

# Arbitrage-free multifactor term structure models: a theory based on stochastic control

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**Abstract**— We present an alternative approach to the pricing of bonds and bond derivatives in a multivariate linear-quadratic model for the term structure that is based on the solution of a linear-quadratic stochastic control problem. We focus on explicit formulas for the computation of bond options in a bivariate factor model which can be easily computed numerically by calculating four line integrals.

**Key words:** Linear-quadratic term structures, bond pricing, stochastic control.

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## I. INTRODUCTION

The use of the Girsanov transformation to obtain a martingale measure has become a fundamental tool of asset and bond pricing. The key feature of this technique is a change of drift which preserves trajectories. However, as it is well known, this is not the only way to change the drift of a stochastic process. In fact, the drift can also be changed by feedback, albeit with a change of the trajectories, but keeping the same measure. It turns out that in the case of a linear-quadratic factor model for the term structure (a term structure where the bond prices are exponentially quadratic in the factors with the latter satisfying linear-Gaussian dynamics, (see (1) and (2) below), a feedback approach resulting from a stochastic control methodology provides the same pricing model, which can be obtained in the traditional manner, without changing the measure at all.

Stochastic control techniques have been adopted quite early in finance (see e.g. [8]), but not in the context of derivative pricing. Here we show that stochastic control techniques can be fruitfully applied also to derivative pricing in the context of term structure models. The feedback approach used in the paper can, according to work in progress, accommodate at least partly also stock option pricing but,

contrary to the term structure, in the stock market one ends up with an infinite-horizon control problem.

In the stochastic control approach the control turns out to be in feedback form, namely as a function of the state/factor process and this leads to a so-called closed-loop model. The trajectories of the factors in a closed-loop model are changed with respect to those of the corresponding open loop model, but for the bond pricing this is quite irrelevant. In fact, since the observed values are eventually the rates and bond prices, it is quite indifferent, as far as pricing is concerned, whether these values are generated by an original open loop model with different trajectories (which we never observe) and the same measure, or same trajectories and a different measure. What is relevant is that they produce the same prices. Moreover the fact that (see the Appendix) Riccati and related equations can be easily computed makes the approach quite appealing and leads also to a computable alternative approach to bond option pricing (see section IV-B)

The paper is structured as follows. In section II, which is based on [6], we recall conditions which the arbitrage-free assumption entails on a quadratic multivariate model for the term structure. Section III is devoted to the linear-quadratic stochastic control approach. In Subsection III-A we first recall some basic facts concerning a generalized version of the classical linear-quadratic Gaussian control problem. These are then applied in subsection III-B to obtain an alternative to bond pricing and in subsection III-C to compute forward prices of a bond. Section IV is aimed at forward measures and bond derivative pricing. To this effect, in subsection IV-A we investigate the relationship between forward measures and our stochastic control approach and apply it to the pricing of general derivatives of the factors. The specific result for bond pricing is then presented in subsection IV-B. We provide explicit formulas for the computation of a call option on a bond in a bivariate factor model for the term structure. This requires to calculate simple line integrals and thus its computational complexity is comparable to that of

the Black and Scholes model. In the Appendix we provide explicit (known) formulas for the computation of the matrix differential Riccati equation needed in the paper.

## II. ARBITRAGE FREE DERIVATION FOR THE TERM STRUCTURE

On a filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$ , consider a model of the form

$$dx(t) = Fx(t)dt + Gdw(t) \quad , \quad t \geq 0 \quad , \quad x_0 = 0$$

$$f(t, T) = a(t, T) + b(t, T)x(t) + x'(t)c(t, T)x(t) \quad (1)$$

where  $x(t)$  has dimension  $n$ ,  $w$  is a  $k$ -dimensional Wiener process w.r. to  $(Q, \mathcal{F}_t)$ , the symbol  $'$  denotes transposition,  $F$  and  $G$  are matrices of appropriate dimensions and  $a(t, T), b(t, T), c(t, T)$  are scalar and matrix-valued functions respectively, differentiable with respect to  $t$  and with  $c(t, T)$  symmetric. Expectation under  $Q$  will in the sequel simply be denoted by  $E$ . In (1), combined with (2) below, we consider an exponentially quadratic output model as an instance of a non-affine model. By imposing the conditions of absence of arbitrage as in Proposition 2.1 below, it can be shown (see [4]) that, for a linear-Gaussian factor model, general exponentially polynomial output models with degree larger than 2 reduce to the quadratic output model, i.e. the coefficients of the powers of  $x$  larger than 2 have to be equal to zero. Notice, furthermore, that the term  $a(t, T)$  in (1) implicitly includes the observed forward rate curve  $f^*(0, T)$  for an initial time  $t = 0$ . Our model (1) is thus a linear factor - nonlinear output model. Dually one could also consider nonlinear factor - affine output models and there is also some equivalence between the two possible settings.

