Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems – MTNS 2010 • 5–9 July, 2010 • Budapest, Hungary

On well-posedness of piecewise affine bimodal dynamical systems

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***Abstract*— The theory of differential inclusions provides certain sufficient conditons for the uniqueness of Filippov solutions such as one-sided Lipschitzian property or maximal monotone condition. When applied to piecewise affine dynamical systems, these conditions impose rather strong conditions. In this paper, we provide less restrictive conditions for uniqueness of Filippov solutions for the bimodal piecewise affine systems.**

I. Introduction

Piecewise affine dynamical models arise in various contexts of system and control theory. When these models are given by differential equations with discontinuous right hand sides, existence and uniqueness of solutions (i.e., well-posedness) become a nontrivial issue. Such models are typically studied in the framework of differential inclusions with the so-called Filippov solution concept. Existence of Filippov solutions require very mild conditions in general. Applied to piecewise affine systems, one can readily guarantee existence of solutions. However, conditions of uniqueness (e.g., one-sided Lipschitz property or monotonicity-type conditions) for general differential inclusions impose quite strong requirements for piecewise affine systems. In this paper, we introduce less restrictive conditions that guarantee uniqueness of solutions.

Importance of well-posedness studies are two folded. On the one hand, conditions for well-posedness cerve as means of model verification. After all, physical phenomena that the model should capture has unique solutions. Naturally, any model should inherit this property. On the other hand, due to well-posedness conditions piecewise affine systems enjoy certain strong structural properties that can be exploited in the context of analysis and design.

II. Piecewise affine bimodal dynamical systems

Throughout the paper, the index i will always belong to the set $\{1, 2\}$.

For given matrices $(A_i, e_i) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n$, $c \in \mathbb{R}^n$ and $f \in \mathbb{R}$ with $c^T \neq 0$, we define set-valued functions F, G : $\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as

$$F(x) = \begin{cases} \{A_1x + e_1\} & \text{if } \; y < 0 \\ \{A_1x + e_1, A_2x + e_2\} & \text{if } \; y = 0 \\ \{A_2x + e_2\} & \text{if } \; y > 0, \end{cases}$$

$$G(x) = \begin{cases} \{A_1x + e_1\} & \text{if } \; y < 0 \\ \operatorname{conv}\left(\{A_1x + e_1, A_2x + e_2\}\right) & \text{if } \; y = 0 \\ \{A_2x + e_2\} & \text{if } \; y > 0 \end{cases}$$

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where $y = c^T x + f$ and $\operatorname{conv}(S)$ denotes the convex hull of the set S.

Consider the bimodal piecewise affine system given by the differential inclusion

$$\dot{x}(t) \in F(x(t)) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state. In case, the implication

$$c^T x + f = 0 \Rightarrow A_1x + e_1 = A_2x + e_2 \tag{2}$$

holds, the set-valued maps F and G boil down to single-valued Lipschitz continuous function. In this paper, we study the general case where (2) may not hold. Various solution concepts exist for differential inclusions (see e.g. [1]).

In this paper, we focus on Carathéodory and Filippov solutions to the system (1).

Definition II.1 An absolutely continuous function $x : \mathbb{R} \to \mathbb{R}^n$ is said to be a solution of the bimodal system (1) for the initial state x_0 in the sense of

- *Carathéodory* if $x(0) = x_0$ and (1) is satisfied for almost all $t \in \mathbb{R}$.
- *forward Carathéodory* if it is a solution in the sense of Carathéodory and for each t^* there exists $\varepsilon_{t^*} > 0$ such that
$$\dot{x}(t) = A_i x(t) + e_i,\ (-1)^{i-1}[c^T x(t) + f] \leqslant 0 \tag{3}$$
for all $t \in (t^*, t^* + \varepsilon_{t^*})$.
- *backward Carathéodory* if it is a solution in the sense of Carathéodory and for each t^* there exists $\varepsilon_{t^*} > 0$ such that
$$\dot{x}(t) = A_i x(t) + e_i,\ (-1)^{i-1}[c^T x(t) + f] \leqslant 0 \tag{4}$$
for all $t \in (t^* - \varepsilon_{t^*}, t^*)$.
- *Filippov* if $x(0) = x_0$ and the differential inclusion
$$\dot{x}(t) \in G(x(t)) \tag{5}$$
is satisfied for almost all $t \in \mathbb{R}$.

Clearly every Carathéodory solution is also Filippov solution. As it is well-known the converse is not true in general.

Existence of Filippov solutions is guaranteed by the following proposition.

Proposition II.2 ([4], Theorem 1, p.77) *For each initial state $x_0 \in \mathbb{R}^n$ there exists a solution of the system* (1) *in the sense of Filippov.*

The main goal of the paper is to investigate uniqueness of Filippov solutions to the differential inclusion (1) and its consequences.

Definition II.3 We say that a solution x for the initial state x_0 is
1) *left-unique* if x' is a solution for the same initial state then $x(t) = x'(t)$ for all $t \leqslant 0$.
2) *right-unique* if x' is a solution for the same initial state then $x(t) = x'(t)$ for all $t \geqslant 0$.
3) *unique* if x is left-unique and right-unique.

The two most common conditions that are employed in the theory of differential inclusions in order to guarantee the uniqueness Filippov solutions are the so-called one-sided Lipschitz and maximal monotonicity property of the set-valued mapping G. For the sake of completeness, we recall the definitions of these two notions.

Definition II.4 A set-valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be

- *one-sided Lipschitz* if there exists L such that for all $x_1, x_2 \in \mathbb{R}^n$ the following inequality holds
$$(y_1 - y_2)^T (x_1 - x_2) \leqslant L||x_1 - x_2||^2 \qquad (6)$$
for all $y_i \in H(x_i)$.
- *monotone* if
$$\langle x_1 - x_2, y_1 - y_2 \rangle \geqslant 0$$
for all $x_i \in \mathbb{R}^n, y_i \in H(x_i)$.
- *maximal monotone* if it is monotone and there is no monotone map H' such that $\operatorname{graph}(H) \subset \operatorname{graph}(H')$.

The following theorem provides a complete characterization of the one-sided Lipschitz and maximal monotonicity properties for the set-valued mapping G.

Theorem II.5 *The following statements are equivalent.*
1) *The set-valued mapping G is one-sided Lipschitz.*
2) *There exist a vector h and a number $\mu \geqslant 0$ such that*
$$A_1 - A_2 = hc^T, \; e_1 - e_2 = hf + \mu c. \qquad (7)$$
3) *There exists λ such that $-G + \lambda I$ is monotone.*
4) *There exists λ such that $-G + \lambda I$ is maximal monotone.*

III. Main results

This theorem shows that the one-sided Lipschitzian and maximal monotonicity properties are closely related in the context of bimodal systems. It also shows that these two properties are quite restrictive. The following theorem constitutes the main results of this paper and provides less restrictive sufficient conditions for uniqueness of solutions as well as necessary conditions.

