Conditions for interpolation of stable polynomials

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Abstract— **This contribution addresses the problem of the interpolation of a set of positive numbers by stable real polynomials. It is shown that the interpolant preserves local positivity, monotonicity, and convexity in order to satisfy stability requirement of the interpolating polynomial. Then this issue is formulated as a nonlinear system carrying on the existence of negative real roots and positive real parameters. By considering an extension of the Farkas's Lemma and the method of Fourier-Motzkin elimination, conditions are explicitly produced for the existence of an Hurwitz polynomial that passes through all the pairs of values to interpolate.**

I. Introduction

A general question in applied mathematics is how to construct specific polynomials from incomplete information about them. Suppose that we have pairs of data (σ_i, β_i) with $i \in \{1, \ldots, n\}$, $\sigma_i \neq \sigma_k$ for $i \neq k$, then f is called an interpolating polynomial if and only if there exists a polynomial f of degree $m \geq n$ defined as

$$f(s) = \sum_{j=0}^{m} \alpha_j s^j \tag{1}$$

where α_j are parameters to find verifying the relation below

$$\forall\, i \in \{1, \ldots, n\}, \quad f(\sigma_i) = \beta_i \tag{2}$$

When $n = m$, this polynomial interpolation problem is equivalent to solve a linear system in the unknown coefficients α_i, referred to a Vandermonde system. One other method for writing this interpolation problem is to use a linear combination of monomial basis $f_j(s)$ verifying

$$f(s) = \sum_{j=0}^{n} a_j f_j(s)$$

with a_j given parameters. This class of methods includes polynomial interpolation with monomial basis as

- Lagrange form where $f_j(s) = \prod_{k=0,k\neq j}^{n} \frac{(s-\sigma_k)}{(\sigma_j-\sigma_k)}$
- Newton form where $f_j(s) = \prod_{k=0}^{j-1}(s - \sigma_k)$

Polynomial interpolation is not restricted to interpolation to point data. One can also interpolate other information as derivative data, for instance the first derivative at two points. In this case, this additional information to interpolate, leads to a scheme called Hermite interpolation, see [6].

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Concerning the interpolation with stable polynomials, according to authors' knowledge, there exists currently no method. Perhaps is due to the existence conditions not quite as straightforward to interpolate data with stable polynomials. It will be shown that this interpolation case provides a non-linear problem difficult to solve. A first way to consider this issue, is to extend the classical polynomial interpolation methods described below to the case of stable polynomials. By the parametrization of the interpolation polynomial, this one becomes the following

$$q(s) = f(s) + n(s)p(s) \tag{3}$$

where we have

- $f(s)$ an interpolation polynomial satisfying $f(\sigma_i) = \beta_i$,
- $n(s)$ a polynomial such that $n(s) = \prod_{i=1}^{n}(s - \sigma_i)$,
- $p(s)$ any real polynomial.

This writing leads to an interpolation algorithm composed of two steps.

1) Firstly, an interpolation polynomial verifying $f(\sigma_i) = \beta_i$ is computed.
2) Secondly, a polynomial $p(s)$ is calculated providing a stable polynomial $q(s)$.

To implement the second step, that is to find a polynomial $p(s)$ such that $q(s)$ is stable, one can make appeal to some methods checking the Hermite criterion or tests based on a Wronskian criterion as given in [4]. This procedure qualifies $q(s)$ for being a stable polynomial satisfying $q(\sigma_i) = \beta_i$. But the solvability of a such design for practical treatments is difficult to establish. Therefore we are directed ourselves towards another approach that takes into account in the same step both the polynomial stability and the data interpolation by a polynomial.

Practical conditions for interpolating data with stable polynomials will be presented in the remainder of this paper as follows: First some necessary conditions are given: it is shown that the interpolant preserves local positivity $(f(s) \geq 0)$, monotonicity $(\dot{f}(s) \geq 0)$ and convexity $(\ddot{f}(s) \geq 0)$. Next, some stability results developed by Hermite-Biehler are discussed for interpolation of stable polynomials. Finally, by considering a modiifed version of Farkas's Lemma, see

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[3] and the Fourier-Motzkin elimination method, necessary and sufficient conditions for existence of an interpolating Hurwitz polynomial will be given in regards to the pairs $(\sigma_i, \beta_i)_{i\in\{1,\dots,n\}}$.

II. Preliminaries

A. Problem formulation

An obvious necessary condition for the existence of stable interpolation polynomials that passes through the pairs of number $(\sigma_i, \beta_i)_{i\in\{1,\dots,n\}}$ where all σ_i are positive, is that all β_i are of same sign. By considering the previous assumptions (1) and (2), we study existence conditions of a real stable univariate polynomial f that interpolates all the positive data points $(\sigma_i, \beta_i)_{i\in\{1,\dots,n\}}$, $\sigma_i \neq \sigma_k$ for $i \neq k$.

B. Necessary conditions

Consider as an example the following pairs of data $(1,2)$ and $(2,1)$ to interpolate with a stable polynomial. Let us mention that these pairs of points may be interpolated by an unstable polynomial, for example by $f(s) = -s+3$. These data may be also easily interpolated by units over RH_∞ see [11] and [8], as the following

$$U(s) = 2\left(\frac{0.1213s + 1.8787}{s+1}\right)^2$$

Recall: an unit over RH_∞ is a rational function formed of two stable real polynomials of same degree.

