

Convergence of Bayesian Posterior Distributions

K. Gillies*, R. Martin**, S. Sun***, and C. Martin**

Abstract

This paper will compare the model of determining the true price of a product to determining the proportion of black balls in a bottomless, rotating urn. In doing this it is seen that as long as the ratio of black balls converge to the true proportion, the Bayesian updating method for the r^{th} moment will converge to the true proportion raised to the r^{th} power. Thus the posterior distributions converge to unit mass at the true proportion. A theory for speeding up the convergence rate for the Bayesian updating method was then tested since it now known that the Bayesian updating method converges.

1. Introduction

The true pricing of a product is something economists and businessmen often struggle with. For instance, if a car company decides to manufacture a new car, they need to determine what the most profitable price would be. If it is priced below or above the optimal price, then the company may lose profits, or even be at a loss.

The true price of any product is determined by people or companies making bids on how much they think a product is worth. The price of a product may start out at what the manufacturer believes is the most profitable price, and will change as bids are being made. What is really being decided is the value of information, [4, 2] and the references contained therein. If an estimate has been made of a true value should the estimate be changed if there is one new bit of information available? If so how much should it be changed and if not how much information should be obtained before the estimate is updated. It was decided to model this process with a simple urn experiment. It is assume that there is an urn with a large number of black and white

balls but the ratio is unknown. A guess is made as to the distribution of the balls and we began by assuming a distribution of the form

$$Ke^{-(w-\mu)^2/\sigma^2}$$

where K is a normalizing constant. The parameter μ is our guess as to the true ratio and σ reflects the confidence we have in the guess. Suppose we have made a guess that $\mu = .75$ and the first ball drawn from the urn is black. Should we change the estimate? What if the first 10 balls drawn are black? Classical Bayesian updating provides an algorithm for the updating and we examined that process. Unfortunately for this process the mean and variance are very hard to evaluate. The integrals are very close to zero and this leads to truly bad numerical properties. Fortunately the mode is easy to evaluate and we needed to know if the mode was converging to a good estimate of the mean. In [1] it was shown that the limiting distribution was point mass at the true value and thus the mode is indeed a good estimate. We are aware that there are sophisticated proof of convergence. The proof supplied here is very elementary and is accessible to any undergraduate with a background in calculus. We feel that it is worth presenting.

In this paper a model for the above pricing scenario will be developed. In the economics literature, other models have been developed, but the development here is thought to be unique. One of the main results is a proof of a convergence of certain estimates to the true value. We feel that the elementary nature of the proof is unique.

With a couple of assumptions, the model for determining true price is similar to the model for determining the true proportion of black and white balls in an urn. The conditions needed are to assume the urn is bottomless and constantly mixing. This will make the drawing of black and white balls unlimited and should there be replacement, random.

We assume an initial distribution given by

$$f(x) = Ke^{-\left(\frac{x-p_0}{\sigma}\right)^2} \text{ where } K = \left(\int_0^1 e^{-\left(\frac{x-p_0}{\sigma}\right)^2} dx\right)^{-1}$$

*** Federal Reserve Board of Governors, Washington, DC USA
The views expressed in this paper are solely the responsibility of the author and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or of any person associated with the Federal Reserve System.: *** University of Texas Arlington, Arlington, Texas *Texas Tech University, Lubbock, Texas (email: clyde.f.martin@ttu.edu)

Here p_0 is the initial estimate of the true proportion of black balls in the urn. The standard deviation, σ , measures the confidence level of the initial guess. For example, taking $\sigma = 2$, which is a low confidence level, gives a very wide distribution, while taking $\sigma = .01$, which is a high confidence level, gives a very narrow distribution. In the above equation the constant K is used to normalize the distribution to give us a proportion in the interval from $[0, 1]$.

2. List of Notation

There are several variables used throughout this paper. They are listed below for an easy reference.

- 1) b will denote the number of black balls drawn
- 2) w will denote the number of white balls drawn
- 3) n will denote the total number of black and white balls drawn
- 4) tp will denote the true proportion of black balls
- 5) p_0 will denote the initial guess of the proportion of black balls
- 6) σ will denote the standard deviation (also known as the confidence level)
- 7) p will denote the proportion of black balls drawn in a sample
- 8) r will denote the moment

3. Bayesian Updating

A process used to update an initial distribution without changing the initial guess is Bayesian Updating. Bayesian Updating will use the amount of white and black balls drawn to update the initial distribution, $f(x)$ [3].