To formulate the theorem, we introduce some nomenclature. For a vector v, we write $v \succeq 0$ if $v = 0$ or its first non-zero entry is positive. We also write $v \succ 0$ meaning that either $v \succeq 0$ and $v \neq 0$. We also write $v \prec 0$ if $-v \succ 0$ and $v \preceq 0$ if $-v \succeq 0$.

Theorem III.1 *Consider the bimodal system* (1). *Suppose that the pairs (c^T, A_i) are observable. Consider the following statements:*

1) *All Filippov solutions are right-unique.*
2) *Any Filippov solution is both forward and backward Carathéodory.*
3) *There exist $0 \leqslant k \leqslant n-1$ and a $(k+1)\times(k+1)$ lower triangular matrix M with positive diagonal elements such that*
$$\begin{bmatrix} c^T \\ c^T A_1 \\ \vdots \\ c^T A_1^k \end{bmatrix} = M \begin{bmatrix} c^T \\ c^T A_2 \\ \vdots \\ c^T A_2^k \end{bmatrix} \; \textit{and} \; \begin{bmatrix} f \\ c^T e_1 \\ \vdots \\ c^T A_1^{k-1} e_1 \end{bmatrix} \succ M \begin{bmatrix} f \\ c^T e_2 \\ \vdots \\ c^T A_2^{k-1} e_2 \end{bmatrix}.$$
4) *There exists an $n \times n$ lower triangular matrix M with positive diagonal elements such that*
$$\begin{bmatrix} c^T \\ c^T A_1 \\ \vdots \\ c^T A_1^{n-1} \end{bmatrix} = M \begin{bmatrix} c^T \\ c^T A_2 \\ \vdots \\ c^T A_2^{n-1} \end{bmatrix} \; \textit{and} \; \begin{bmatrix} f \\ c^T e_1 \\ \vdots \\ c^T A_1^{n-2} e_1 \end{bmatrix} = M \begin{bmatrix} f \\ c^T e_2 \\ \vdots \\ c^T A_2^{n-2} e_2 \end{bmatrix}.$$
5) *There exists an $(n+1) \times (n+1)$ lower triangular matrix M with positive diagonal elements such that*
$$\begin{bmatrix} c^T \\ c^T A_1 \\ \vdots \\ c^T A_1^{n} \end{bmatrix} = M \begin{bmatrix} c^T \\ c^T A_2 \\ \vdots \\ c^T A_2^{n} \end{bmatrix} \; \textit{and} \; \begin{bmatrix} f \\ c^T e_1 \\ \vdots \\ c^T A_1^{n-1} e_1 \end{bmatrix} = M \begin{bmatrix} f \\ c^T e_2 \\ \vdots \\ c^T A_2^{n-1} e_2 \end{bmatrix}.$$
6) *There exists a 2×2 lower triangular matrix M with positive diagonal elements such that*
$$\begin{bmatrix} c^T \\ c^T A_1 \end{bmatrix} = M \begin{bmatrix} c^T \\ c^T A_2 \end{bmatrix} \; \textit{and} \; \begin{bmatrix} f \\ c^T e_1 \end{bmatrix} \succ M \begin{bmatrix} f \\ c^T e_2 \end{bmatrix}.$$

The following implications hold:
i. *1 ⇒ 3 or 4*
ii. *5 ⇒ 2*
iii. *5 ⇒ 1*
iv. *6 ⇒ 1*

IV. Proof of Theorem II.5

In this section, we prove Theorem II.5.

A. $1 \Rightarrow 2$

Suppose that G satisfies the one-sided Lipschitz condition. Let
$$S_- = \{x \mid c^T x + f \leqslant 0\} \text{ and } S_+ = \{x \mid c^T x + f \geqslant 0\}.$$
Take $x_1 \in S_-, x_2 \in S_+$ and take $\bar{x}$ such that $c^T \bar{x} + f = 0$. For $\alpha \in (0, 1]$ define
$$x_1' = \alpha x_1 + (1-\alpha)\bar{x}, x_2' = \alpha x_2 + (1-\alpha)\bar{x}.$$
It is easy to check that $x_1' \in S_-, x_2' \in S_+$. Since G satisfies the one-sided Lipschitz condition, one has
$$[(A_1 x_1' + e_1) - (A_2 x_2' + e_2)]^T (x_1' - x_2') \leqslant L||x_1' - x_2'||^2$$
or equivalently
$$\frac{(1-\alpha)}{\alpha}[(A_1 - A_2)\bar{x} + (e_1 - e_2)]^T (x_1 - x_2) + [(A_1 x_1 + e_1) - (A_2 x_2 + e_2)]^T (x_1 - x_2) \leqslant L||x_1 - x_2||^2.$$

By taking sufficient small α, we can conclude that

$$[(A_1 - A_2)\bar{x} + (e_1 - e_2)]^T(x_1 - x_2) \leqslant 0$$

for all $x_1 \in S_-, x_2 \in S_+$. It implies that

$$(A_1 - A_2)\bar{x} + (e_1 - e_2) \in (S_- - S_+)^o \tag{8}$$

for any $\bar{x}$ satisfying $c^T\bar{x}+f = 0$ where the notation ^{o} denotes the polar cone. Then, one gets

$$\begin{aligned}(A_1 - A_2)(\ker c^T) + (A_1 - A_2)\bar{x} + (e_1 - e_2)\\ \subseteq (S_- - S_+)^o = \{\alpha c \mid \alpha \geqslant 0\}\end{aligned}$$

for fixed $\bar{x}$ satisfying $c^T\bar{x} + f = 0$. Since the left hand side is an affine set and the right hand side is a cone, we can conclude that $(A_1 - A_2)(\ker c^T) = \{0\}$. So $A_1 - A_2 = hc^T$ for some h. Then it follows from (8) that

$$e_1 - e_2 - hf \in (S_- - S_+)^o.$$

Note that

$$S_- - S_+ = \{x \mid c^T x \leqslant 0\}.$$

Hence,

$$(S_- - S_+)^o = \{\alpha c \mid \alpha \geqslant 0\}.$$

This means that $e_1 - e_2 = hf + \alpha c$ for some $\alpha \geqslant 0$.