The next propositions give necessary conditions for interpolation of stable polynomials. The first set question is this of existence of a stable interpolant that passes through two pairs of data.

Proposition 1: Let two pairs of positive numbers be (σ_a, β_a) and (σ_b, β_b) with $0 < \sigma_a < \sigma_b$. If f defined by (1) is a stable polynomial that interpolates (σ_a, β_a) and (σ_b, β_b), then

$$0 \leq \frac{\beta_b - \beta_a}{\sigma_b - \sigma_a} \tag{4}$$

Proof. It follows

$$f(\sigma_b) - f(\sigma_a) = \alpha_m\,(\sigma_b^m - \sigma_a^m) + \alpha_{m-1}\,(\sigma_b^{m-1} - \sigma_a^{m-1}) \\ \dots + \alpha_2\,(\sigma_b^2 - \sigma_a^2) + \alpha_1(\sigma_b - \sigma_a)$$

We know that

$$\sigma_b^m - \sigma_a^m = (\sigma_b - \sigma_a)\left(\sum_{i=0}^{m-1} \sigma_b^i\,\sigma_a^{m-1-i}\right)$$

This yields to

$$f(\sigma_b) - f(\sigma_a) = \alpha_m(\sigma_b - \sigma_a)\left(\sum_{i=0}^{m-1} \sigma_b^i\,\sigma_a^{m-1-i}\right) + \dots \\ + \alpha_2(\sigma_b - \sigma_a)\left(\sum_{i=0}^{1} \sigma_b^i\,\sigma_a^{1-i}\right) + \alpha_1(\sigma_b - \sigma_a)$$

Equivalently, we have

$$\frac{\beta_b - \beta_a}{\sigma_b - \sigma_a} = \alpha_m\left(\sum_{i=0}^{m-1} \sigma_b^i\,\sigma_a^{m-1-i}\right) + \dots \\ + \alpha_2\left(\sum_{i=0}^{1} \sigma_b^i\,\sigma_a^{1-i}\right) + \alpha_1$$

Thus if f is a stable polynomial, then all the coefficients α_i have the same sign. Moreover the pairs $(\sigma_i, \beta_i)_{i=1,n}$ to interpolate are positive then all the coefficients α_i are positive. As $0 < \sigma_a < \sigma_b$, we have $\sum_{i=0}^{j-1} \sigma_b^i\,\sigma_a^{j-1-i} > 0$ with $i \leq j-1 \leq m-1$. Hence (4) is verified. ■

Condition (4) of Proposition 1, may be reduced to $\beta_b \geq \beta_a$. Let us remark that there does not exist stable polynomials of interpolation for the two pairs of numbers $(1,2)$ and $(2,1)$.

Examine now the case of three pairs of data to interpolate.

Proposition 2: Let three pairs of positive numbers be (σ_a, β_a), (σ_b, β_b) and (σ_c, β_c) with $0 < \sigma_a < \sigma_b < \sigma_c$. If f defined by (1) is a stable polynomial that interpolates (σ_a, β_a), (σ_b, β_b) and (σ_c, β_c) then

$$0 \leq \frac{\beta_b - \beta_a}{\sigma_b - \sigma_a} \leq \frac{\beta_c - \beta_a}{\sigma_c - \sigma_a} \leq \frac{\beta_c - \beta_b}{\sigma_c - \sigma_b} \tag{5}$$

Proof. The proof is immediate. For all integers i, we have

$$0 < \sigma_a^i < \sigma_b^i < \sigma_c^i \tag{6}$$

For all integers i, j, $(i \leq j-1)$, relation (6) yields to

$$0 < \sum_{i=0}^{j-1} \sigma_b^i \sigma_a^{j-1-i} < \sum_{i=0}^{j-1} \sigma_c^i \sigma_a^{j-1-i}$$

This implies that

$$0 \leq \alpha_m\left(\textstyle\sum_{i=0}^{m-1} \sigma_b^i\,\sigma_a^{m-1-i}\right) + \dots \\ + \alpha_2\left(\textstyle\sum_{i=0}^{1} \sigma_b^i\,\sigma_a^{1-i}\right) + \alpha_1 \leq \alpha_m\left(\textstyle\sum_{i=0}^{m-1} \sigma_c^i\,\sigma_a^{m-1-i}\right) + \\ \dots + \alpha_2\left(\textstyle\sum_{i=0}^{1} \sigma_c^i\,\sigma_a^{1-i}\right) + \alpha_1 \tag{7}$$

Note that inequality (7) is not strict since the cases $m = 1$ or $m = 0$ cannot be excluded. From Proposition 1 and relation (7), we deduce

$$0 \leq \frac{\beta_b - \beta_a}{\sigma_b - \sigma_a} \leq \frac{\beta_c - \beta_a}{\sigma_c - \sigma_a}$$

Since (6) holds, it follows the two relationships hereafter

$$0 < \sum_{i=0}^{j-1} \sigma_c^i \sigma_a^{j-1-i} < \sum_{i=0}^{j-1} \sigma_c^i \sigma_b^{j-1-i}$$

$$0 < \sum_{i=0}^{j-1} \sigma_b^i \sigma_a^{j-1-i} < \sum_{i=0}^{j-1} \sigma_c^i \sigma_b^{j-1-i}$$

It is straightforward to see that

$$0 \leq \frac{\beta_c - \beta_a}{\sigma_c - \sigma_a} \leq \frac{\beta_c - \beta_b}{\sigma_c - \sigma_b}$$

$$0 \leq \frac{\beta_b - \beta_a}{\sigma_b - \sigma_a} \leq \frac{\beta_c - \beta_b}{\sigma_c - \sigma_b}$$

Thus, if f is a stable polynomial, then relation (5) holds. ■

From Proposition 2, we conclude that the interpolating polynomial $f(\sigma)$ is locally convex on the real line defined by $\sigma \in]0, \infty[$. This result may be seen as a corollary of the three chords lemma, see [7], applied to Proposition 2. Consequently, as f verifies local convex properties, the following assertions hold.