3.1. Basic Example of Bayesian Updating

To get a better understanding of how Bayesian updating works, consider the following example:

John wants to know the true proportion of landing on heads when flipping a coin. He has a non-normal distribution, $h(x) = x - x^2$ and has tossed the coin 13 times and the coin landed on heads for 3 of the flips. He wants to know what the updated distribution should be.

There are three parts to the Bayesian updating formula. First is the original distribution, second is the updater, and last is the constant which will normalize the updated distribution.

Since, the original distribution is $h(x) = x - x^2$, the first part of the updated distribution formula is complete.

The updater or second part of the updated distribution has the form $x^i(1-x)^j$ where i and j represent the

number of successes and failures respectively. In John's problem a success is considered to be landing on heads, so there are 3 successes and 10 failures. This means the updater will be $x^3(1-x)^{10}$

The constant which normalizes the new formula is $\left(\int_0^1 x^3(1-x)^{10}(x-x^2)dx\right)^{-1}$ which equals 21840.

Combining the three steps, it is concluded the new distributions using Bayesian updating is

$$j(x) = 21840x^4(1-x)^{11}.$$

3.2. Moments of a Bayesian Updated Distribution

With a better understanding of Bayesian updating consider the moments of a distribution $h(x)$ which is given by the formula

$$\int_0^1 x^r h(x) dx \quad \text{where} \quad r \in \mathbb{N} \cup \{0\}.$$

Since the mean is the first moment, the mean of an initial distribution $h(x)$ will be

$$\int_0^1 xh(x)dx.$$

Using this formula the following are the steps to finding the mean updated distribution from the earlier example. Since the mean is $\int_0^1 xg(x)dx$, the following is obtained by replacing $g(x)$ with the updated distribution.

$$\int_0^1 x(21840(x^4(1-x)^{11}))dx,$$

which simplifies to

$$21840 \int_0^1 x^5(1-x)^{11}dx,$$

which is equal to $\frac{5}{17}$ or .29.

Applying Bayesian updating to the initial distribution of the urn problem, $f(x)$, gives

$$Cx^b(1-x)^w e^{-\left(\frac{x-p_0}{\sigma}\right)^2} \tag{1}$$

where

$$C = \left(\int_0^1 x^b(1-x)^w e^{-\left(\frac{x-p_0}{\sigma}\right)^2} dx \right)^{-1}. \tag{2}$$

Furthermore each moment will be calculated by

$$\frac{\int_0^1 x^{b+r}(1-x)^w e^{-\left(\frac{x-p_0}{\sigma}\right)^2} dx}{\int_0^1 x^b(1-x)^w e^{-\left(\frac{x-p_0}{\sigma}\right)^2} dx} \quad \text{where} \quad r \in \mathbb{N} \cup \{0\}.$$

To calculate the true proportion, the first moment will be calculated as the number of black and white balls goes to infinity.

4. Giving Bayesian Updating a Faster Convergence Rate

A problem with using Bayesian updating to determine the true proportion is it has a very slow convergence rate. Theoretically updating the initial guess only when the proportion of black balls in a sample is within a certain range of the initial guess should converge to the true proportion without drawing numerous balls. In doing this, the issues with updating the initial guess with the data from a bad draw will be eliminated.¹ In fixing the problem of updating the initial guess data from a bad draw, one must consider the problem of the initial guess not being close to the true proportion. An idea of fixing this situation is to update for a certain amount of balls a small number of times. This should bring the updated initial guess closer to the true proportion.

5. The Convergence of Bayesian Posterior Distributions

5.1. Initial Test

The first step to testing the ideas for a faster convergence was to create a program where the number of black balls drawn was based off of the true proportion. The program updates p_0 every 10 draws and will draw up to 1000 balls.² After drawing 1000 balls it will repeat the process over again, resetting b, w and n . Note, the second run does not reset p_0 back to the initial guess. Table ?? ran the program with $tp = .55$, $p_0 = .75$ and $\sigma = .316$. From Table ?? it is seen that not only are the two runs identical, but they do not converge to true proportion, .55. This suggests something is wrong with the program or the first moment of the Bayesian updating does not converge to the true proportion. The program was then changed to show the first ten results along with the fiftieth and the one hundredth update to further investigate the situation. Notice how on Table ?? the first result from the first run and second run are not equal but the other results are. This suggests one of two problems, either the program has rounding errors or the first moment does not converge.