B. $2 \Rightarrow 3$

Let $\lambda = \frac{1}{2}\max\{\lambda_{\max}(A_1 + A_1^T), \lambda_{\max}(A_2 + A_2^T)\}$ where $\lambda_{\max}(A)$ denotes the largest eigenvalue of A. It is easy to check that $A_{i,\lambda} := A_i - \lambda I \leqslant 0$. Define $G_\lambda := -G + \lambda I$. Note that

$$G_\lambda(x) = \begin{cases}\{G_\lambda^1(x)\} & \text{if } c^T x + f < 0\\ \text{conv}\{G_\lambda^1(x), G_\lambda^2(x)\} & \text{if } c^T x + f = 0\\ \{G_\lambda^2(x)\} & \text{if } c^T x + f > 0\end{cases}$$

where $G_\lambda^i(x) := -A_{i,\lambda}x - e_i$. To prove G_λ being monotone, consider the set-valued mapping

$$\tilde{G}_\lambda(x) = \begin{cases}\{G_\lambda^1(x) + \frac{\mu c}{2}\} & \text{if } y < 0\\ \text{conv}\{G_\lambda^1(x) + \frac{\mu c}{2}, G_\lambda^2(x) - \frac{\mu c}{2}\} & \text{if } y = 0\\ \{G_\lambda^2(x) - \frac{\mu c}{2}\} & \text{if } y > 0\end{cases}$$

where $y = c^T x + f$. Due to (7), $\tilde{G}_\lambda$ is singleton and continuous. The generalized Jacobian of $\tilde{G}_\lambda$ at x is

$$\partial(\tilde{G}_\lambda)(x) = \begin{cases}-A_{1,\lambda} & \text{if } c^T x + f < 0\\ \text{conv}\{-A_{1,\lambda}, -A_{2,\lambda}\} & \text{if } c^T x + f = 0\\ -A_{2,\lambda} & \text{if } c^T x + f > 0.\end{cases}$$

Since $A_{i,\lambda} \leqslant 0$, it is easy to see that each element of $\partial(\tilde{G}_\lambda)(x)$ is positive semidefinite. By [6], Proposition 2.1, $\tilde{G}_\lambda$ is monotone. Now for every $x_i \in \mathbb{R}^n$, $y_i \in G_\lambda(x_i)$, we can see that $y_i = \tilde{y}_i - (2\alpha_i - 1)\frac{\mu c}{2}$ for some $\tilde{y}_i \in \tilde{G}_\lambda(x_i)$ and $\alpha_i \in [0, 1]$ with $\alpha_i(c^T x_i + f) = 0$. Thus we have

$$\langle x_1 - x_2, y_1 - y_2\rangle = \langle x_1 - x_2, \tilde{y}_1 - \tilde{y}_2\rangle + (\alpha_2 - \alpha_1)\mu\langle x_1 - x_2, c\rangle.$$

Observe that $(\alpha_2 - \alpha_1)\mu\langle x_1 - x_2, c\rangle \geqslant 0$ for all x_1, x_2. Thus, monotonicity of G_λ follows from monotonicity of $\tilde{G}_\lambda$.

C. $3 \Rightarrow 4$

To prove that G_λ is maximal monotone, we invoke the following lemma which is a result of [5, Thm. 3.4] and [3, Exercise 1.3].

Lemma IV.1 *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map with convex compact values and $dom(F) = \mathbb{R}^n$. Suppose that F satisfies linear growth condition;that is, there exist positive constants γ and c such that $y \in F(x) \implies ||y|| \leqslant \gamma||x|| + c$. Then F is maximal monotone if and only if it is monotone and upper semicontinuous.*

In view of this lemma, it suffices to prove that G_λ is upper semicontinuous, i.e., for each x and for every open set N containing $G(x)$ there exists a neighborhood M of x such that $G(M) \subseteq N$. Let $x_0 \in \mathbb{R}^n$ and N be an open set containing $G_\lambda(x_0)$. Existence a neighborhood M of x_0 such that $G_\lambda(M) \subseteq N$ is shown as follows: If x_0 satisfies $(-1)^{i+1}(c^T x_0 + f) < 0$ then $G_\lambda(x_0) = \{G_\lambda^i(x_0)\}$. Existence of M follows from the continuity of the function

$$\{x \in \mathbb{R}^n \mid (-1)^{i+1}(c^T x + f) < 0\} \to \mathbb{R}^n,\ x \mapsto G_\lambda^i(x).$$

If x_0 satisfies $c^T x_0 + f = 0$ then there exists a neighborhood N_i of $G_\lambda^i(x_0)$ such that $\text{conv}(\{N_1, N_2\}) \subseteq N$. Due to the continuity of the function $G_\lambda^i(x)$, there exists a neighborhood M_i of x_0 such that $G_\lambda^i(M_i) \subseteq N_i$. Then $M := M_1 \cap M_2$ is a neighborhood of x_0 satisfying

$$G_\lambda(M) \subseteq \text{conv}(G_\lambda^1(M_1), G_\lambda^2(M_2)) \subseteq \text{conv}(\{N_1, N_2\}) \subseteq N.$$

D. $4 \Rightarrow 1$

For any $x_1, x_2 \in \mathbb{R}^n, y_i \in G(x_i)$, there exists $\bar{y}_i \in G_\lambda(x_i)$ such that $y_i = -\bar{y}_i + \lambda x_i$. It follows that

$$\begin{aligned}\langle x_1 - x_2, y_1 - y_2\rangle = -\langle x_1 - x_2, \bar{y}_1 - \bar{y}_2\rangle\\ + \langle x_1 - x_2, \lambda(x_1 - x_2)\rangle \leqslant \lambda||x_1 - x_2||^2\end{aligned}$$

because of the maximal monotonicity of G_λ. Thus G is one-sided Lipschitz. ■

V. Proof of Theorem III.1

This section is devoted to the proof of Theorem III.1. First, we introduce some simplifying notations.

Consider the affine dynamical system $\Sigma = \Sigma(A, e, c^T, f)$

$$\dot{x}(t) = Ax(t) + e \tag{9a}$$

$$y(t) = c^T x(t) + f \tag{9b}$$

where $x \in \mathbb{R}^n$ is the state and $y \in \mathbb{R}$ is the output.

Let $x(t, \xi)$ and $y(t, \xi)$ denote, respectively, the state and the output of the system for the initial state ξ. Define the sets

$$\begin{aligned}\mathcal{W}_\Sigma^- &= \{\xi \mid \exists\varepsilon > 0 \text{ such that } \quad y(t, \xi) < 0\ \forall t \in (0, \varepsilon)\},\\ \mathcal{W}_\Sigma^0 &= \{\xi \mid \exists\varepsilon > 0 \text{ such that } \quad y(t, \xi) = 0\ \forall t \in (0, \varepsilon)\},\\ \mathcal{W}_\Sigma^+ &= \{\xi \mid \exists\varepsilon > 0 \text{ such that } \quad y(t, \xi) > 0\ \forall t \in (0, \varepsilon)\}.\end{aligned}$$

In order to characterize these sets, we need to introduce some notations.

For $k \in \mathbb{N}$,

$$T_\Sigma^k = \begin{bmatrix} c^T \\ c^T A \\ \vdots \\ c^T A^k \end{bmatrix}, \quad \mathbf{e}_\Sigma^k = \begin{bmatrix} f \\ c^T e \\ \vdots \\ c^T A^{k-1} e \end{bmatrix}.$$

Note that for each initial state ξ the output $y(t, \xi)$ is an analytic function. As such, it is completely determined by the values of its higher order derivatives at $t = 0$. This observation together with Cayley-Hamilton theorem leads to the following characterization of the $\mathcal{W}$-sets.