***Corollary** 1: If f is a stable polynomial that interpolates the set of positive pairs $(\sigma_i, \beta_i)_{i\in\{1,\ldots,n\}}$ then*

i) *f is convex on $]0, \infty[$,*
ii) *f is increasing on $]0, \infty[$,*
iii) *f is positive on $]0, \infty[$.*

Proof. Assertions i), ii) and iii) are direct consequences of Proposition 1 and Proposition 2. ■

According to Corollary 1, a characteristic of this interpolation problem is to assure positivity ($f(s) > 0$), monotonicity ($\dot{f}(s) > 0$) and convexity ($\ddot{f}(s) > 0$) of the interpolant. Moreover as all coefficients α_i of f are positive, then all successive derivatives of f are also positive.

Proposition 1 and Proposition 2 give respectively conditions for two pairs and three pairs of interpolating data. The following result produces a condition for four pairs of interpolating data to interpolate.

Proposition 3: Let $(\sigma_i, \beta_i)_{i\in\{1,\ldots,n\}}$ be a set of n pairs of positive real numbers with $0 < \sigma_1 < \ldots < \sigma_n$ and $n \geq 4$. If f defined by (1) is a stable polynomial that interpolates the pairs (σ_i, β_i) then

$$\forall i \in \{1, \ldots, n-3\},$$
$$\begin{bmatrix}\sigma_{i+3}^2 & \sigma_{i+3} & 1\end{bmatrix} \begin{bmatrix}\sigma_i^2 & \sigma_i & 1\\ \sigma_{i+1}^2 & \sigma_{i+1} & 1\\ \sigma_{i+2}^2 & \sigma_{i+2} & 1\end{bmatrix}^{-1} \begin{bmatrix}\beta_i\\ \beta_{i+1}\\ \beta_{i+2}\end{bmatrix} \leq \beta_{i+3} \tag{8}$$

Proof. Consider $P_i(s) = as^2 + bs + c$, a real polynomial that interpolates three pairs of data (σ_i, β_i), $(\sigma_{i+1}, \beta_{i+1})$, $(\sigma_{i+2}, \beta_{i+2})$. By applying successively the Rolle's Lemma to the polynomial function $f - P_i$ and to its derivative $(f - P_i)'$, it can be shown that $(f - P_i)'$ has at least one zero strictly between σ_i and σ_{i+1} and at least one zero strictly between σ_{i+1} and σ_{i+2}. Therefore $(f - P_i)''$ has at least one zero strictly between σ_i and σ_{i+2}. This yields to

$$\forall \sigma > 0, \quad (f - Pi)'''(\sigma) \geq 0 \tag{9a}$$
$$(9a) \Rightarrow \quad \forall \sigma > \sigma_{i+2}, \quad (f - P_i)''(\sigma) \geq 0 \tag{9b}$$
$$(9b) \Rightarrow \quad \forall \sigma > \sigma_{i+2}, \quad (f - P_i)'(\sigma) \geq 0 \tag{9c}$$
$$(9c) \Rightarrow \quad \forall \sigma > \sigma_{i+2}, \quad (f - P_i)(\sigma) \geq 0 \tag{9d}$$
$$(9d) \Rightarrow \quad \beta_{i+3} = f(\sigma_{i+3}) \geq P_i(\sigma_{i+3}) \tag{9e}$$

Moreover as P_i interpolates the data (σ_i, β_i), $(\sigma_{i+1}, \beta_{i+1})$ and $(\sigma_{i+2}, \beta_{i+2})$, the real coefficients a, b, c satisfy the following matrix relation

$$\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}\sigma_i^2 & \sigma_i & 1\\ \sigma_{i+1}^2 & \sigma_{i+1} & 1\\ \sigma_{i+2}^2 & \sigma_{i+2} & 1\end{bmatrix}^{-1} \begin{bmatrix}\beta_i\\ \beta_{i+1}\\ \beta_{i+2}\end{bmatrix} \tag{10}$$

From relations (10) and (9e), condition (8) is deduced. ■

The inequalities given by Propositions 1, 2 and 3 are not sufficient conditions. The numerous necessary conditions of the polynomial stability allows to provide others additional requirements on the data to interpolate. The purpose of this note is not to examine each necessary condition that preserves polynomial stability for this interpolation issue, e.g. [10], [9], [1] but to derive a general approach to construct stable interpolating polynomials.

III. Roots interlacing and interpolation of stable polynomials

A. Link between polynomial stability and zeroes interlacing

Any polynomial $f(s)$ can be written as composed of two parts given by $f^e(s^2)$ and $f^o(s^2)$ such that

$$f(s) = f^e(s^2) + sf^o(s^2) \tag{11}$$

According to the Hermite-Biehler's Theorem, see [5], if $f(s)$ is a stable polynomial then the roots of $f^e(s^2)$ and $f^o(s^2)$ interlace as below-noted.