Model (1) for the forward rates implies for the short rate  $r(t) = r(t, x)$  and the zero-coupon bond prices  $p(t, T) = p(t, T, x)$  the representations

$$\begin{aligned} r(t) &= a(t) + b(t)x(t) + x'(t)c(t)x(t) \\ p(t, T, x) &= \exp \left\{ - \int_t^T r(u, x)du \right\} \\ &= \exp [-A(t, T) - B(t, T)x(t) - x'(t)C(t, T)x(t)] \end{aligned} \quad (2)$$

with  $a(t) = a(t, t), A(t, T) = \int_t^T a(t, u)du$  and, analogously, for  $b(t), c(t), B(t, T), C(t, T)$ . Notice that, by the above definitions,  $p(t, T, x)$  means that the bond price depends also on the factor process  $x$ , evaluated at time  $t$ ; analogously for  $r(t, x)$ . Although obvious in the present context, this meaning of notation will be important in the sequel (see Section III-C). In what follows, whenever it does not create confusion, we shall use the shorthand notations  $r(t)$  and  $p(t, T)$ .

So far  $F, G$  and  $a(t, T), b(t, T), c(t, T)$  in (1) appear as parameters in our model and the latter induce the parametric functions  $a(t), A(t, T), \dots, c(t), C(t, T)$  in (2). However, while  $F$  and  $G$  are essentially free, in order to exclude the possibility of arbitrage,  $a(t, T), b(t, T), c(t, T)$  cannot be chosen arbitrarily. We shall therefore impose on them conditions for absence of arbitrage, that can equivalently be

imposed on their integrated variants  $A(t, T), \dots, C(t, T)$  in (2). We have

*Proposition 2.1:* A sufficient condition for the term structure model (1), (2) to be arbitrage-free is that the coefficients  $A(t, T), B(t, T), C(t, T)$  in (2) satisfy the system of differential equations in  $t$

$$\begin{aligned} \frac{\partial}{\partial t} C(t, T) + F' C(t, T) + C(t, T)F \\ - 2C(t, T)GG' C(t, T) + c(t) &= 0 \\ \frac{\partial}{\partial t} B(t, T) + B(t, T)F \\ - 2B(t, T)GG' C(t, T) + b(t) &= 0 \\ \frac{\partial}{\partial t} A(t, T) + \text{tr}(G' C(t, T)G) \\ - \frac{1}{2} B(t, T)GG' B'(t, T) + a(t) &= 0 \end{aligned} \quad (3)$$

with terminal conditions  $A(T, T) = 0, B(T, T) = 0, C(T, T) = 0$ . The functions  $b(t), c(t)$  are here to be considered as parameters, while for  $a(t)$  we have

$$a(t) = f^*(0, t) + \frac{1}{2} \int_0^t \beta_T(s, t) ds \quad (4)$$

having put

$$\beta(t, T) := B(t, T)GG' B'(t, T) - 2 \text{tr}(G' C(t, T)G) \quad (5)$$

and where  $f^*(0, t)$  is the observed initial forward rate curve and the subscript  $T$  denotes partial differentiation with respect to the second variable.

For the proof see [6].

Concerning the solutions  $C(t, T)$  and  $B(t, T)$  in system (3) see the Appendix A.

Notice that, under conditions (3), the given measure  $Q$  is a martingale measure for the numeraire given by the money market account  $B(t) := \exp \left\{ \int_0^t r(s, x) ds \right\}$ .

## III. THE LINEAR-QUADRATIC STOCHASTIC CONTROL APPROACH

### A. Basic facts from linear-quadratic stochastic control

Here we recall some basic facts concerning the linear-quadratic regulator problem (see e.g. [3], [9], see also chapter 19 in [2]) in the particular form that will be used in the sequel.