Proposition V.1 *The following statements hold.*

1) $\mathcal{W}_\Sigma^0 = \{\xi \mid T_\Sigma^n \xi + \mathbf{e}_\Sigma^n = 0\}$.
2) $\mathcal{W}_\Sigma^- = \{\xi \mid T_\Sigma^n \xi + \mathbf{e}_\Sigma^n \prec 0\}$.
3) $\mathcal{W}_\Sigma^+ = \{\xi \mid T_\Sigma^n \xi + \mathbf{e}_\Sigma^n \succ 0\}$.
4) $\mathcal{W}_\Sigma^- \cup \mathcal{W}_\Sigma^0 \cup \mathcal{W}_\Sigma^+ = \mathbb{R}^n$.

Now, we turn our attention to the bimodal system (1) and define $\Sigma_i = \Sigma_i(A_i, e_i, c^T, f)$. We also define

$$\begin{aligned} T_i^k &:= T_{\Sigma_i}^k & \mathbf{e}_i^k &:= \mathbf{e}_{\Sigma_i}^k \\ \mathcal{W}_1^0 &:= \mathcal{W}_{\Sigma_1}^0 & \mathcal{W}_2^0 &:= \mathcal{W}_{\Sigma_2}^0 \\ \mathcal{W}_1^- &:= \mathcal{W}_{\Sigma_1}^- & \mathcal{W}_2^+ &:= \mathcal{W}_{\Sigma_2}^+ \\ \mathcal{W}_1 &:= \mathcal{W}_{\Sigma_1}^- \cup \mathcal{W}_{\Sigma_1}^0 & \mathcal{W}_2 &:= \mathcal{W}_{\Sigma_2}^+ \cup \mathcal{W}_{\Sigma_2}^0. \end{aligned}$$

A. $1 \Rightarrow 3$ *or* 4

To prove this statement, we first present some consequences of right-uniqueness of solutions in terms of the above defined $\mathcal{W}$-sets.

Theorem V.2 *If all Filippov solutions of the differential inclusion* (1) *is right-unique then*
1) $\mathcal{W}_1^- \cap \mathcal{W}_2 = \varnothing$.
2) $\mathcal{W}_1 \cap \mathcal{W}_2^+ = \varnothing$.
3) *for any $\xi \in \mathcal{W}_1^0 \cap \mathcal{W}_2^0$, if x_i is a solution of the system $\dot{x}_i = A_i x_i + e_i, x_i(0) = \xi$ then $x_1(t) = x_2(t)$ for all $t \geqslant 0$.*

Proof. For the first statement, arguing by contradiction, assume that $\mathcal{W}_1^- \cap \mathcal{W}_2 \neq \varnothing$. Take $\xi \in \mathcal{W}_1^- \cap \mathcal{W}_2$ and let x_i be the solution of the system Σ_i with $x_i(0) = \xi$. Due to $\xi \in \mathcal{W}_1^-$ there exists $\varepsilon_1 > 0$ such that $y_1(t, \xi) = c^T x_1(t) + f < 0$ for almost all $t \in (0, \varepsilon_1)$. This shows that x_1 is a Filippov solution of (1) on $(0, \varepsilon_1)$ with the initial state ξ. Since $\xi \in \mathcal{W}_2$, there exists $\varepsilon_2 > 0$ such that $y_2(t, \xi) = c^T x_2(t) + f \geqslant 0$ for almost all $t \in (0, \varepsilon_2)$. It shows that $x_2(t)$ is a Filippov solution of (1) on $(0, \varepsilon_2)$ with the initial state ξ. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Because of the right-uniqueness of Filippov solution of system (1), one has $x_1(t) = x_2(t)$ for all $t \in (0, \varepsilon)$. It implies $0 \leqslant y_2(t, \xi) = y_1(t, \xi) < 0$ for almost all $t \in (0, \varepsilon)$, which is a contradiction.

The second statement can be proven similarly. For the last statement, note that the solution x_i of equation $\dot{x}(t) = A_i x(t) + e_i, x(0) = \xi$ satisfies $y = c^T x_i(t) + f = 0$ for all $t \geqslant 0$. Hence, both x_1 and x_2 are Filippov solutions for the initial state ξ of (1). Right-uniqueness implies $x_1(t) = x_2(t)$ for all $t \geqslant 0$. ■

As suggested by Proposition V.1, lexicographic inequalities play an important role in characterizing the $\mathcal{W}$-sets. In order to complete the proof, we first present five lemmas that deal with sets of lexicographic inequalities.

Lemma V.3 *Let $P \in \mathbb{R}^{m \times n}$ be a full row rank matrix and $\alpha, \beta \in \mathbb{R}^n$. If the implication $Px \prec \alpha \Rightarrow Px \preceq \beta$ holds, then $\alpha \preceq \beta$.*

Proof. Arguing by contradiction, assume that $\beta \prec \alpha$. Then there exists γ such that $\beta \prec \gamma \prec \alpha$. Since P is full row rank, there exists $\bar{x}$ such that $P\bar{x} = \gamma$. It follows that $\beta \prec P\bar{x} \prec \alpha$. This is a contradiction. ■

Lemma V.4 *Suppose that $p_1, p_2 \in \mathbb{R}^n$ with $p_1 \neq 0, p_2 \neq 0$ and $M \in \mathbb{R}^{m \times n}$ be such that*

$$\begin{bmatrix} M \\ p_2^T \end{bmatrix}$$

is of full row rank. Then the following statements are equivalent
1) $x \in \ker M, p_1^T x \geqslant 0 \implies p_2^T x \geqslant 0$.
2) $p_2^T = r^T M + \alpha p_1^T$ *for some* $\alpha > 0$ *and* $r \in \mathbb{R}^n$.