***Definition** 1:* [5], Zeroes Interlacing Property.
Let $f^e(u)$ be of degree k and $f^o(u)$ of degree $k-1$ (or k), two real polynomials. Let us assume the roots of these polynomials defined by the following sets

$$root(f^e(u)) = \{a_1, \ldots, a_k\}$$
$$root(f^o(u)) = \{b_1, \cdots, b_{k-1}\}$$
$$(or\ root(f^o(u)) = \{b_1, \cdots, b_k\})$$

Then $f^e(u)$ and $f^o(u)$ interlace iff

- *The roots of $f^e(u)$ and $f^o(u)$ are real and negative and distinct and simple.*
- *The leading coefficients of $f^e(u)$ and $f^o(u)$ have the same sign,*
- *The k roots of $f^e(u)$ alternate with the $k-1$ (or k) roots of $f^o(u)$ as follows*

$$a_1 < b_1 < a_2 < b_2 \ldots < a_{k-1} < b_{k-1} < a_k < 0$$
$$(or\ b_1 < a_1 < b_2 \quad \ldots < b_k < a_k < 0)$$

The relationship between Zeroes Interlacing Property and Hurwitz stability has been emphasized by the Hermite-Biehler's Theorem, which states that the following conditions are equivalent.

***Theorem** 1:* [2], Hermite-Biehler's Theorem.
The three following assertions are equivalent :

i) *The real polynomial $f(s)$ is Hurwitz (or stable),*
ii) *All the real parts of the roots of $f(s)$ are strictly negative,*

iii) *The polynomials $f^e(u)$ and $f^o(u)$ given by (11) verify the Zeroes Interlacing Property.*

In the next section, the existence conditions of negative real roots $a_{j,j=1,\dots,k}$ is discussed for interpolation of stable polynomials, such that defined by the Zeroes Interlacing Property.

B. Necessary and sufficient conditions for interpolation of stable polynomials

A rational function may be associated to the polynomials $f^o(u)$ and $f^e(u)$ by using the Cauchy index, see [5]. In the next theorem a simple condition is given for the existence of a stable polynomial f from this rational function.

***Theorem** 2:* Existence conditions of stable polynomials. *Consider $m = 2k$. Then f is stable iff the two following conditions hold*

- *The roots $\{a_1, \dots, a_k\}$ of f^e are simple and negative.*
- *All real c_j in (12) are positive.*

$$f^o(u) = \sum_{j=1}^{k} c_j \frac{f^e(u)}{u - a_j} \tag{12}$$

Proof. Following [5], if $m = 2k$ then $f^e(u)$ and $f^o(u)$ interlace iff we can write

$$\frac{f^o(u)}{f^e(u)} = \sum_{j=1}^{k} \frac{c_j}{u - a_j} \tag{13}$$

where

$$c_j = \frac{f^o(a_j)}{\frac{df^e}{du}(a_j)} > 0$$

Then Theorem 2 follows from Theorem 1. ■

Relation (11) yields to

$$\beta_i = f^e(\sigma_i^2) + \sigma_i \, f^o(\sigma_i^2) \tag{14}$$

By considering Theorem 2, equation (14) becomes

$$\beta_i = f^e(\sigma_i^2) + \ \sigma_i \sum_{j=1}^{k} \ c_j \frac{f^e(\sigma_i^2)}{\sigma_i^2 - a_j} \tag{15}$$

This relation (15) implies the following equations

$$\begin{cases} \dfrac{\beta_1}{\prod_{j=1}^{k}(\sigma_1^2 - a_j)} - 1 = c_1 \dfrac{\sigma_1}{\sigma_1^2 - a_1} + \dots + c_k \dfrac{\sigma_1}{\sigma_1^2 - a_k} \\ \qquad \vdots \qquad\qquad \vdots \\ \dfrac{\beta_n}{\prod_{j=1}^{k}(\sigma_n^2 - a_j)} - 1 = c_1 \dfrac{\sigma_n}{\sigma_n^2 - a_1} + \dots + c_k \dfrac{\sigma_n}{\sigma_n^2 - a_k} \end{cases} \tag{16}$$

Therefore, the interpolation problem of stable polynomials is expressed as system of equations (16) to solve where the unknown variables are $a_{j,j\in\{1\dots,k\}}$ the set of distinct negative roots of $f^e(u)$ and $c_{j,j\in\{1\dots,k\}}$, a set of positive parameters. We can observe that these equations (16) are non-linear functions of $a_{j,j\in\{1\dots,k\}}$ and linear functions of $c_{j,j\in\{1\dots,k\}}$.