Given always a filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$ , let the state  $x(t) \in \mathbb{R}^n$  of a controlled system evolve according to

$$dx(t) = [F(t)x(t) + H(t) + G u(t)] dt + G dw(t) \quad (6)$$

where the coefficients are given matrices/vectors,  $u(t) \in \mathbb{R}^k$  is the control and  $w(t)$  is a  $(Q, \mathcal{F}_t)$ -Wiener process with values in  $\mathbb{R}^k$  for the same  $k$  as for  $u(t)$ . For what follows we may without loss of generality assume that the controls  $u(t)$  are of the feedback form, namely

$$u(t) = u(t, x(t)) \quad (7)$$

and we shall call them *admissible* if equation (6) has a unique solution for  $u(t) = u(t, x(t))$ . Denote by  $\mathcal{U}$  the class of *admissible controls*.

Given a time horizon  $T$ , we shall take as criterion, on the basis of which to select the control  $u \in \mathcal{U}$ , the minimization of the expected total quadratic cost

$$J(x, u) := E_x \left\{ \int_0^T \left( x'(s)c(s)x(s) + b(s)x(s) + a(s) + \frac{1}{2}u'(s)u(s) \right) ds + x'(T)Cx(T) + Bx(T) + A \right\} \quad (8)$$

where  $x = x_0$  is the initial condition of the process  $x(t)$  in (6) and  $c(t), b(t), a(t), C, B, A$  are given functions/constants with  $c(t) > 0$  and  $C > 0$ .

Define the expected cost-to-go when in state  $x$  at time  $t$  and using a control  $u$  as

$$J(t, x, u) := E_x \left\{ \int_t^T \left( x'(s)c(s)x(s) + b(s)x(s) + a(s) + \frac{1}{2}u'(s)u(s) \right) ds + x'(T)Cx(T) + Bx(T) + A \right\} \quad (9)$$

and the expected optimal cost-to-go as

$$W(t, x) := \inf_{u \in \mathcal{U}} J(t, x, u) \quad (10)$$

It follows that  $W(0, x) = \inf_{u \in \mathcal{U}} J(x, u)$ . Sometimes we shall write  $J(t, T, x, u)$  to make explicit the dependence on the horizon.

To determine  $W(t, x)$  and an optimal  $u^*$  if it exists, i.e. such that  $W(0, x) = J(x, u^*)$ , we shall follow the *Dynamic Programming* approach. To this effect recall (see e.g. [3]) that the Hamilton-Jacobi-Bellman (HJB) equation for  $W(t, x)$  defined in (10) with  $x(t)$  according to (6) is

$$\begin{cases} \frac{\partial W}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^k} \left\{ x'c(t)x + b(t)x + a(t) + \frac{1}{2}u'u + \nabla_x W(t, x)[F(t)x + H(t) + Gu] + \frac{1}{2}tr(\nabla_{xx} WGG') \right\} = 0 \\ W(T, x) = x'Cx + Bx + A \end{cases} \quad (11)$$

By the so-called *Verification theorem* we then have that

- i) if  $W(t, x)$  is a  $\mathcal{C}^{1,2}$ -solution of (11),
- ii) if  $u^* = u^*(t, x) \in \mathbb{R}^k$  attains the infimum in (11)

then the control  $u^* \in \mathcal{U}$  that corresponds to the  $u^*(t, x)$  that are the minimizers in (11) is an optimal control in the sense that

$$J(x, u^*) = W(0, x) = \inf_{u \in \mathcal{U}} J(x, u) \quad \forall x \in \mathbb{R}^n \quad (12)$$

The procedure to apply the Verification theorem consists of two steps that in the case of (11) are

1. Fix  $(t, x) \in [0, T] \times \mathbb{R}^n$  and solve the static optimization problem

$$\inf_{u \in \mathbb{R}^k} \left\{ x'c(t)x + b(t)x + a(t) + \frac{1}{2}u'u + \nabla_x W(t, x)[F(t)x + H(t) + Gu] \right\} \quad (13)$$

The solution, if it exists, depends on  $(t, x)$  and also on the yet unknown function  $W$ , i.e.  $u^* = u^*(t, x; W)$ .

2. Substitute the so obtained expression for  $u = u^*$  into (11) and solve the resulting PDE.

**For Step 1.** we have the following

*Lemma 3.1:* The minimizing  $u$  in (13), and thus also in (11), is

$$u^*(t, x; W) = -G'(\nabla_x W(t, x))' \quad (14)$$

The proof is obtained by applying standard optimal control techniques; in fact, if  $b(t) \equiv 0$  (see [7]), the minimization problem becomes then the well known Linear Regulator Problem (see [1]).

