomial matrix as the SLRA problem. Finally we discuss a numerical algorithm to compute the nearest SLRA of a given matrix.

##### A. SLRA formulation

Let  $\Omega \subset \mathbb{R}^{p \times q}$  denote the subspace of matrices with a given structure. Let  $B = \{B_1, B_2, \dots, B_N\}$  be a basis of  $\Omega$ . Now we define SLRA problem as it is defined in [12].

*Problem Statement 4.1:* Given  $\Omega \subset \mathbb{R}^{p \times q}$ , the subspace of matrices with the given structure, and  $X \in \Omega$  such that  $\text{rank}(X) = k$  for  $k \leq \min\{p, q\}$ , find a matrix  $Y$  such that

$$\min_{Y \in \Omega, \text{rank}(Y)=k-1} \|X - Y\|_F.$$

We now give an algorithm to compute the nearest noncoprime polynomial matrix to the given polynomial matrix using the SLRA formulation discussed above. Let  $R(s) \in \mathbb{R}^{g \times (g+1)}[s]$  be a given polynomial matrix with degree  $n$  and normal rank  $g$ . Construct the sequence of structured matrices  $\{X_i\}_{i=0,1,\dots}$  as in equation (6). Let  $n_0 \in \mathbb{N}$  be such that  $n_0$  is the degree of the nullspace  $\mathcal{N}$ . Consider  $X_{n_0-1} \in \mathbb{R}^{(n+n_0)g \times n_0(g+1)}$ . Let  $\Omega \subset \mathbb{R}^{(n+n_0)g \times n_0(g+1)}$  be the subspace of all the matrices with same structure as that of  $X_{n_0-1}$ . Then we compute the nearest SLRA  $\tilde{X}_{n_0-1} \in \Omega$  of  $X_{n_0-1}$ . We construct polynomial matrix  $\tilde{R}(s)$  from  $\tilde{X}_{n_0-1}$ .  $\tilde{R}(s)$  is the nearest noncoprime polynomial matrix.

*Algorithm 4.2:* Algorithm to Compute nearest noncoprime polynomial matrix

Input: Polynomial matrix  $R(s) \in \mathbb{R}^{g \times (g+1)}$  of degree  $n$  and normal rank  $g$ .

Output:  $\tilde{R}(s)$ , the nearest noncoprime polynomial matrix to  $R(s)$ .

Step 1: Compute  $n_0$ , the degree of  $\mathcal{N}$ .

Step 2: Construct the structured matrix  $X_{n_0-1}$ .

Step 3: Obtain the nearest SLRA  $\tilde{X}_{n_0-1}$  of  $X_{n_0-1}$ .

Step 4: Construct polynomial matrix  $\tilde{R}(s)$  from  $\tilde{X}_{n_0-1}$ .

In the following subsection we discuss an algorithm to compute the nearest SLRA of a given linearly structured matrix based on Structured Total Least Squares (STLS) approach.

##### B. An Algorithm to Compute the nearest SLRA

The problem of computing the nearest SLRA of a given structured matrix is well studied in the literature (see [12], [13]). Here we adopt the method discussed in [14] to the structured matrices with the structure as described in Section III. We explain the Structured Total Least Norm (STLN) algorithm in this subsection.

We state the following theorem which justifies the formulation of the Structured Total Least Squares (STLS) problem in the sequel.

*Theorem 4.3:* Let  $R(s) \in \mathbb{R}^{g \times (g+1)}[s]$  be a given polynomial matrix with degree  $n$ . Let  $n_0$  be the degree of  $\mathcal{N}$ , the nullspace of  $R(s)$ . Let  $X = X_{n_0-1} \in \mathbb{R}^{(n+n_0)g \times n_0(g+1)}$  be constructed as in Section III. Partition  $X$  as  $X = [A \ y]$  where  $A \in \mathbb{R}^{(n+n_0)g \times n_0(g+1)-1}$  and  $y \in \mathbb{R}^{n_0(g+1)}$ . Then the polynomial matrix  $R(s)$  is not left coprime if and only if  $Ax = y$  has a nontrivial solution for  $x$ .