These relationships are summarized by the following matrix system

$$\Lambda(a_1, \dots, a_k) = \Phi(a_1, \dots, a_k)\Gamma \tag{17}$$

where

$$\Lambda^T(a_1, \dots, a_k) = \left[\frac{\beta_1}{\prod_{j=1}^{k}((\sigma_1)^2 - a_j)} - 1, \ \dots \ \frac{\beta_n}{\prod_{j=1}^{k}((\sigma_n)^2 - a_j)} - 1 \right]$$

$$\Gamma^T = [c_1 \ \dots, c_k]$$

$$\Phi(a_1, \dots, a_k) = \begin{bmatrix} \dfrac{\sigma_1}{(\sigma_1)^2 - a_1} & \cdots & \dfrac{\sigma_1}{(\sigma_1)^2 - a_k} \\ \vdots & & \vdots \\ \dfrac{\sigma_n}{(\sigma_n)^2 - a_1} & \cdots & \dfrac{\sigma_n}{(\sigma_n)^2 - a_k} \end{bmatrix}$$

By considering equation (17), the next proposition presents a necessary and sufficient condition for interpolation of stable polynomials.

Proposition 4: Let n pairs of numbers be (σ_i, β_i), with $i \in \{1, \dots, n\}$. Then f is a stable polynomial that interpolates all (σ_i, β_i) iff there exists a set of positive numbers $c_{j,j\in\{1\dots,k\}}$ such that (17) holds where $a_{j,j\in\{1\dots,k\}}$ is a set of negative distinct real numbers.

Proof. See Theorem 2 and equation (15). ■

Expression (17) is a nonlinear system with respect to negative distinct real numbers $a_{j,j\in\{1\dots,k\}}$ to find.

C. Feasibility and infeasibility analysis of system (17).

Assume that the set of negative distinct reals $a_{j,j\in\{1\dots,k\}}$ in (17) is a parameter to be determined. Consequently, we denote expression (17) by $\Lambda = \Phi\Gamma$. In this part, are examined conditions affecting existence and non-existence of positive parameters $c_{j,j\in\{1\dots,k\}}$ function of the data (σ_i, β_i), with $i \in \{1, \dots, n\}$ and verifying $\Lambda = \Phi\Gamma$. If there exist solutions for this algebraic problem, there are precisely in a strictly convex polyhedral cone defined by $C = \{\Lambda = \Phi\Gamma : \Gamma > 0\}$. The notation $\Gamma > 0$ means that all $c_{j,j\in\{1\dots,k\}}$ are positive. To study such a system of equations with inequality constraints, it is easier to transform the original system (17) into another system with only inequalities. It is the sense of the next result, that is a modified version of Farkas' Theorem, see [3], that takes into account the strict inequality $\Gamma > 0$.

***Theorem** 3:* : Theorem of alternatives relative to $\Gamma > 0$. *Exactly one of the two following statements is true.*

i) *The system $\Lambda = \Phi\Gamma$ has a solution $\Gamma > 0$*
ii) *There exists a vector Y such that $\Phi^T Y > 0$ and $\Lambda^T Y \leq 0$.*

Proof. First, show that both statement i) and ii) cannot be true. Assume that Γ satisfies i) and that Y satisfies ii). This yields to

$$0 \geq \Lambda^T Y = Y^T \Lambda = Y^T(\Phi\Gamma) = (\Phi^T Y)\Gamma > 0$$

that is an obvious contradiction.

Secondly let us prove that at least one of assertions i) and ii) has a solution. Suppose that statement i) has no solution. Then we have

$$\Lambda \notin \mathcal{C} = \{z = \Phi\Gamma,\ \Gamma > 0\} \tag{18}$$

Moreover, $\mathcal{C}$ is convex and open. Then from the Separating Hyperplane Theorem, we deduce that there exists an hyperplane that separates properly Λ and $\mathcal{C}$, i.e. there exists a vector $Y \neq 0$ such that

$$\forall z \in \mathcal{C},\ \Lambda^T Y < z^T Y \tag{19}$$

As $0 \in \overline{\mathcal{C}}$, then we deduce from (19) that

$$\Lambda^T Y \leq 0^T Y = 0 \tag{20}$$

Now let us show that $\Phi^T Y > 0$. Suppose that $\Phi^T Y \leq 0$. Then for any $\lambda > 0$ and for any $\Gamma > 0$, we have

$$\lambda\Gamma^T\Phi^T Y \leq 0.$$

As $\Lambda^T Y \leq 0$, we can choose λ great enough such that

$$\lambda\Gamma^T\Phi^T Y = (\Phi\lambda\Gamma)^T Y \leq \Lambda^T Y$$

which is in contradiction with (19). ■

The previous problem described by relation (17) where the unknown variables are $c_{j,j\in\{1\ldots,k\}}$, is transformed in a new problem, see Theorem 3 where the unknown variables are the vector coordinates Y. In order to determine if there exists infeasible solutions to the original problem (17), let us study the existence conditions of Y given in Theorem 3 with

$$Y^T = [y_1\ \ldots\ y_n]$$

This yields to write $\Phi^T Y > 0$ and $\Lambda^T Y \leq 0$ as it follows

$$\begin{cases} \left(\frac{\sigma_1}{(\sigma_1)^2 - a_1}\right) y_1 + \quad \ldots \quad + \left(\frac{\sigma_n}{(\sigma_n)^2 - a_1}\right) y_n > 0 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ \left(\frac{\sigma_1}{(\sigma_1)^2 - a_k}\right) y_1 + \quad \ldots \quad + \left(\frac{\sigma_n}{(\sigma_n)^2 - a_k}\right) y_n > 0 \\ \left(\frac{\beta_1}{\prod_{j=1}^{k}((\sigma_1)^2 - a_j)} - 1\right) y_1 + \ldots \\ \qquad\qquad + \left(\frac{\beta_n}{\prod_{j=1}^{k}((\sigma_n)^2 - a_j)} - 1\right) y_n \leq 0 \end{cases} \tag{21}$$

Let us consider the Fourier-Motzkin method to eliminate variables in (21). This procedure allows to verify if there exists a solution Y for these inequalities (21). Basic idea of variable elimination of this approach is the following

- Pick one variable,
- Eliminate it by surrounding the variable by a lower bound and an upper bound,
- Continue until all variables but one are eliminated.