**For Step 2.** one makes the usual Ansatz by putting

$$W(t, x) = x'C(t)x + B(t)x + A(t) \quad (15)$$

where  $C(t), B(t)$  are deterministic matrix functions with  $C(t)$  symmetric and  $A(t)$  is a scalar function. It can then be shown that equations (3) can be obtained from the Hamilton-Jacobi-Bellman equation (the terminal conditions will be specified below).

### B. Bond pricing

With bond prices described by  $p(t, T, x)$ , the Term Structure Equation (see [2]), under the assumption that the dynamics of  $x(t)$  are given by (1), can be written as:

$$\begin{aligned} \frac{\partial}{\partial t} p(t, T, x) + x'(t)F'(\nabla_x p(t, T, x))' \\ + \frac{1}{2}tr GG' \nabla_{xx} p(t, T, x) - p(t, T, x)r(t, x) = 0 \end{aligned} \quad (16)$$

with terminal condition  $p(T, T, x) = 1$ . It is well known [5] (and it can be easily verified by direct computation) that (16) can be transformed, putting  $W(t, T, x) := -\ln p(t, T, x)$  (and dropping the variables) into

$$\begin{aligned} \frac{\partial}{\partial t} W + x'F'(\nabla_x W)' - \frac{1}{2}\nabla_x WGG'(\nabla_x W)' \\ + \frac{1}{2}tr GG' \nabla_{xx} W + r = 0 \end{aligned} \quad (17)$$

Consider now the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} \frac{\partial}{\partial t} W + \inf_{u \in \mathbb{R}^k} \left\{ (x'F' + u'G') \cdot (\nabla_x W)' \right. \\ \left. + \frac{1}{2}tr(GG' \nabla_{xx} W) + \frac{1}{2}u'u + r \right\} = 0 \end{aligned} \quad (18)$$

that has as solution  $u(t, x) := -G'\nabla_x W(t, x)$  and notice that, by substituting this solution into (18), this latter equation becomes (17). On the other hand, equation (18) is the HJB equation associated with a linear-quadratic regulator problem as discussed in the previous subsection. More precisely, this corresponds to a dynamics of  $x(t)$  of the form (6) where  $F(t) \equiv F$  with  $F$  as in (1) and  $H(t) = 0$ . Furthermore, the expected cost-to-go  $J(t, T, x, u)$ , where we now indicate explicitly the dependence on  $T$ , is here given by (9) where the expectation is with respect to the measure  $Q$ ,  $a(t), b(t), c(t)$  are those in (2) and  $A = B = C = 0$  (the terminal cost is here in fact  $J(T, T, x, u) = W(T, T, x) = -\log p(T, T, x) = 0$ .) It can then be show that the solution  $W(t, T, x)$  of (18) is given by (15). It is then not too difficult to show that  $W(t, T, x)$  can be rewritten here as

$$W(t, T; x) = x'C(t, T)x + B(t, T)x + A(t, T) \quad (19)$$

where, for each  $T$ ,  $C(t, T)$ ,  $B(t, T)$ ,  $A(t, T)$  satisfy the equations in (3) respectively with  $C(T, T) = 0$ ,  $B(T, T) = 0$ ,  $A(T, T) = 0$ ; this last condition is then seen to imply (4). In other words, the no arbitrage conditions (3) and (4) can be derived entirely by an optimal control argument (which, in view of the Kolmogorov backward equation, is not too surprising). The interest of this approach, though, will be clear below, where we will use the system theoretic approach to derive relatively simple formulas for computing derivative prices. We have now

*Theorem 3.2:* Let  $r(t)$  be defined as in (2) and let the bond price  $p(t, T, x)$  be given by

$$p(t, T, x) = E_{t,x} \left\{ e^{-\int_t^T r(s) ds} \right\}$$

Then

$$p(t, T, x) = e^{-W(t, T, x)} \quad (20)$$

where  $W$  is, for each  $T$ , of the form (19) and  $a(t)$  is determined by the boundary condition

$$W(0, T, 0) = \int_0^T f^*(0, s) ds \quad (21)$$

where  $f^*(0, t)$  is the initially observed forward rate.