Proof. The first statement is equivalent to $Mx = 0, p_1^T x \geqslant 0$ and $p_2^T x < 0$ has no solution. By Motzkin's alternative theorem, the last one is equivalent to existence of $\beta > 0$ and $\gamma \geqslant 0$ such that

$$\beta p_2^T = \gamma p_1^T + r^T M. \tag{10}$$

We claim that $\gamma \neq 0$. Suppose that $\gamma = 0$. Then it follows from (10) that

$$\begin{bmatrix} r^T & -\beta \end{bmatrix} \begin{bmatrix} M \\ p_2^T \end{bmatrix} = 0.$$

Since $\operatorname{col}(M, p_2^T)$ is of full row rank, it follows that $r = 0$ and $\beta = 0$. This is a contradiction. So we have $p_2^T = \alpha p_1^T + r^T M$ where $\alpha = \dfrac{\gamma}{\beta} > 0$. ■

Lemma V.5 *Suppose that $p_1, p_2 \in \mathbb{R}^n$ with $p_1 \neq 0, p_2 \neq 0$. Then the following statements are equivalent*

1) $p_1^T x < r_1 \Longrightarrow p_2^T x \leqslant r_2$
2) $p_1^T = \alpha p_2^T$ *and* $r_1 \leqslant \alpha r_2$ *for some* $\alpha > 0$
3) $p_1^T x \leqslant r_1 \Longrightarrow p_2^T x \leqslant r_2$

Proof.
1) ⇒ 2): Firstly, we will prove that $p_1^T x \geqslant 0$ implies $p_2^T x \geqslant 0$. Indeed, take $\xi \in \mathbb{R}^n$ such that $p_1^T \xi < r_1$. Then we have $p_1^T(\xi + tx) < r_1$ for all $t \leqslant 0$. It follows from the hypothesis that $p_2^T \xi + t p_2^T x = p_2^T(\xi + tx) \leqslant r_2$ for all $t \leqslant 0$. This yields $p_2^T x \geqslant 0$. Now, by Lemma V.4 there exists $\alpha > 0$ such that $p_1^T = \alpha p_2^T$. Then, $p_2^T x < \dfrac{r_1}{\alpha}$ implies $p_2^T x \leqslant r_2$. By Lemma V.3, we have $\dfrac{r_1}{\alpha} \leqslant r_2$, i.e. $r_1 \leqslant \alpha r_2$.

The implications 2) $\Rightarrow$ 3) and 3) $\Rightarrow$ 1) are obvious. ∎

Let $P \in \mathbb{R}^{m\times n}$ be a matrix. For $1 \leqslant k \leqslant m$, denote the submatrix that is consisted of the first k rows of P by $P^{[k]}$. The set of all $k \times k$ lower-triangular real matrices with positive diagonal elements will be denoted by $\mathcal{L}^k_+$.

Lemma V.6 *Given $P_i \in \mathbb{R}^{m\times n}, q_i \in \mathbb{R}^m$ with $m \leqslant n$ and $\operatorname{rank}(P_i) = m$. Then the following two statements are equivalent*
1) *$P_1x \prec q_1$ implies $P_2x \prec q_2$*
2) *either*
i) there exist $l \leqslant m$ and $M \in \mathcal{L}^l_+$ such that $P_1^{[l]} = MP_2^{[l]}$ and $q_1^{[l]} \prec Mq_2^{[l]}$,
or
ii) there exists $M \in \mathcal{L}^m_+$ such that $P_1 = MP_2$ and $q_1 = Mq_2$.

Proof.
1) $\Rightarrow$ 2): The proof is based on induction on the number of rows of the matrix P. The case $k = 1$ follows from Lemma V.5. Suppose that it holds for all $k \leqslant r < m$. We want to prove that claim holds for $k = r+1$. The matrices P_i, q_i can be written as

$$P_i = \begin{bmatrix} P_i^{[r]} \\ p_i^T \end{bmatrix}, \quad q_i = \begin{bmatrix} q_i^{[r]} \\ r_i \end{bmatrix}.$$

Then, we have the implication $P_1^{[r]}x \prec q_1^{[r]} \implies P_2^{[r]}x \preceq q_2^{[r]}$. By the induction hypothesis, either there exist $l \leqslant r$ and a matrix $M_1 \in \mathcal{L}^l_+$ such that

$$P_1^{[l]} = M_1P_2^{[l]} \text{ and } q_1^{[l]} \prec M_1q_2^{[l]}, \tag{11}$$

or there exists $M \in \mathcal{L}^m_+$ such that

$$P_1^{[r]} = M_2P_2^{[r]} \text{ and } q_1^{[r]} = M_2q_2^{[r]}. \tag{12}$$

If (11) holds then the claim immediately follows. If (12) holds then P_1, q_1 can be written as

$$P_1 = \begin{bmatrix} M_2P_2^{[r]} \\ p_1^T \end{bmatrix}, \quad q_1 = \begin{bmatrix} M_2q_2^{[r]} \\ r_1 \end{bmatrix}.$$

We will prove that $p_1^T = s^TP_2^{[r]} + \alpha p_2^T$ for some $\alpha > 0$. Take x_0 such that $M_2P_2^{[r]}x_0 = M_2q_2^{[r]}$ and $p_1^Tx_0 < r_1$. Then $P_1x_0 \prec q_1$ and $P_2^{[r]}x_0 = q_2^{[r]}$. Thus we have $p_2^Tx_0 \leqslant r_2$. Now for any $\xi \in \ker P_2^{[r]} = \ker P_1^{[r]}$, if $p_1^T\xi \geqslant 0$ then $p_1^T(x_0+\lambda\xi) < r_1$ for all $\lambda \leqslant 0$. It follows that $p_2^T(x_0+\lambda\xi) \leqslant r_2$ for all $\lambda \leqslant 0$. This implies $p_2^T\xi \geqslant 0$. By Lemma V.5, there exist $\alpha > 0$ and $s \in \mathbb{R}^r$ such that $p_1^T = s^TP_2^{[r]} + \alpha p_2^T$. Thus we have

$$P_1 = \begin{bmatrix} M_2 & 0 \\ s^T & \alpha \end{bmatrix} P_2 = MP_2$$

where $M := \begin{bmatrix} M_2 & 0 \\ s^T & \alpha \end{bmatrix} \in \mathcal{L}^r_+$. Note that $MP_2x \prec q_1 \iff P_2x \prec M^{-1}q_1$. Thus from 1) we get the implication $P_2x \prec M^{-1}q_1 \implies P_2x \preceq q_2$. By Lemma V.3 we have $M^{-1}q_1 \preceq q_2$. It follows that $q_1 \preceq Mq_2$.

2) $\Rightarrow$ 1): If i) occurs then $P_1x \prec q_1 \Rightarrow P_1^{[l]}x \preceq q_1^{[l]} \Rightarrow MP_2^{[l]}x \prec Mq_2^{[l]} \Rightarrow P_2^{[l]}x \prec q_2^{[l]} \Rightarrow P_2x \prec q_2$. If ii) holds then $P_1x \prec q_1 \Rightarrow MP_2x \prec Mq_2 \Rightarrow P_2x \prec q_2$. ∎

Lemma V.7 *Given $P_i \in \mathbb{R}^{m\times n}, q_i \in \mathbb{R}^m$ with $m \leqslant n$ and $\operatorname{rank}(P_i) = m$. Then the following two statements are equivalent:*
1) *$P_1x \prec q_1$ implies $P_2x \prec q_2$*
2) *$P_1x \prec q_1$ implies $P_2x \preceq q_2$*