At each step, a new system is got. Solutions to the new system can be used to determine solutions to the original system and feasibility to the new system can be used to determine infeasibility to the original system (and conversely). This iterative procedure is developed below in order to highlight conditions on Y reported as conditions on the set of data $(\sigma_i, \beta_i)_{i\in\{1,\ldots,n\}}$. Only the two first iterations will be described hereafter because of the complexity of inequalities to itemize. This formulation gives non-existence conditions of a stable polynomial relatively to the two first pairs of data (σ_1, β_1) and (σ_2, β_2) to interpolate.

First iteration

The system (21) is equivalent to the following writing

$$\begin{cases} y_1 & > & Min_{1\text{-}y_1} \\ \vdots & & \vdots \\ y_1 & > & Min_{k\text{-}y_1} \\ y_1 & \leq & Max_{y_1} \end{cases} \tag{22}$$

where $a_j,\ j \in \{1, \ldots, k\}$ are negative real numbers and

$$\begin{aligned} Min_{1\text{-}y_1} = &- \left(\frac{(\sigma_1)^2 - a_1}{\sigma_1}\right)\left(\frac{\sigma_2}{(\sigma_2)^2 - a_1}\right) y_2 \\ \ldots \quad &- \left(\frac{(\sigma_1)^2 - a_1}{\sigma_1}\right)\left(\frac{\sigma_n}{(\sigma_n)^2 - a_1}\right) y_n \end{aligned}$$

$$\vdots$$

$$\begin{aligned} Min_{k\text{-}y_1} = &- \left(\frac{(\sigma_1)^2 - a_k}{\sigma_1}\right)\left(\frac{\sigma_2}{(\sigma_2)^2 - a_k}\right) y_2 \\ \ldots \quad &- \left(\frac{(\sigma_1)^2 - a_k}{\sigma_1}\right)\left(\frac{\sigma_n}{(\sigma_n)^2 - a_k}\right) y_n \end{aligned}$$

$$\begin{aligned} &Max_{y_1} = \\ &-\left(\frac{\prod_{j=1}^{k}((\sigma_1)^2 - a_j)}{\beta_1 - \prod_{j=1}^{k}((\sigma_1)^2 - a_j)}\right)\left(\frac{\beta_2 - \prod_{j=1}^{k}((\sigma_2)^2 - a_j)}{\prod_{j=1}^{k}((\sigma_2)^2 - a_j)}\right) y_2 \\ &\ldots \\ &-\left(\frac{\prod_{j=1}^{k}((\sigma_1)^2 - a_j)}{\beta_1 - \prod_{j=1}^{k}((\sigma_1)^2 - a_j)}\right)\left(\frac{\beta_n - \prod_{j=1}^{j=k}((\sigma_n)^2 - a_j)}{\prod_{j=1}^{k}((\sigma_n)^2 - a_j)}\right) y_n \end{aligned}$$

Let us set

$$Min_{y_1} = max\left(Min_{1\text{-}y_1}, \ldots Min_{k\text{-}y_1}\right)$$

These inequalities (22) are rewritten as a lower bound and an upper bound on the variable y_1 as it follows

$$Min_{y_1} < y_1 \leq Max_{y_1} \tag{23}$$

Relation (23) will be checked after elimination of all variables. If there exists y_1, the system (22) becomes the following

$$\begin{cases} Min_{1\text{-}y_1} & < & Max_{y_1} \\ \vdots & & \vdots \\ Min_{k\text{-}y_1} & < & Max_{y_1} \end{cases} \tag{24}$$

Second iteration

Equations (24) are linear inequalities with variables $y_2, \ldots, y_k$. The inequalities for this second iteration depends on the sign of the coefficients of y_2. These coefficients are the following

$$\begin{cases} Coeff_{1\text{-}y_2} = -\left(\frac{(\sigma_1)^2 - a_1}{\sigma_1}\right)\left(\frac{\sigma_2}{(\sigma_2)^2 - a_1}\right) \\ +\left(\frac{\beta_2 - \prod_{j=1}^{k}((\sigma_2)^2 - a_j)}{\prod_{j=1}^{k}((\sigma_2)^2 - a_j)}\right)\left(\frac{\prod_{j=1}^{k}((\sigma_1)^2 - a_j)}{\beta_1 - \prod_{j=1}^{k}((\sigma_1)^2 - a_j)}\right) \\ \quad\vdots \qquad\qquad \vdots \\ Coeff_{k\text{-}y_2} = -\left(\frac{(\sigma_1)^2 - a_k}{\sigma_1}\right)\left(\frac{\sigma_2}{(\sigma_2)^2 - a_k}\right) \\ +\left(\frac{\beta_2 - \prod_{j=1}^{k}((\sigma_2)^2 - a_j)}{\prod_{j=1}^{k}((\sigma_2)^2 - a_j)}\right)\left(\frac{\prod_{j=1}^{k}((\sigma_1)^2 - a_j)}{\beta_1 - \prod_{j=1}^{k}((\sigma_1)^2 - a_j)}\right) \end{cases} \tag{25}$$

A) Examine a first case: $0 < \beta_2 < \beta_1$.