### C. Forward Prices

We compute now the forward price of a bond, that is the value  $E_{t,x}^{Q_\tau} p(\tau, T, x)$  for a given  $\tau$  with  $t \leq \tau \leq T$ , and where  $Q_\tau$  is the *forward measure* with respect to the numeraire  $p(t, \tau, x)$ . Since prices expressed in units of  $p(t, \tau, x)$  are  $Q_\tau$ -martingales, we have

$$E_{t,x}^{Q_\tau} p(\tau, T, x) = \frac{p(t, T, x)}{p(t, \tau, x)} \quad (22)$$

Recall that in  $p(t, \tau, x)$ , the dependence on the factor process  $x$  is through its value at time  $t$ . We claim that we can derive this forward price as an expectation with respect to  $Q$  assuming however that, instead of (1), the factors satisfy

$$dx^\tau(t) = [(F - 2GG'C(t, \tau))x^\tau(t) - GG'B'(t, \tau)]dt + Gdw(t) \quad (23)$$

where  $F$  and  $G$  are as in (1) and  $C(t, \tau)$ ,  $B(t, \tau)$  correspond to those in (19) for  $T = \tau$ . For this purpose introduce the following quantity

$$\begin{aligned} p^\tau(t, T, x) &:= E_{t,x} p(\tau, T, x^\tau) \\ &= E_{t,x} e^{-(x^\tau(\tau))'C(\tau, T)x^\tau(\tau) - B(\tau, T)x^\tau(\tau) - A(\tau, T)} \end{aligned} \quad (24)$$

where the second equality follows from (20) and (19). In a manner completely similar to that used in Theorem 3.2 for  $p(t, T, x)$ , it can be easily shown that from the Term Structure Equation we can derive a HJB equation for  $W^\tau(t, T, x) = -\ln p^\tau(t, T, x)$  and, making the equivalent Ansatz of (15) for  $W^\tau$ , using the Verification theorem we can find  $C^\tau$ ,  $B^\tau$ ,  $A^\tau$  such that the following holds:

*Proposition 3.3:* The forward price at time  $t$  of a bond is given by

$$E_{t,x}^{Q_\tau} p(\tau, T, x) = p^\tau(t, T, x) \quad (25)$$

where

$$\begin{aligned} p^\tau(t, T, x) &= E_{t,x} \exp\{-(x^\tau(\tau))'C(\tau, T)x^\tau(\tau) \\ &\quad - B(\tau, T)x^\tau(\tau) - A(\tau, T)\} \\ &= \exp\{-(x^\tau(t))'C^\tau(t, T)x^\tau(t) \\ &\quad - B^\tau(t, T)x^\tau(t) - A^\tau(t, T)\} \end{aligned} \quad (26)$$

The process  $p^\tau(t, T, x)$  has dynamics

$$dp^\tau(\tau, T, x) = -[2(x^\tau(t))'C^\tau(t, \tau) + B^\tau(t, \tau)]Gdw(t) \quad (27)$$

The proof is omitted.

## IV. FORWARD MEASURES AND BOND OPTION PRICING

### A. Forward measures

The results of section III-C suggest that a deeper connection exists between the forward prices  $p^\tau(t, T)$  and the usual forward measure  $Q_\tau$  which is normally used to compute (22). We shall now derive this connection and show that pricing with the forward measure can be made equivalent to a pricing approach under the standard martingale measure  $Q$  by using the forward prices  $p^\tau(t, T)$ . To this effect

- Let  $x(\tau)$  be the value in  $\tau$  of the solution to (1) with initial condition  $x(t) = x$ .
- Let  $x^\tau(\tau)$  be the value in  $\tau$  of the solution to (23) with initial condition  $x(t) = x$

*Proposition 4.1:* Given  $\tau$ , the two random variables  $x(\tau)$  and  $x^\tau(\tau)$  have the same Gaussian distribution, the first under the forward measure  $Q_\tau$  (with numeraire  $p(t, \tau, x)$ ), the second under the standard martingale measure  $Q$  (with numeraire  $B(t)$ ).