Proof. Clearly, the former implies the latter. To see the reverse implication, let $\bar{x}$ be such that $P_1\bar{x} + q_1 \prec 0$. Then, $P_2\bar{x} + q_2 \preceq 0$. Suppose that $P_2\bar{x} + q_2 = 0$. Then, there exists an integer $0 \leqslant k < m$ such that $P_1^{[k]}\bar{x} + q_1^{[k]} = 0$ and $P_1^{[k+1]}\bar{x} + q_1^{[k+1]} \prec 0$. Let $x' \in \ker P_1^{[k]}$. Then, $P_1^{[k]}(\bar{x} + \alpha x') + q_1^{[k]} = 0$ and $P_1^{[k+1]}(\bar{x} + \alpha x') + q_1^{[k+1]} \prec 0$ for all $\alpha \in [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Hence, it must hold that $P_2(\bar{x} + \alpha x') + q_2 \preceq 0$ for all $\alpha \in [-\varepsilon, \varepsilon]$. This means that $x' \in \ker P_2$. Therefore, $\ker P_1^{[k]} \subseteq \ker P_2$. This, however, is a contradiction as both P_1 and P_2 are of full row rank and $k < m$. Consequently, $P_2\bar{x} + q_2 \prec 0$. ∎

With all these preparations, we are now ready to complete the proof. It follows from Theorem V.2 that $\mathcal{W}_1^- \cap \mathcal{W}_2 = \varnothing$. In view of Proposition V.1, the following implication holds:

$$T_1^nx + \mathbf{e}_1^n \prec 0 \quad \Rightarrow \quad T_2^nx + \mathbf{e}_2^n \prec 0.$$

This implies that

$$T_1^{n-1}x + \mathbf{e}_1^n \prec 0 \quad \Rightarrow \quad T_2^{n-1}x + \mathbf{e}_2^{n-1} \preceq 0.$$

In view of Lemma V.7, we get

$$T_1^{n-1}x + \mathbf{e}_1^n \prec 0 \quad \Rightarrow \quad T_2^{n-1}x + \mathbf{e}_2^{n-1} \prec 0.$$

By applying Lemma V.6, we can conclude that either the statement 3 or 4 of Theorem III.1. ∎

B. $5 \Rightarrow 2$

Note that $\mathcal{W}_1^0 = \mathcal{W}_2^0 =: \mathcal{W}^0$ due to the condition 5 and Proposition V.1. Since (c^T, A_i) are observable pairs, $\mathcal{W}^0$ is a singleton, say $\mathcal{W}^0 = \{\xi\}$. First, we claim that the implication

$$\dot{x}_i = A_ix_i + e_i, x_i(0) = \xi \Rightarrow x_1(t) = x_2(t) \; \forall t \in \mathbb{R} \tag{13}$$

holds. Since $\xi \in \mathcal{W}^0$, $A_i\xi + e_i \in \langle \ker c^T | A_i \rangle$. It follows from observability of (c^T, A_i) that $A_i\xi + e_i = 0$ and hence $x_i^{(k)}(0) = 0$ for all $k \geqslant 1$. Since x_i is an analytic function, we get $x_i(t) = \xi$ for all t.

This shows that the Filippov solution with the initial state $\xi \in \mathcal{W}_0$ is both backward and forward Carathéodory. Next, we will show that the same holds for $\xi \notin \mathcal{W}^0$. To do so, we need to introduce some nomenclature and a number of auxiliary results. For $\lambda \in [0, 1]$, define

$$A(\lambda) := \lambda A_1 + (1 - \lambda)A_2$$
$$e(\lambda) := \lambda e_1 + (1 - \lambda)e_2.$$

Also define

$$\mathbb{G}_0 = \mathbb{H}_0 := \{I\}$$

and for $k \geqslant 1$

$$\mathbb{G}_k := \Big\{A_i G' \mid G' \in \mathbb{G}_{k-1}\Big\}$$
$$\mathbb{H}_k := \Big\{A(\lambda)H' \mid \lambda \in [0,1] \text{ and } H' \in \mathbb{H}_{k-1}\Big\}.$$

Note that

$$\mathbb{G}_k \subseteq \mathbb{H}_k$$

for all $k \geqslant 0$.

Define $e_I := f$. For $k \geqslant 1$ and every $H \in \mathbb{H}_k$ with

$$H = A(\lambda_k)A(\lambda_{k-1}) \ \cdots A(\lambda_1)$$

define

$$e_H := c^T A(\lambda_k)A(\lambda_{k-1})...A(\lambda_2)e(\lambda_1)$$

Also define

$$\mathcal{G}_k := \{(G, e_G) \mid G \in \mathbb{G}_k\}$$
$$\mathcal{H}_k := \{(H, e_H) \mid H \in \mathbb{H}_k\}.$$

Note that

$$\mathcal{G}_k \subseteq \mathcal{H}_k \tag{14}$$

for all $k \geqslant 0$.

Lemma V.8 *For every $k \geqslant 0$, $\operatorname{conv}(\mathcal{H}_k) = \operatorname{conv}(\mathcal{G}_k)$.*

Proof. We prove the statement by induction on k. For $k = 0$, it is obvious. Suppose that $\operatorname{conv}(\mathcal{G}_k) = \operatorname{conv}(\mathcal{H}_k)$ holds for all $k = 0, 1, \ldots, m$. Then we need to prove that $\operatorname{conv}(\mathcal{G}_{m+1}) = \operatorname{conv}(\mathcal{H}_{m+1})$. It follows from (14) that $\operatorname{conv}(\mathcal{G}_{m+1}) \subseteq \operatorname{conv}(\mathcal{H}_{m+1})$. To claim the reverse inclusion, take $H \in \mathbb{H}_{m+1}$. Then, it is of the form $H = A(\lambda)H' = \lambda A_1 H' + (1-\lambda)A_2 H'$ for some $H' \in \mathbb{H}_m$. If we write $e_{H'} = c^T R_{H'}$ then

$$e_H = c^T A(\lambda)R_{H'} = \lambda c^T A_1 R_{H'} + (1-\lambda)c^T A_2 R_{H'}$$
$$= \lambda e_{A_1 H'} + (1-\lambda)e_{A_2 H'}.$$

So $(H, e_H) \in \operatorname{conv}(A_1 H', e_{A_1 H'}), (A_2 H', e_{A_2 H'}))$ for some $H' \in \mathbb{H}_m$. By the assumption of the induction, one has $(A_i H', e_{A_i H'}) \in \operatorname{conv}(A_i \mathbb{G}_m, e_{A_i \mathbb{G}_m}) \subseteq \operatorname{conv}(\mathcal{G}_{m+1})$ where $(A_i \mathbb{G}_m, e_{A_i \mathbb{G}_m}) := \{(A_i G, e_{A_i G}) \mid G \in \mathbb{G}_m\}$. Thus (H, e_H) is in $\operatorname{conv}((A_1 H', e_{A_1 H'}), (A_2 H', e_{A_2 H'})) \subseteq \operatorname{conv}(\mathcal{G}_{m+1})$ for any $H \in \mathbb{H}_{m+1}$. It implies that $\mathcal{H}_{m+1} \subseteq \operatorname{conv}(\mathcal{G}_{m+1})$ and further more $\operatorname{conv}(\mathcal{H}_{m+1}) \subseteq \operatorname{conv}(\mathcal{G}_{m+1})$. ■