¹An example of a bad draw is if the true proportion of black balls is .1 and all black balls are drawn.

²Refer to Appendix A for the program

5.2. Simplifying the Integrals

Since both of the integrals of the first moment have $e^{-\left(\frac{x-p_0}{\sigma}\right)^2}$ they are impossible to compute by hand, so to get a better understanding of the convergence of the first moment consider

$$\frac{\int_0^1 x^{2b+1}(1-x)^b dx}{\int_0^1 x^{2b}(1-x)^b dx}, \tag{3}$$

which should converge to $\frac{2}{3}$.

First consider

$$\int_0^1 x^{2b+1}(1-x)^b dx. \tag{4}$$

Let $u = (1-x)^b$ and $dv = x^{2b+1} dx$. Therefore $du = -b(1-x)^{b-1} dx$ and $v = \frac{1}{2b+2} x^{2b+2}$. Therefore (4) is equal to

$$\left[\frac{-b}{2b+2} x^{2b+2}(1-x)^{b-1} \right]_0^1 - \frac{-b}{2b+2} \int_0^1 x^{2b+2}(1-x)^{b-1} dx,$$

which simplifies to

$$\frac{b}{2b+2} \int_0^1 x^{2b+2}(1-x)^{b-1} dx.$$

Now Let $u = (1-x)^{b-1}$ and $dv = x^{2b+2} dx$. Therefore $du = -(b-1)(1-x)^{b-2} dx$ and $v = \frac{1}{2b+3} x^{2b+3}$. Therefore (4) is equal to

$$\left[\frac{b(b-1)}{(2b+2)(2b+3)} x^{2b+3}(1-x)^{b-2} \right]_0^1 - \frac{-b(b-1)}{(2b+2)(2b+3)} \int_0^1 x^{2b+3}(1-x)^{b-2} dx,$$

which simplifies to

$$\frac{b(b-1)}{(2b+2)(2b+3)} \int_0^1 x^{2b+3}(1-x)^{b-2} dx.$$

If this process is repeated $b-2$ more times it is seen that (4) is equal to

$$\frac{b!}{(3b)(3b-1)\dots(2b+2)} \int_0^1 x^{3b} dx = \tag{5}$$

$$\frac{b!}{(3b+1)(3b)\dots(2b+2)} =$$

$$\frac{b!(2b+1)!}{(3b+1)!}.$$

Applying the same method to $\int_0^1 x^{2b}(1-x)^b dx$ we get

$$\frac{b!(2b)!}{(3b)!}. \tag{6}$$

Now with combining (3) with (5) and (6) we have (3) is equal to

$$\frac{\frac{b!(2b+1)!}{(3b+1)!}}{\frac{b!(2b)!}{(3b)!}},$$

which is equal to

$$\frac{b!(2b+1)!(3b)!}{b!(2b)!(3b+1)!},$$

which is equal to

$$\frac{(2b+1)}{(3b+1)}.$$

In considering the limit it is seen that

$$\lim_{b \rightarrow \infty} \frac{(2b+1)}{(3b+1)} = \frac{2}{3}.$$

Since the proof for showing

$$\lim_{(b+w) \rightarrow \infty} \frac{\int_0^1 x^{b+1}(1-x)^w dx}{\int_0^1 x^b(1-x)^w dx} = \lim_{(b+w) \rightarrow \infty} \frac{b}{b+w}$$

is similar to the proof above, the question that remains is for what conditions on a function $h(x)$, with $b, r, w \in \mathbb{N} \cup \{0\}$ will

$$\lim_{(b+w) \rightarrow \infty} \frac{\int_0^1 x^{b+r}(1-x)^w h(x) dx}{\int_0^1 x^b(1-x)^w h(x) dx} = \lim_{(b+w) \rightarrow \infty} \left(\frac{b}{b+w}\right)^r. \tag{7}$$

After considering the case $h(x) = x^3 + x^2 + x + 1$ a more generalized proof was created by letting $h(x) = \sum_{k=0}^{\infty} a_k x^k$ be an absolutely convergent power series on $[0, 1 + \varepsilon]$, where $\varepsilon > 0$. Since $h(x)$ is an absolutely convergent series the summation could be factored out of the integrals allowing the convergence of (7) to be shown. Now to consider the following theorem.