This theorem justifies the partition of the matrix  $X$  that we introduce below. We describe the algorithm for a general structure where this partition is justified. Let  $\Omega \subset \mathbb{R}^{p \times q}$  be the space of all structured matrices with a given structure. For a given  $X \in \Omega$  with rank  $r$  we need to compute the

nearest  $Y \in \Omega$  with rank  $r-1$ . We partition  $X = [A \ b]$ , where  $A \in \mathbb{R}^{p \times (q-1)}$  and  $b \in \mathbb{R}^{p \times 1}$ . Then the problem of computing the nearest SLRA can be formulated as

$$\begin{aligned} \min_{H, h, x} & \| [H \ h] \|_F & (9) \\ \text{subject to} & \\ (A+H)x & = (b+h) \end{aligned}$$

where  $H \in \mathbb{R}^{p \times (q-1)}$ ,  $h \in \mathbb{R}^{p \times 1}$  are such that  $[H \ h] \in \Omega$  and  $x \in \mathbb{R}^{q-1}$ . Note that this problem is similar to the Total Least Squares (TLS) problem with an additional constraint on the structure of perturbation matrices  $H$  and  $h$ , hence the name Structured Total Least Squares (STLS) problem.

Let  $\Delta X = [H \ h] \in \Omega$ . We introduce a vector  $\alpha = [\alpha_1 \ \alpha_2 \ \dots \ \alpha_N]^T \in \mathbb{R}^N$  such that the matrix  $\Delta X$  is represented as  $\Delta X = \sum_{i=1}^N \alpha_i B_i$ . We call  $\alpha$  the representation of  $\Delta X$ . Let  $P \in \mathbb{R}^{p \times N}$  be the matrix of zeros and ones such that  $h = P\alpha$ . Then the structured minimization problem as in equation (9) can be stated as follows:

$$\begin{aligned} \min_{\alpha, x} & \| D\alpha \|_2 & (10) \\ \text{subject to} & \\ \hat{r} & = 0 \end{aligned}$$

where the structured residual  $\hat{r} = \hat{r}(\alpha, x) = b + P\alpha - (A+H)x$  and  $D$  is a positive definite weight matrix. In our case  $D = I_N$  and hence we do not consider the weight matrix in the following discussion. The above problem can be solved using the penalty method in the following way.

$$\min_{\alpha, x} \left\| \begin{bmatrix} \omega \hat{r}(\alpha, x) \\ \alpha \end{bmatrix} \right\|_2, \quad (11)$$

where  $\omega$  is a very large positive constant. Typically in numerical simulations  $\omega$  is taken in the range of  $10^8$  to  $10^{10}$ . As proposed in [14] we linearize the structured residual as follows:

$$\begin{aligned} \hat{r}(\alpha + \Delta\alpha, x + \Delta x) & = b + P(\alpha + \Delta\alpha) \\ & \quad - (A + H + \Delta H)(x + \Delta x) \\ & \approx b + P\alpha + P\Delta\alpha - (A + H)x \\ & \quad - (A + H)\Delta x - \Delta Hx. \end{aligned}$$

Let  $S \in \mathbb{R}^{p \times N}$  be a matrix such that  $S\Delta\alpha = \Delta Hx$ . The structure of  $S$  is similar to that of  $H$ . The entries in  $S$  depend on the entries of the vector  $x$ . Then (11) can be approximated by

$$\min_{\Delta\alpha, \Delta x} \left\| \begin{bmatrix} \omega(S-P) & \omega(A+H) \\ I_N & 0 \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta x \end{bmatrix} + \begin{bmatrix} -\omega\hat{r} \\ \alpha \end{bmatrix} \right\|_2 \quad (12)$$

We now summarize the algorithm.

*Algorithm 4.4: STLN Algorithm*

Input: Matrices  $A, b$  and tolerance  $\varepsilon$ .

Output: Error matrix  $\Delta X$  such that  $\Delta X \in \Omega$ ,

vector  $x$  and STLN error

Step 1: Choose a large number  $\omega$ .

Step 2: Set  $H = 0$ ,  $h = 0$  and compute  $x$  from

$$\min_x \| b - Ax \|_2 \text{ and } S \text{ from } x.$$

Step 3: Set  $\hat{r} = b - Ax$ .

Step 4: Repeat

(a) Solve the minimization problem in (12).

(b) Set  $x := x + \Delta x$ ,  $\alpha := \alpha + \Delta\alpha$ .