Note that if x is a Filippov solution of the system (1) then for almost all t there exists $\lambda(t) \in [0,1]$ such that

$$A(\lambda(t)) \in \mathbb{H}_1 \text{ and } \dot{x}(t) = A(\lambda(t))x(t) + e(\lambda(t)). \tag{15}$$

Lemma V.9 *Let x be a Filippov solution of* (1) *for some initial state x^0. Let $t^* \geqslant 0$ and suppose that there exist non-negative integers m, p and a positive number ε such that*
1) $c^T Hx(t^*) + e_H = 0$ for all $H \in \mathbb{H}_k$ and $0 \leqslant k \leqslant m$,
2) $(-1)^p[c^T Hx(t) + e_H] > 0$ for all $H \in \mathbb{H}_{m+1}$ and $t \in (t^*, t^* + \varepsilon)$.
Then $(-1)^p[c^T x(t) + f] > 0$ for all $t \in (t^*, t^* + \varepsilon)$.

Proof. Let $H \in \mathbb{H}_m$. For almost all $t \geqslant 0$ there exists $A(\lambda(t)) \in \mathbb{H}_1$ such that

$$\frac{d}{dt}[c^T Hx(t) + e_H] = c^T H[A(\lambda(t))x(t) + e(\lambda(t))]$$
$$= c^T HA(\lambda(t))x(t) + c^T He(\lambda(t)) = c^T H'(t)x(t) + e_{H'(t)}$$

where $H'(t) \in \mathbb{H}_{m+1}$. Then it follows from 2) that

$$\frac{d}{dt}(-1)^p\{c^T Hx(t) + e_H\} > 0$$

for all $t \in (t^*, t^* + \varepsilon)$. This shows that

$$(-1)^p\{c^T Hx(t) + e_H\} > (-1)^p\{c^T Hx(t^*) + e_H\} = 0$$

for all $H \in \mathbb{H}_m$ and $t \in (t^*, t^* + \varepsilon)$. By repeating similar arguments, after m steps, we obtain

$$(-1)^p[c^T x(t) + f] > 0$$

for all $t \in (t^*, t^* + \varepsilon)$. ■

Lemma V.10 *Suppose that the condition 5 of Theorem III.1 holds. Let x be a Filippov solution of the differential inclusion* (1) *for some initial x_0 and let $m, q \in \mathbb{N}$. If $T_i^q x(t^*) + \mathbf{e}_i^q = 0$ and $(-1)^m\{c^T A_i^{q+1} x(t^*) + c^T A_i^q e_i\} > 0$ then*
1) $c^T Hx(t^*) + e_H = 0$ for all $H \in \mathbb{H}_k$ and $0 \leqslant k \leqslant q$.
2) $(-1)^m\{c^T Hx(t^*) + e_H\} > 0$ for all $H \in \mathbb{H}_{q+1}$.

Proof. First we prove that

$$c^T Gx(t^*) + e_G = 0 \tag{16}$$

for all $G \in \mathbb{G}_k, 0 \leqslant k \leqslant q$ by induction on k and $c^T Gx(t^*) + e_G > 0$ for all $G \in \mathbb{G}_{q+1}$. It is easy to see that (16) holds with $k = 0$. Suppose that (16) holds for all $0 \leqslant k \leqslant p < q$, i.e.,

$$c^T Gx(t^*) + e_G = 0, \forall\ G \in \mathbb{G}_k, 0 \leqslant k \leqslant p.$$

We prove that (16) holds for $k = p+1$. Indeed, by the assumption we see that $c^T G_i x(t^*) + e_{G_i} = 0$ where $G_i = A_i^{p+1}$. Taking any $G \in \mathbb{G}_{p+1}$, it is of the form $G = A_i G'$ for some $G' \in \mathbb{G}_p$. We write $e_{G'}$ in the form $e_{G'} = c^T R_{G'}$. Then we get $e_G = c^T A_i R_{G'}$ and

$$c^T Gx(t^*) + e_G = c^T A_i G' x(t^*) + c^T A_i R_{G'}.$$

Since $G' \in \mathbb{G}_p$, it is of the form $G' = A_j G''$ for some $G'' \in \mathbb{G}_{p-1}$. We claim that

$$c^T A_i A_j G'' x(t^*) + c^T A_i A_j R_{G''}$$
$$= \alpha\{c^T A_j A_j G'' x(t^*) + c^T A_j^2 R_{G''}\}$$

for some $\alpha > 0$. The claim is obvious if $i = j$. In the case $i \neq j$ note that we always have relations

$$T_i^1 = MT_j^1, \mathbf{e}_i^1 = M\mathbf{e}_j^1$$

for some $M \in \mathcal{L}_+^2$. It follows that

$$\begin{aligned} &c^T A_i A_j G'' x(t^*) + c^T A_i A_j R_{G''} \\ &= m_{21} c^T A_j G'' x(t^*) + m_{22} c^T A_j A_j G'' x(t^*) + c^T A_i A_j R_{G''} \\ &= m_{22} c^T A_j A_j G'' x(t^*) + c^T A_i A_j R_{G''} - m_{21} c^T A_j R_{G''} \\ &= m_{22}\{c^T A_j A_j G'' x(t^*) + c^T A_j^2 R_{G''}\}. \end{aligned}$$

Thus we have

$$c^T G x(t^*) + e_G = \alpha\{c^T A_i^2 G'' x(t^*) + c^T A_i^2 R_{G''}\}$$

for some $G'' \in \mathbb{G}_{p-1}$ and $\alpha > 0$. By applying the same arguments, we obtain

$$c^T G x(t^*) + e_G = \alpha\{c^T G_i x(t^*) + e_{G_i}\} = 0.$$

Therefore, the formula (16) is proved. Moreover, through the proof we also see that

$$c^T G x(t^*) + e_G = \gamma_G\{c^T G_i x(t^*) + e_{G_i}\}$$

for all $G \in \mathbb{G}_{q+1}$ and for some $\gamma_G > 0$. So it follows from the second assumption that

$$(-1)^m\{c^T G x(t^*) + e_G\} = (-1)^m\{c^T G_1 x(t^*) + e_{G_1}\} > 0$$

for all $G \in \mathbb{G}_{q+1}$. Now let $H \in \mathbb{H}_k$. Since $\mathcal{H}_k \subseteq \operatorname{conv}(\mathcal{H}_k) = \operatorname{conv}(\mathcal{G}_k)$, there exists a positive integer number $s = s(k, H)$ depending on H such that

$$(H, e_H) = \sum_{j=1}^{s} \lambda_j (G^j, e_{G^j}).$$

where $\lambda_j \in [0,1]$, $\lambda_1 + \cdots + \lambda_s = 1$ and $G^j \in \mathbb{G}_k, j = 1, 2, \ldots, s$.