5.3. Convergence of Bayesian Posterior Distributions

In this section Theorem 1, which shows the convergence of Bayesian posterior distributions, will be proved. **Theorem 1:** If the following are true:

1. If a power series $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely on $(0, 1 + \varepsilon)$ where $\varepsilon > 0$
2. b, w and $r \in \mathbb{N} \cup \{0\}$
3. $(b+w) \rightarrow \infty$ and $b \rightarrow \infty$ simultaneously

then:

$$\lim_{(b+w) \rightarrow \infty} \frac{\int_0^1 x^{b+r}(1-x)^w \sum_{k=0}^{\infty} a_k x^k dx}{\int_0^1 x^b(1-x)^w \sum_{k=0}^{\infty} a_k x^k dx} = \lim_{(b+w) \rightarrow \infty} \left(\frac{b}{b+w}\right)^r.$$

Before proving Theorem 1, consider the following lemmas.

Lemma 1: If a power series $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely on $[0, 1 + \varepsilon]$ where $\varepsilon > 0$ then

$$\int_0^1 x^{b+r}(1-x)^w \sum_{k=0}^{\infty} a_k x^k dx = \sum_{k=0}^{\infty} \frac{a_k w!(b+k+r)!}{(b+k+r+w+1)!}$$

where r, b and $w \in \mathbb{N} \cup \{0\}$.

Lemma 2: If the following are true (i) The power series $\sum_{k=0}^{\infty} a_k x^k$ converges on $[a, b]$ where $a, b \in \mathbb{R}$ and $a \leq 1 \leq b$

(ii) $\{b_h\}_{h=0}^{\infty}$ converges to some $y \in \mathbb{R}$ where $b_h \in \mathbb{R}$ for every $h \in \mathbb{N} \cup \{0\}$

Then $\sum_{k=0}^{\infty} a_k b_k$ converges.

Lemma 3: If $p, q \in \mathbb{N} \cup \{0\}$ then $p!q! \leq (p+q)!$.

Corollary 1: If a power series $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely on $[0, 1 + \varepsilon]$ where $\varepsilon > 0$ then for some fixed r, b and $w \in \mathbb{N} \cup \{0\}$

$$\sum_{k=0}^{\infty} \frac{a_k w!(b+k+r)!}{(b+k+r+w+1)!}$$

converges.

We are now able to prove Theorem 1.

Proof: Consider

$$\int_0^1 x^b(1-x)^w \sum_{k=0}^{\infty} a_k x^k dx.$$

By Lemma 1, with $r=0$

$$\int_0^1 x^b(1-x)^w \sum_{k=0}^{\infty} a_k x^k dx = \sum_{k=0}^{\infty} \frac{a_k w!(b+k)!}{(b+k+w+1)!}.$$

With applying Lemma 1 to

$$\frac{\int_0^1 x^{b+r}(1-x)^w \sum_{k=0}^{\infty} a_k x^k dx}{\int_0^1 x^b(1-x)^w \sum_{k=0}^{\infty} a_k x^k dx}, \quad (8)$$

it is seen that (8) equivalent to

$$\frac{\sum_{k=0}^{\infty} \frac{a_k w!(b+k+r)!}{(b+k+r+w+1)!}}{\sum_{k=0}^{\infty} \frac{a_k w!(b+k)!}{(b+k+w+1)!}},$$

which is equivalent to the following by multiplying the numerator and denominator each by 1

$$\frac{\frac{(b+r)!(b+r+w+1)!}{(b+r)!(b+r+w+1)!} \sum_{k=0}^{\infty} \frac{a_k w!(b+k+r)!}{(b+k+r+w+1)!}}{\frac{b!(b+w+1)!}{b!(b+w+1)!} \sum_{k=0}^{\infty} \frac{a_k w!(b+k)!}{(b+k+w+1)!}}.$$

Since $\frac{1}{(b+r)!}$, $(b+w+r+1)!$, $(b+w+1)!$ and $\frac{1}{b!}$ are constants they can be distributed into the sum to obtain [5]

$$\frac{\frac{(b+r)!}{(b+r+w+1)!} \sum_{k=0}^{\infty} \frac{a_k w!(b+k+r)!(b+w+r+1)!}{(b+r)!(b+k+r+w+1)!}}{\frac{b!}{(b+w+1)!} \sum_{k=0}^{\infty} \frac{a_k w!(b+k)!(b+w+1)!}{b!(b+k+w+1)!}}. \quad (9)$$