**Proof :** For the numeraire  $p(t, \tau, x)$  we have, under  $Q$ ,

$$\begin{aligned} dp(t, \tau, x) &= p(t, \tau, x)[r(t)dt - (2x'(t)C(t, \tau) \\ &\quad + B(\tau, T))Gdw(t) \end{aligned} \quad (28)$$

For the Radon-Nikodym derivative

$$L(t) := \frac{dQ_\tau}{dQ} = \frac{p(t, \tau, x)}{B(t)}$$

one then has (it is easily checked that the Novikov condition holds)

$$dL(t) = -[2x'(t)C(t, \tau) + B(t, \tau)]GL(t)dw(t) \quad (29)$$

It follows that the process  $w^\tau(t)$  defined by

$$dw^\tau(t) = dw(t) + [2G'C(t, \tau)x(t) + G'B'(t, \tau)]dt \quad (30)$$

is a Wiener process under  $Q_\tau$ .

For  $x(t)$  satisfying (1) under  $Q$  one then has, under  $Q_\tau$ ,

$$dx(t) = [(F - 2GG'C(t, \tau))x(t) - GG'B'(t, \tau)]dt + Gdw^\tau(t) \quad (31)$$

Since the dynamics in (31) is identical to those in (23) and  $x(t) = x$  in both cases, the distribution of  $x(\tau)$  under  $Q_\tau$  and  $x^\tau(\tau)$  under  $Q$  are the same and given by a Gaussian. ■

*Remark 4.2:* An alternative approach to obtain the same result as in the previous Proposition could be to show that from the following equality, that derives from (25), namely

$$\begin{aligned} E_{t,x}^{Q_\tau} \{ \exp[-x'(\tau)C(\tau,T)x(\tau) - B(\tau,T)x(\tau) - A(\tau,T)] \} \\ = E_{t,x} \{ \exp[-(x^\tau(\tau))'C(\tau,T)x^\tau(\tau) \\ - B(\tau,T)x^\tau(\tau) - A(\tau,T)] \} \end{aligned} \quad (32)$$

follows the equality of the two Gaussian distributions, that of  $x(\tau)$  under  $Q_\tau$  and that of  $x^\tau(\tau)$  under  $Q$ , given that  $x(t) = x^\tau(t) = x$ .

Coming now to pricing, we have the following result, which generalizes a scalar result in [10]:

*Proposition 4.3:* Given a maturity  $\tau$  and an integrable claim  $H(x(\tau))$ , its arbitrage free price at  $t < \tau$  is

$$\begin{aligned} \Pi_t &= E_{t,x} \left\{ e^{-\int_t^\tau r(s)ds} H(x(\tau)) \right\} \\ &= p(t, \tau, x) E_{t,x}^{Q_\tau} \{ H(x(\tau)) \} \\ &= e^{-W(t,\tau,x)} E_{t,x} \{ H(x^\tau(\tau)) \} \end{aligned} \quad (33)$$

**Proof :** The first equality follows from the definition of  $Q$  as martingale measure for the numeraire  $B(t)$ , the second from the definition of the forward measure  $Q_\tau$  and the third follows from (20) and the equality of the distributions of  $x(\tau)$  under  $Q_\tau$  and of  $x^\tau(\tau)$  under  $Q$ . ■

### B. Pricing of a bond derivative

We derive now an explicit formula for the pricing of a bond option that is based on (33) and on the representation of the factor process  $x^\tau$  in section III-C. First we have

*Remark 4.4:* If  $\Phi_\tau(\tau, t)$  denotes the fundamental solution of (23), we immediately see that, for  $\tau > t$  with  $\tau < T$  the conditional mean of the Gaussian process  $x^\tau$  given  $x^\tau(t) = x$  is expressed by:

$$E_{t,x} x^\tau(\tau) = \Phi_\tau(\tau, t)x - \int_t^\tau \Phi_\tau(\tau, s)GG'B'(s, \tau)ds \quad (34)$$

and its conditional variance by

$$\begin{aligned} E_{t,x} [x^\tau(\tau) - E_{t,x} x^\tau(\tau)] [(x^\tau(\tau))' - E_{t,x} (x^\tau(\tau))'] \\ = E \int_t^\tau \Phi_\tau(\tau, s)Gdw(s)dw'(s)G'\Phi_\tau(\tau, s)' \\ = \int_t^\tau \Phi_\tau(\tau, s)GG'\Phi_\tau(\tau, s)'ds \end{aligned} \quad (35)$$