It follows from (16) that

$$c^T H x(t^*) + e_H = \sum_{j=1}^{s} \lambda_j \{c^T G^j x(t^*) + e_{G^j}\} = 0$$

for all $H \in \mathbb{H}_k, 0 \leqslant k \leqslant q$ and

$$\begin{aligned} &(-1)^m\{c^T H x(t^*) + e_H\} \\ &\quad = \sum_{j=1}^{s(q+1,H)} \lambda_j (-1)^m \{c^T G^j x(t^*) + e_{G^j}\} > 0 \end{aligned}$$

for all $H \in \mathbb{H}_{q+1}$. ■

With all these preparations, we are ready to complete the proof. Let x be a solution of the system (1) with the initial state x_0 in the sense of Filippov. Let $t^* \in \mathbb{R}$. If $x(t^*) \in \mathcal{W}^0$ then the claim follows as shown before. Consider the case that $x(t^*) \notin \mathcal{W}^0$. First, we want to show that x is a forward Carathéodory solution, i.e. there exists $\varepsilon_{t^*} > 0$ such that at least one of relations (3) holds for all $t \in (t^*, t^* + \varepsilon_{t^*})$. Note that the continuity of x readily implies the claim if $c^T x(t^*) + f \neq 0$. Suppose that $c^T x(t^*) + f = 0$. Since $x(t^*) \notin \mathcal{W}^0$, it follows from condition 5 of Theorem III.1 that there exists a nonnegative integer q such that $T_i^q x(t^*) + \mathbf{e}_i^q = 0$ and $(-1)^p\{c^T A^{(q+1)} x(t^*) + c^T A_i^q e_i\} > 0$. By Lemma V.10, we obtain $c^T H x(t^*) + e_H = 0$ for all $H \in \mathbb{H}_k,\ 0 \leqslant k \leqslant q$ and $(-1)^p\{c^T H x(t^*) + e_H\} > 0$ for all $H \in \mathbb{H}_{q+1}$. Since x is continuous, for each $H \in \mathbb{G}_{q+1}$ there exists a positive number ε_H such that $(-1)^p\{c^T H x(t^*) + e_H\} > 0$ for all $t \in (t^*, t^* + \varepsilon_H)$. Because $\mathbb{G}_{q+1}$ is a finite set, we can define $\varepsilon_{t^*} := \min_{H \in \mathbb{G}_{q+1}} \{\varepsilon_H\}$. Since the set $\mathcal{H}_{q+1}$ is contained in the convex hull of the set $\mathcal{G}_{q+1}$, we can conclude that $(-1)^p\{c^T H x(t^*) + e_H\} > 0$ for all $t \in (t^*, t^* + \varepsilon_{t^*})$ and for all $H \in \mathbb{H}_{q+1}$. By Lemma V.9 we get

$$(-1)^p\{c^T x(t) + f\} > 0$$

for all $t \in (t^*, t^* + \varepsilon_{t^*})$. Thus we x is a forward Carathéodory solution as claimed. Since the condition 5 of Theorem III.1 is invariant under time reversal, we can conclude that x is also backward Carathéodory solution.

C. $5 \Rightarrow 1$

We have just shown that any Filippov solution is forward Carathéodory. To prove this statement, we show that all forward Carathéodory solutions are right-unique. Let x_1 and x_2 be two forward Carathéodory solutions for the initial state x_0. If these solutions are not identical, then there exist $t^* \geqslant 0$ and $\varepsilon_{t^*} > 0$ such that $x_1(t) = x_2(t)$ for all $t \in [0, t^*]$ and

$$\begin{aligned} \dot{x}_1(t) &= A_i x_1(t) + e_i,\ (-1)^i[c^T x_1(t) + f] \geqslant 0 \\ \dot{x}_2(t) &= A_j x_2(t) + e_j,\ (-1)^j[c^T x_2(t) + f] \geqslant 0 \end{aligned}$$

for all $t \in [t^*, t^* + \varepsilon_{t^*})$ with $i \neq j$. Without loss generality, we can assume that $i = 1$ and $j = 2$. This implies that $\mathcal{W}_1 \ni x_1(t^*) = x_2(t^*) \in \mathcal{W}_2$. Since 5 implies that

$$\mathcal{W}_1^- \cap \mathcal{W}_2 = \varnothing \quad \text{and} \quad \mathcal{W}_1 \cap \mathcal{W}_2^+ = \varnothing, \tag{17}$$

we get $x_1(t^*) = x_2(t^*) \in \mathcal{W}^0 = \mathcal{W}_1^0 = \mathcal{W}_2^0$. Then, it follows from (13) that $x_1(t) = x_2(t)$ on $[t^*, \infty)$. This shows $x_1(t) = x_2(t)$ for all $t \geqslant 0$. By reversing the time and using the backward Carathéodory property, one can conclude that $x_1(t) = x_2(t)$ for all $t \in \mathbb{R}$. Therefore, all Filippov solutions are right-unique.

D. $6 \Rightarrow 1$

This immediately follows from Theorems 2.10.1 and 2.10.2 of [4].

VI. Conclusions

In this paper, we provided a set of necessary and a set of sufficient conditions for uniqueness of solutions to a piecewise affine bimodal dynamical system with possibly discontinuous vector field. These conditions are less restrictive than those of general differential inclusions such as one-sided Lipschitzian or monotonicity-type conditions.

Further research concerns extensions to multimodal case on the one hand and to the case where external inputs present on the other hand.

REFERENCES

[1] B. A. Bacciotti, On several notions of generalized solutions for discontinuous differential equations and their relationships, *Preprint of ICTP, Trieste*, 2003.

[2] M. K. Camlibel, Well-posed bimodal piecewise linear systems do not exhibit Zeno behavior, *Conference paper of 17th IFAC World Congress on Automatic Control*, 2008.

[3] F. H. Clarke and Y. S. Ledyaev and R. J. Stern and P. R. Wolenski, "Nonsmooth Analysis and Control Theory", *Graduate Texts in Mathematics*, 1998.

[4] A. F. Filippov, "Differential equations with discontinuous righthand sides", *Mathematics and Its applications, Prentice-Hall, Dordrecht*, 1988.

[5] A. Lohne, "A characterization of maximal monotone operators", *Set-Valued Analysis*, Vol. 16, 693-700, 2008.

[6] D. T. Luc and S. Schaible, "Generalized Monotone Nonsmooth Maps", *Journal of Convex Analysis*, Vol. 3, No. 2,195-205, 1996.

[7] A. Y. Pogromsky and W. P. M. H. Heemels and H. Nijmeijer, "On solution concepts and well-posedness of linear relay systems", *Automica*, Vol. 39, 2139-2147, 2003.