Now consider

$$\frac{\frac{(b+r)!}{(b+r+w+1)!}}{\frac{b!}{(b+w+1)!}},$$

which is equal to

$$\frac{(b+r)!(b+w+1)!}{b!(b+w+r+1)!},$$

which is equal to

$$\left(\frac{(b+r)(b+r-1)\dots(b+1)b!}{b!} \right) \times \left(\frac{(b+w+1)!}{(b+w+r+1)(b+w+r)\dots(b+w+2)(b+w+1)!} \right),$$

which is equal to

$$\frac{(b+r)(b+r-1)\dots(b+1)}{(b+w+r+1)(b+w+r)\dots(b+w+2)},$$

which can be written as

$$\frac{\prod_{t=0}^{r-1} (b+r-t)}{\prod_{t=0}^{r-1} (b+w+1+r-t)},$$

which by using the laws of multiplication can be written as

$$\prod_{t=0}^{r-1} \frac{(b+r-t)}{(b+w+1+r-t)}.$$

(9) can now be written as

$$\left(\prod_{t=0}^{r-1} \frac{(b+r-t)}{(b+w+1+r-t)} \right) \times \quad (10)$$

$$\left(\frac{\sum_{k=0}^{\infty} \frac{a_k w!(b+k+r)!(b+w+r+1)!}{(b+r)!(b+k+r+w+1)!}}{\sum_{k=0}^{\infty} \frac{a_k w!(b+k)!(b+w+1)!}{b!(b+k+w+1)!}} \right).$$

Therefore (8) is equal to (10).

Now with (10) it will be easier to find the limit of (8). First consider

$$\sum_{k=0}^{\infty} \frac{a_k (b+k+r)!(b+w+r+1)!}{(b+r)!(b+k+w+r+1)!} - \sum_{k=0}^{\infty} \frac{a_k (b+w+1)!(b+k)!}{b!(b+k+w+1)!}. \quad (11)$$

By Corollary 1 both $\sum_{k=0}^{\infty} \frac{a_k (b+k+r)!(b+w+r+1)!}{(b+r)!(b+k+w+r+1)!}$

and $\sum_{k=0}^{\infty} \frac{a_k (b+w+1)!(b+k)!}{b!(b+k+w+1)!}$ converge.

Therefore (11) can be written as

$$\sum_{k=0}^{\infty} \frac{a_k (b+k+r)!(b+w+r+1)!}{(b+r)!(b+k+w+r+1)!} - \frac{a_k (b+w+1)!(b+k)!}{b!(b+k+w+1)!}$$

[5],

which after factoring out the appropriate expressions becomes

with simplification

$$\sum_{k=0}^{\infty} \frac{a_k (b+k)!(b+w+1)!}{b!(b+w+k+1)!} \times$$

$$\left[\frac{\prod_{t=1}^r (b+k+t) \prod_{t=1}^r (b+w+1+t)}{\prod_{t=1}^r (b+t) \prod_{t=1}^r (b+k+w+1+t)} - 1 \right].$$

Therefore expression (11) is equal to

$$\sum_{k=0}^{\infty} \frac{a_k(b+k)!(b+w+1)!}{b!(b+w+k+1)!} \times \left[\prod_{t=1}^r \frac{(b+k+t)(b+w+1+t)}{(b+t)(b+k+w+1+t)} - 1 \right].$$

Since $\lim_{(b+w) \rightarrow \infty} \frac{b}{b+w} \neq 0$,

$$\lim_{(b+w) \rightarrow \infty} \frac{(b+k+t)}{(b+t)} = 1 \text{ and } \lim_{(b+w) \rightarrow \infty} \frac{(b+w+1+t)}{(b+w+k+1+t)} = 1$$

where k is fixed.

This gives

$$\lim_{(b+w) \rightarrow \infty} \frac{(b+k+t)(b+w+1+t)}{(b+t)(b+k+w+1+t)} = 1$$

[5].

Therefore, by the same type of reasoning as above

$$\lim_{(b+w) \rightarrow \infty} \prod_{t=1}^r \frac{(b+k+t)(b+w+1+t)}{(b+t)(b+k+w+1+t)} = 1 \quad (12)$$

[5].