In this case, all the coefficients of y_2 are negative. Then there exists y_2 such that

$$\begin{cases} Min_{1\text{-}y_2} & < & y_2 \\ \vdots & & \vdots \\ Min_{k\text{-}y_2} & < & y_2 \end{cases} \tag{26}$$

Since by assumption $0 < \sigma_1 < \sigma_2$ and for any $a_j < 0$, we have

$$0 < \left(\frac{(\sigma_1)^2 - a_j}{(\sigma_2)^2 - a_j}\right) < 1$$

Hence this yields to

$$0 < \prod_{j=1}^{k}((\sigma_1)^2 - a_j) < \prod_{j=1}^{k}((\sigma_2)^2 - a_j)$$

Consequently

$$0 < \left(\frac{\beta_2 - \prod_{j=1}^{k}((\sigma_2)^2 - a_j)}{\beta_1 - \prod_{j=1}^{k}((\sigma_1)^2 - a_j)}\right) < 1 \tag{27}$$

Finally, for all $a_j < 0$ where $j \in \{1, \ldots, k\}$, we get (26). That is a case of infeasibility of system (17) if inequalities (23) hold.

B) Examine now the second case: $0 < \beta_1 < \beta_2$.

Several subcases may be considered. Only one is considered hereafter.

$\beta_2 - \beta_1 < \sigma_2 - \sigma_1$ and $\sigma_2 \geq 1$.

In this subcase, for any $a_j < 0$, we have

$$0 < \beta_2 - \prod_{j=1}^{k}(\sigma_2^2 - a_j) < \beta_1 - \prod_{j=1}^{k}(\sigma_1^2 - a_j) \tag{28}$$

This inequality (28) is true since $a_i < 0$ with $i \in \{1, \ldots, k\}$. Then $f^e(u)$ is Hurwitz and the coefficients of $f^e(u)$ are all of the same sign. That is

$$\begin{aligned} f^e(\sigma^2) &= (\sigma^2)^k + \alpha_{2k-2}\,(\sigma^2)^{2k-2} \cdots + \alpha_4\,(\sigma^2)^2 + \alpha_2\,(\sigma^2) + \alpha_0 \\ f^e(\sigma^2) &= (\sigma^2 - a_1)(\sigma^2 - a_2)\cdots(\sigma^2 - a_k) \end{aligned} \tag{29}$$

where α_{2i} with $i \in \{0, \ldots, k-1\}$ satisfies the relations below

$$\begin{cases} \alpha_{2k-2} & = -\sum_{i=1}^{k} a_i > 0 \\ \alpha_{2k-4} & = \sum_{0 \leq i < j \leq k} a_i a_j > 0 \\ & \vdots \\ \alpha_{2(k-l)} & = (-1)^l \sum_{1 \leq i_1 < \ldots < i_l \leq k} a_{i_1} a_{i_2} \ldots a_{i_l} > 0 \\ & \vdots \\ \alpha_0 & = (-1)^k a_1 a_2 \ldots a_k > 0 \end{cases}$$

That implies that the relation above holds

$$\begin{aligned} &\prod_{j=1}^{k}(\sigma_2^2 - a_j) - \prod_{j=1}^{k}(\sigma_1^2 - a_j) = (\sigma_2^2 - \sigma_1^2)\sum_{i=0}^{k-1}\sigma_2^{2i}\sigma_1^{2(k-1-i)} \\ &+(\sigma_2^2 - \sigma_1^2)\left(\sum_{j=1}^{k-1}\alpha_{2j}\sum_{i=0}^{j-1}\sigma_2^{2i}\sigma_1^{2(j-1-i)}\right) > 0 \end{aligned} \tag{30}$$

For $\sigma_2 \geq 1$, we get

$$0 < \sigma_2 - \sigma_1 \leq \prod_{j=1}^{k}(\sigma_2^2 - a_j) - \prod_{j=1}^{k}(\sigma_1^2 - a_j)$$

Straighforward

$$0 < \left(\frac{\beta_2 - \prod_{j=1}^{k}((\sigma_2)^2 - a_j)}{\beta_1 - \prod_{j=1}^{k}((\sigma_1)^2 - a_j)}\right) < 1$$

Consequently for all $a_p < 0$ where $p \in \{1, \ldots, k\}$, we have

$$\begin{aligned} &\left(\frac{\sigma_1}{\sigma_2}\right)\left(\frac{(\sigma_2)^2 - a_p}{(\sigma_1)^2 - a_p}\right)\left(\frac{\prod_{j=1}^{k}((\sigma_1)^2 - a_j)}{\prod_{j=1}^{k}((\sigma_2)^2 - a_j)}\right) \\ &\cdot\left(\frac{\beta_2 - \prod_{j=1}^{k}((\sigma_2)^2 - a_j)}{\beta_1 - \prod_{j=1}^{k}((\sigma_1)^2 - a_j)}\right) - 1 < 0 \end{aligned} \tag{31}$$

Hence, all the coefficients of y_2 are negative. Finally, we get also (26). That is a case of infeasibility of system (17) if inequalities (23) hold.