(c) Construct  $[H \ h]$  from  $\alpha$  and  $S$  from  $x$ .

(d) Compute  $\hat{r} = (b + P\alpha) - (A + H)x$ .

until  $(\|x\|, \|\Delta\alpha\| \leq \varepsilon)$ .

## V. NUMERICAL EXAMPLES

In this section we consider examples to illustrate the algorithm to compute the radius of controllability.

*Example 5.1:* Let  $R(s) \in \mathbb{R}^{2 \times 3}[s]$  with degree 2 be given polynomial matrix which represents minimal kernel representation of a given behavior. The given behavior is controllable.

$$R(s) = R_0 + R_1s + R_2s^2, \quad (13)$$

where

$$\begin{aligned} R_0 & = \begin{bmatrix} 2.8996 & 1.7865 & 3.5071 \\ -1.8148 & -0.0482 & 0.6056 \end{bmatrix}, \\ R_1 & = \begin{bmatrix} 0.3406 & -3.1029 & 4.2961 \\ -4.1005 & -0.0499 & 1.9667 \end{bmatrix}, \\ R_2 & = \begin{bmatrix} -3.8829 & -3.5239 & 0.8279 \\ -3.6371 & -4.4503 & 3.1540 \end{bmatrix}. \end{aligned}$$

We construct the sequence of structured matrices and observe that  $d_4 = 1$  is full rank matrix. This proves that the McMillan degree of the behavior is 4. In order to compute the nearest uncontrollable system, we obtain the nearest noncoprime matrix, say  $Q(s)$ , to  $R(s)$ . In order to do so, we compute the nearest SLRA of  $X_3$ , say  $\tilde{X}_3$ . We use the algorithm that is discussed in the previous section. The matrix  $Q(s)$  obtained from  $\tilde{X}_3$  can be written as

$$Q(s) = Q_0 + Q_1s + Q_2s^2, \quad (14)$$

where

$$\begin{aligned} Q_0 & = \begin{bmatrix} 2.8875 & 1.8192 & 3.5363 \\ -1.8052 & -0.0739 & 0.5826 \end{bmatrix}, \\ Q_1 & = \begin{bmatrix} 0.3058 & -3.0123 & 4.3879 \\ -4.0752 & -0.1151 & 1.8976 \end{bmatrix}, \\ Q_2 & = \begin{bmatrix} -4.0446 & -3.1042 & 1.2208 \\ -3.5101 & -4.7768 & 2.8442 \end{bmatrix}. \end{aligned}$$

The real radius of controllability  $r_c$  is given by  $r_c = \|R(s) - Q(s)\|_* = 0.7786$  and the complex number  $\lambda$  at which the matrix  $Q(\lambda)$  is not full row rank is  $\lambda = 3.6639$ .

*Example 5.2:* In case of SISO systems, computing the real radius of controllability is same as computing an approximate GCD of univariate polynomials. In [6], this fact is used to compute the nearest uncontrollable system to a given system. We consider an example given in [6] and obtain the real radius of controllability. Let  $R(s) = [a(s) \ b(s)]$  where

$$\begin{aligned} a(s) & = s^5 + 3.6220s^4 + 4.7510s^3 + 2.7450s^2 + 0.6840s + 0.0580 \\ b(s) & = -1.6040s^5 - 6.0920s^4 - 8.3810s^3 - 5.1490s^2 - 1.4080s - 0.1340 \end{aligned}$$

The radius of controllability  $r_c$  is equal to  $7.5748 \times 10^{-4}$  and the complex number  $\lambda$  at which the system loses controllability is  $\lambda = -0.8904$ . Our results match with those given in [6].

#### VI. CONCLUDING REMARKS

In this paper we considered the problem of computing the radius of controllability for SIMO systems which are described using polynomial matrices. The problem of computing the radius of controllability is shown to be equivalent to computing the nearest noncoprime polynomial matrix to the matrix describing the system. This problem in turn is shown to be equivalent to full rank property of some linearly structured resultant matrix obtained from the polynomial matrix. Thus the nearest SLRA of this structured resultant matrix gives the nearest noncoprime matrix and in turn the radius of controllability. As a particular case of the problem addressed in this paper, we solve the problem of computation of approximate GCD of two univariate polynomials.

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