Now consider the following

$$\lim_{(b+w) \rightarrow \infty} \sum_{k=0}^{\infty} \frac{a_k(b+k)!(b+w+1)!}{b!(b+w+k+1)!} \times \left[\prod_{t=1}^r \frac{(b+k+t)(b+w+1+t)}{(b+t)(b+k+w+1+t)} - 1 \right].$$

By (12)

$$\lim_{(b+w) \rightarrow \infty} \sum_{k=0}^{\infty} \frac{a_k(b+k)!(b+w+1)!}{b!(b+w+k+1)!} \times \left[\prod_{t=1}^r \frac{(b+k+t)(b+w+1+t)}{(b+t)(b+k+w+1+t)} - 1 \right] = 0.$$

Hence

$$\lim_{(b+w) \rightarrow \infty} \left[\sum_{k=0}^{\infty} \frac{a_k(b+k+r)!(b+w+r+1)!}{(b+r)!(b+k+w+r+1)!} - \sum_{k=0}^{\infty} \frac{a_k(b+w+1)!(b+k)!}{b!(b+k+w+1)!} \right] = 0.$$

Therefore

$$\lim_{(b+w) \rightarrow \infty} \sum_{k=0}^{\infty} \frac{a_k(b+k+r)!(b+w+r+1)!}{(b+r)!(b+k+w+r+1)!} = \lim_{(b+w) \rightarrow \infty} \sum_{k=0}^{\infty} \frac{a_k(b+w+1)!(b+k)!}{b!(b+k+w+1)!}.$$

Therefore

$$\lim_{(b+w) \rightarrow \infty} \frac{\sum_{k=0}^{\infty} \frac{a_k(b+k+r)!(b+w+r+1)!}{(b+r)!(b+k+w+r+1)!}}{\sum_{k=0}^{\infty} \frac{a_k(b+w+1)!(b+k)!}{b!(b+k+w+1)!}} = 1.$$

Since both the limits in the numerator and denominator exist the above equation can be written as

$$\lim_{(b+w) \rightarrow \infty} \frac{\sum_{k=0}^{\infty} \frac{a_k w!(b+k+r)!(b+w+r+1)!}{(b+r)!(b+k+r+w+1)!}}{\sum_{k=0}^{\infty} \frac{a_k w!(b+k)!(b+w+1)!}{b!(b+k+w+1)!}} = 1. \quad (13)$$

Now consider

$$\lim_{(b+w) \rightarrow \infty} \frac{(b+r-t)}{(b+w+1+r-t)}.$$

Since t and r are fixed

$$\lim_{(b+w) \rightarrow \infty} \frac{(b+r-t)}{(b+w+1+r-t)} = \lim_{(b+w) \rightarrow \infty} \frac{b}{(b+w)}.$$

Therefore

$$\lim_{(b+w) \rightarrow \infty} \prod_{t=0}^{r-1} \frac{(b+r-t)}{(b+w+1+r-t)} = \lim_{(b+w) \rightarrow \infty} \prod_{t=0}^{r-1} \frac{b}{(b+w)}$$

[5].

Therefore

$$\lim_{(b+w) \rightarrow \infty} \prod_{t=0}^{r-1} \frac{(b+r-t)}{(b+w+1+r-t)} = \lim_{(b+w) \rightarrow \infty} \left(\frac{b}{(b+w)} \right)^r. \quad (14)$$

Since the limits exist for

$$\prod_{t=0}^{r-1} \frac{(b+r-t)}{(b+w+1+r-t)}$$

and

$$\frac{\sum_{k=0}^{\infty} \frac{a_k w!(b+k+r)!(b+w+r+1)!}{(b+r)!(b+k+r+w+1)!}}{\sum_{k=0}^{\infty} \frac{a_k w!(b+k)!(b+w+1)!}{b!(b+k+w+1)!}}$$

we have

$$\lim_{(b+w) \rightarrow \infty} \left(\prod_{t=0}^{r-1} \frac{(b+r-t)}{(b+w+1+r-t)} \right) \times \left(\frac{\sum_{k=0}^{\infty} \frac{a_k w!(b+k+r)!(b+w+r+1)!}{(b+r)!(b+k+r+w+1)!}}{\sum_{k=0}^{\infty} \frac{a_k w!(b+k)!(b+w+1)!}{b!(b+k+w+1)!}} \right),$$