IV. Application

Apply the proposed approach to the case of two pairs of data to interpolate $(1, 2)$ and $(2, 1)$. By using the Theorem of alternatives, see Theorem 3, the question of existence of an interpolating polynomial for these two pairs of data becomes the following: can you find a vector $Y = [y_1\ , y_2]$ such that $\Phi^T Y > 0$ and $\Lambda^T Y \leq 0$?

By considering relations (21) with the negative real number a_i, $i \in \{1, \ldots, k\}$, we get

$$\begin{cases} \left(\frac{1}{1-a_1}\right) y_1 + \left(\frac{2}{4-a_1}\right) y_2 > 0 \\ \vdots \qquad\qquad \vdots \\ \left(\frac{1}{1-a_k}\right) y_1 + \left(\frac{2}{4-a_k}\right) y_2 > 0 \\ \left(\frac{2}{\prod_{j=1}^{j=k}(1-a_j)} - 1\right) y_1 + \left(\frac{1}{\prod_{j=1}^{j=k}(4-a_j)} - 1\right) y_2 \leq 0 \end{cases} \tag{32}$$

Then we get

$$Min_{y_1} = Max\left(-2\left(\frac{1-a_1}{4-a_1}\right)y_2, \dots, -2\left(\frac{1-a_k}{4-a_k}\right)y_2\right)$$

$$Max_{y_1} = -\frac{\left(1-\prod_{j=1}^{j=k}(4-a_j)\right)}{\left(2-\prod_{j=1}^{j=k}(1-a_j)\right)} \frac{\left(\prod_{j=1}^{j=k}(1-a_j)\right)}{\left(\prod_{j=1}^{j=k}(4-a_j)\right)} y_2$$

If there exists y_1, the system (32) becomes relations (24), which is satisfied iff all the coefficients of y_2 given by (25) have the same sign :

$$Coeff_{p\text{-}y_2} = -2\left(\frac{1-a_p}{4-a_p}\right) + \frac{\left(1-\prod_{j=1}^{j=k}(4-a_j)\right)}{\left(2-\prod_{j=1}^{j=k}(1-a_j)\right)} \frac{\left(\prod_{j=1}^{j=k}(1-a_j)\right)}{\left(\prod_{j=1}^{j=k}(4-a_j)\right)}$$

It is easy to show that we have

$$\forall\, a_j < 0,\ j \in \{1,\dots,k\}, \quad -2\left(\frac{1-a_j}{4-a_j}\right) < -\frac{1}{2} \tag{33}$$

Now, let us prove the following inequality (34).

$$\forall\, a_j < 0,\ j \in \{1,\dots,k\}, \quad \frac{\left(1-\prod_{j=1}^{j=k}(4-a_j)\right)}{\left(2-\prod_{j=1}^{j=k}(1-a_j)\right)} \frac{\left(\prod_{j=1}^{j=k}(1-a_j)\right)}{\left(\prod_{j=1}^{j=k}(4-a_j)\right)} < \frac{1}{2} \tag{34}$$

Inequality (34) may be written as two others inequalities to prove as it follows
$\forall\, a_j < 0,\ j \in \{1,\dots,k\}$,

$$0 < \frac{\prod_{j=1}^{j=k}(1-a_j)}{\prod_{j=1}^{j=k}(4-a_j)} < 1 \tag{35a}$$

$$0 < \frac{1-\prod_{j=1}^{j=k}(4-a_j)}{2-\prod_{j=1}^{j=k}(1-a_j)} < \frac{1}{2} \tag{35b}$$

Inequality (35a) is true so we have

$$\forall\, a_j < 0,\ j \in \{1,\dots,k\}, \quad 0 < \frac{1-a_j}{4-a_j} < 1$$

Notice, we have

$$\beta_i - f^e(\sigma_i^2) = \sigma_i f^o(\sigma_i^2) > 0$$

For any $a_1, \dots, a_k < 0$, this yields to

$$2 - \prod_{j=1}^{j=k}(1-a_j) > 0 \tag{36}$$

and then

$$0 < \left(\frac{\prod_{j=1}^{j=k}(1-a_j)}{\prod_{j=1}^{j=k}(4-a_j)}\right) < 2 \tag{37}$$

As the inequality (37) is verified and (36) is true, then inequality (35b) holds.

As inequalities (34) and (33) hold then it can be deduced that for any integer p, $Coeff_{p\text{-}y_2} < 0$. Therefore any $y_2 > 0$ is a suitable solution. Moreover we have

$$\begin{cases} Min_{y_1} \leq -\frac{1}{2} \\ -\frac{1}{2} \leq Max_{y_1} \end{cases}$$

We conclude that, for any negative real numbers a_i, $i \in \{1,\dots,k\}$, there exists y_1 satisfying the following relationship

$$Min_{y_1} \leq y_1 \leq Max_{y_1} \tag{38}$$

Consequently, there exists a vector Y. Equivalently, a stable polynomial that interpolate these points does not exist.

V. Conclusion

In this paper we face the question of interpolation of stable polynomials. This issue is more complex than the unit interpolation over RH_∞ since it provides conditions more restrictive. We have shown that the stable interpolation polynomial is necessarily convex, positive and monotone on the positive real set. Then a necessary and sufficient condition to interpolate stable polynomials has been highlighted. Finally a method to generate these polynomials has been provided by the use of the Fourier-Motzkin approach.

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