is equal to

$$\left(\lim_{(b+w) \rightarrow \infty} \prod_{t=0}^{r-1} \frac{(b+r-t)}{(b+w+1+r-t)} \right) \times \left(\lim_{(b+w) \rightarrow \infty} \frac{\sum_{k=0}^{\infty} \frac{a_k w! (b+k+r)! (b+w+r+1)!}{(b+r)! (b+k+r+w+1)!}}{\sum_{k=0}^{\infty} \frac{a_k w! (b+k)! (b+w+1)!}{b! (b+k+w+1)!}} \right) \quad (15)$$

[5].

Therefore by (13), (14) and (15)

$$\lim_{(b+w) \rightarrow \infty} \left(\prod_{t=0}^{r-1} \frac{(b+r-t)}{(b+w+1+r-t)} \right) \times \left(\frac{\sum_{k=0}^{\infty} \frac{a_k w! (b+k+r)! (b+w+r+1)!}{(b+r)! (b+k+r+w+1)!}}{\sum_{k=0}^{\infty} \frac{a_k w! (b+k)! (b+w+1)!}{b! (b+k+w+1)!}} \right) = \left(\lim_{(b+w) \rightarrow \infty} \frac{b}{(b+w)} \right)^r$$

Therefore by (10) for any absolutely convergent power series

$$\lim_{(b+w) \rightarrow \infty} \frac{\int_0^1 x^{b+r} (1-x)^w \sum_{k=0}^{\infty} a_k x^k dx}{\int_0^1 x^b (1-x)^w \sum_{k=0}^{\infty} a_k x^k dx} = \lim_{(b+w) \rightarrow \infty} \left(\frac{b}{b+w} \right)^r.$$

6. Summary

Since Theorem 1 holds for any absolutely convergent power series, then Theorem 1 holds for any finitely differentiable function. This means Theorem 1 holds for $e^{-\left(\frac{x-p_0}{\sigma}\right)^2}$.

Consider the following theorem. **Theorem 2:** (*Fréchet and Scohat*). Let the distribution functions F_m possess finite moments $\alpha_k^{(m)} = \int t^k dF_m(t)$ for $k = 1, 2, \dots$ and $m = 1, 2, \dots$. Assume that the limits $\alpha_k = \lim_m \alpha_k^{(m)}$ exist (finite), each k . Then (i) the limits $\{\alpha_k\}$ are the moments of a distribution function F ; (ii) if the F given by (i) is unique, then $F_m \Rightarrow F$ [6].

Corollary 2 The r^{th} moment of the distribution

$$C x^b (1-x)^w e^{-\left(\frac{x-p_0}{\sigma}\right)^2}$$

where

$$C = \left(\int_0^1 x^b (1-x)^w e^{-\left(\frac{x-p_0}{\sigma}\right)^2} dx \right)^{-1}$$

is equal to $(tp)^r$, where tp is the true proportion, if

$$\lim_{(b+w) \rightarrow \infty} \frac{b}{b+w} = tp.$$

7. Creating a New Program

By proving the first moment converges to the true proportion, it is concluded that the problems with the program were rounding errors. To fix this, the finite

Taylor expansion is used to represent $e^{-\left(\frac{x-p_0}{\sigma}\right)^2}$. The following is the binomial expansion of the finite Taylor series expansion for $e^{-\left(\frac{x-p_0}{\sigma}\right)^2}$

$$\sum_{k=0}^{10} \frac{1}{k! \sigma^{2k}} \sum_{l=0}^{2k} \binom{2k}{l} x^l (-p_0)^{2k-l}. \quad (16)$$

Combining (16) with (17) and applying Lemma 1, we obtain (17) is equal to

$$\frac{\sum_{k=0}^{10} \frac{1}{k! \sigma^{2k}} \sum_{i=0}^{2k} \frac{(2k)! w! (b+i+1)! (-p_0)^{2k-i}}{(2k-i)! i! (b+i+w+2)!}}{\sum_{k=0}^{10} \frac{1}{k! \sigma^{2k}} \sum_{i=0}^{2k} \frac{(2k)! w! (b+i)! (-p_0)^{2k-i}}{(2k-i)! i! (b+w+1+i)}}$$

Using this information a new program was created with the ideal situation and was tested. We found that p_0 converges up to 140 balls, then rounding errors start to occur. Since the new program worked for the ideal situation, the program was changed to draw 140 random balls, updating after every ten draws. We obtained results for two different runs with the true proportion set at .55 and the initial guess starting at .75. Even though it is clear p_0 is beginning to converge to the true proportion, not enough balls are being drawn to get complete convergence. The next idea was to run the program fifty times and instead of letting $p_0 = .75$ at the beginning of each of the draws it will remain what it was on the 140th ball from the previous run, For this strategy p_0 never converges to the true proportion but stays within about the range of (.48, .61). The question then became, is there a way to only update when p falls with in or out of a certain range of p_0 .

8. Conclusion

This paper discussed using a different model, which is thought to be unique, to determine the true price of a product. The model used was determining the true proportion of black and white balls in an urn that is

bottomless and constantly mixing. This made the drawing of black and white balls unlimited and should there have been replacement, random. We assumed the initial distribution was given by

$$f(x) = Ke^{-\left(\frac{x-p_0}{\sigma}\right)^2}$$

where

$$K = \left(\int_0^1 e^{-\left(\frac{x-p_0}{\sigma}\right)^2} dx \right)^{-1}$$

Where p_0 is the initial guess of the proportion and σ measured how confident we were in our guess. Also the constant K is used to normalize the distribution, to give us a proportion in the interval from $[0,1]$. Bayesian updating was used to update the initial distribution. The updated distribution was

$$Cx^b(1-x)^we^{-\left(\frac{x-p_0}{\sigma}\right)^2} \tag{17}$$

where

$$C = \left(\int_0^1 x^b(1-x)^we^{-\left(\frac{x-p_0}{\sigma}\right)^2} dx \right)^{-1} .$$

Furthermore each moment will be calculated by

$$\frac{\int_0^1 x^{b+r}(1-x)^we^{-\left(\frac{x-p_0}{\sigma}\right)^2} dx}{\int_0^1 x^b(1-x)^we^{-\left(\frac{x-p_0}{\sigma}\right)^2} dx} \quad \text{where } r \in \mathbb{N} \cup \{0\} .$$

Using Theorem 1, which showed the convergence of the Bayesian posterior distributions for any infinitely differentiable function on $[0,1]$, and Theorem 2, *Fréchet and Scohat* Theorem, we were able to see if $\lim_{(b+w) \rightarrow \infty} \frac{b}{b+w} = tp$ then the r^{th} moment of the distribution was equal to $(tp)^r$. Thus the posterior distributions converged to unit mass, which shows the convergence of the r^{th} moment is the value that takes on the maximum of the r^{th} moment. In showing this, if the $\lim_{(b+w) \rightarrow \infty} \frac{b}{b+w} = tp$ then the value that takes on the maximum at the first moment is the true proportion.

Also this paper show that while there might be a method to make the Bayesian updating method converge to the true proportion faster, it was not the method of updating when proportion of black balls drawn in a sample fell within a certain range of updated initial guess. The reason being is, with out loss of generality, if the true proportion is less than the updated initial guess and after a round of drawing balls, the proportion of black balls is greater than the updated initial guess

but still less than the updated initial guess added to ϵ . The program then updates the initial guess, causing it to become larger, and ϵ , causing it to become smaller. Even if on the next round of drawing the proportion of black balls is equal to the true proportion, it might not be in the range of the updated initial guess and the updated initial guess will not be updated back to towards the true proportion.

References

- [1] Kendall Gillies, Convergence of Bayesian Posterior Distributions, Master’s Thesis, Texas Tech University, 2009
- [2] Y.C. Ho and F. K. Sun, Value of information in two-team zero-sum problems. Collection of articles dedicated to Magnus R. Hestenes. J. Optimization Theory Appl. 14 (1974), 557–571.
- [3] Christopher D. Manning and Hinrich Schtze, Foundations of Statistical Natural Language Processing,
- [4] Clyde F. Martin and Y.C. Ho, Value of information in the Polya urn process. Inform. Sci. 147 (2002), no. 1-4, 65–90.
- [5] Walter Rudin, Principles of Mathematical Analysis, McGraw-Hill, Inc., USA, 1953
- [6] Robert J. Serfling, Approximation Theorems of Mathematical Statistics, Wiley-Interscience, New York, 1980