Integral (34) and (35) can be computed numerically, using the explicit representations of  $\Phi_\tau(\tau, s)$  and  $B(s, t)$  (see (54) and (56) in the Appendix). In fact, from (54) with  $X(\tau) = I, Y(\tau) = 0$ ,

$$\Phi_\tau(\tau, s) = \Phi_\tau(s, \tau)^{-1} = \left( [I, 0] e^{H(s-\tau)} \begin{bmatrix} I \\ 0 \end{bmatrix} \right)^{-1} \quad (36)$$

and from the fact that  $B(\tau, \tau) = 0$  (and assuming, for simplicity, that  $b$  is constant)

$$\begin{aligned} B(t, \tau) &= \left[ \int_\tau^t b\phi_\tau(s, \tau)ds \right] \phi_\tau(t, \tau)^{-1} \\ &= \left[ \int_\tau^t b[I, 0]e^{H(s-\tau)} \begin{bmatrix} I \\ 0 \end{bmatrix} ds \right] \\ &\quad \cdot \left( [I, 0]e^{H(t-\tau)} \begin{bmatrix} I \\ 0 \end{bmatrix} \right)^{-1} \\ &= \left[ b[I, 0]H^{-1} \left( e^{H(t-\tau)} - I \right) \begin{bmatrix} I \\ 0 \end{bmatrix} \right] \\ &\quad \cdot \left( [I, 0]e^{H(t-\tau)} \begin{bmatrix} I \\ 0 \end{bmatrix} \right)^{-1} \end{aligned} \quad (37)$$

The mean (34) can now be calculated from (36) and (37) by numerical computation of a line integral. As for the variance (35), we can use numerical integration using again (36).

In view of the previous section, we can express the arbitrage free price of a claim by formula (33) where now  $x^\tau$  is the Gaussian process with mean  $\mu$  given by (34) and variance  $\Sigma$  given by (35). In particular, the value of a call option with strike price  $K$  and expiration  $\tau$  on a bond with maturity  $T$  will be

$$\begin{aligned} \Pi_t &= e^{-W(t,\tau,x)} E_{t,x} \max \{ 0, p(\tau, T, x) - K \} \\ &= e^{-W(t,\tau,x)} E_{t,x} \max \left\{ 0, e^{-W(\tau,T,x)} - K \right\} \end{aligned} \quad (38)$$

where

$$\begin{aligned} E_{t,x} \max \{ 0, e^{-W(\tau,T,x)} - K \} \\ = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \\ \int_{-W(\tau,T,x) > \ln K} e^{-\frac{1}{2}(\xi-\mu)'\Sigma^{-1}(\xi-\mu)} e^{-\xi'C(\tau,T)\xi - B'(\tau,T)\xi - A(\tau,T)} d\xi_1 \dots d\xi_n \\ - \frac{K}{\sqrt{(2\pi)^n \det \Sigma}} \int_{-W(\tau,T,x) > \ln K} e^{-\frac{1}{2}(\xi-\mu)'\Sigma^{-1}(\xi-\mu)} d\xi_1 \dots d\xi_n \end{aligned} \quad (39)$$

Assume now that  $\Sigma$  has full rank (it can be shown that this is always the case if the pair  $(F, G)$  is controllable, that is if the matrix  $G, FG, \dots, F^{n-1}G$  has full column rank). We perform then a suitable change of variables remembering that: a) two positive definite matrices can be simultaneously diagonalized by congruence and the transformation can be chosen so that one is the identity; b) if  $\Sigma$  is invertible, we can complete the squares. In fact, the quadratic form  $F(\xi) := \xi' C \xi + B' \xi + A$  can be written as

$$F(\xi) := \xi' C \xi + B' \xi + A = (\xi' - \nu') C (\xi - \nu) - \nu' C \nu + A \quad (40)$$

where we have set

$$\nu := -\frac{1}{2} C^{-1} B' \quad (41)$$

To clarify the procedure we introduce the following quadratic functions  $F$  and  $F_D$ , which will represent the integrand and define the domain  $D$  of integration, respectively:

*Lemma 4.5:* Let  $\xi \in \mathbb{R}^n$  and let  $A, B, C$  and  $A_D, B_D, C_D$  be coefficients of suitable size for the quadratic functions

$$F(\xi) := \xi' C \xi + B' \xi + A \quad F_D(\xi) := \xi' C_D \xi + B'_D \xi + A_D \quad (42)$$

with  $C$  and  $C_D$  positive definite and let

$$D := \{\xi : F_D(\xi) \leq 0\}$$

Then

$$\int_D e^{-\frac{1}{2}F(\xi)} d\xi_1 \dots d\xi_n = \tilde{\lambda} \int_{\tilde{D}} e^{-\frac{1}{2}|\tilde{\xi}|^2} d\tilde{\xi}_1 \dots d\tilde{\xi}_n \quad (43)$$

where, with  $L'L = C$  and  $\nu$  as in (41)

$$\tilde{\xi} = L(\xi - \nu) \quad (44)$$

$$\tilde{\lambda} = e^{\nu' C \nu - A} |\det L|^{-1} \quad (45)$$

$$\tilde{D} = \{\tilde{\xi} : F_D(L^{-1}\tilde{\xi} + \nu) \leq -\ln K\} \quad (46)$$

**Proof** The substitution (44) immediately yields  $d\tilde{\xi}_1 \dots d\tilde{\xi}_n = \det L d\xi_1 \dots d\xi_n$  and:

$$F(\xi) = \xi' C \xi + B' \xi + A = (\xi' - \nu') C (\xi - \nu) - \nu' C \nu + A = |\tilde{\xi}|^2 - \nu' C \nu + A$$

from which (43) immediately follows. Notice that

$$\tilde{D} = \{\tilde{\xi} : \tilde{\xi}' L^{-T} C_D L^{-1} \tilde{\xi} + (2\nu' C_D + B_D) L^{-1} \tilde{\xi} + \nu' C_D \nu + B_D \nu + A_D \leq -\ln K\} \quad (47)$$

■

Thus, in principle, derivative prices can be computed in a standard manner. Nevertheless, these are multiple integrals and thus their actual computation is quite demanding. In the case of  $n = 2$ , though, the formulas, although complicated, can be reduced to calculating the value of two single integrals and require therefore a computational effort comparable with that of the Black and Scholes formula.

In fact, let:

$$\begin{aligned} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} &:= L^{-T} C_D L^{-1} \\ [\kappa_1, \kappa_2] &:= (2\nu' C_D + B_D) L^{-1} \\ \alpha &:= \nu' C_D \nu + B_D \nu + A_D + \ln K \end{aligned}$$

Then, using a standard substitutions in (47), that is  $\tilde{\xi}_1 = \rho \cos \theta$  and  $\tilde{\xi}_2 = \rho \sin \theta$ , we see that the set  $D_{\xi_1 \xi_2} := \{(\xi_1, \xi_2); F_D(\xi_1, \xi_2) \leq 0\}$  is mapped into the set

$$D_{\rho, \theta} := \{(\rho, \theta); p(\rho, \theta) \leq 0, \rho \geq 0\}$$

where we have set (using the above substitution):

$$p(\rho, \theta) := \rho^2 (\sigma_{11} \cos^2 \theta + \sigma_{22} \sin^2 \theta + 2\sigma_{12} \sin \theta \cos \theta) + \rho (\kappa_1 \cos \theta + \kappa_2 \sin \theta) + \alpha \quad (48)$$

As a function of  $\rho$ , the above is a second degree polynomial with roots  $\rho_1(\theta), \rho_2(\theta)$  in  $D_{\rho, \theta}$ . Notice that, if  $p(\rho, \theta) = 0$ , also  $p(-\rho, \theta + \pi) = 0$ . Therefore (as long as the discriminant  $\Delta$  of (48) is non negative), the set  $\Gamma$  of points satisfying (48) with  $\rho > 0$  is non empty (it is an ellipse). There are two possibilities:

- The ellipse  $\Gamma$  does not contain the origin so, if they are real, both roots  $\rho_1(\theta), \rho_2(\theta)$  have the same sign. Denote therefore by  $I_\theta$  the subinterval of  $[0, 2\pi)$  where the discriminant  $\Delta$  of (48) is non negative and  $\rho \geq 0$ . If we make the convention that  $\rho_1(\theta) \leq \rho_2(\theta)$ , we can write the integral in (43) as:

$$\begin{aligned} \int_{D_{\xi_1 \xi_2}} e^{-\frac{1}{2}(\xi_1^2 + \xi_2^2)} d\xi_1 d\xi_2 &= \int_{D_{\rho, \theta}} e^{-\frac{1}{2}\rho^2} \rho d\rho d\theta \\ &= - \int_{I_\theta} e^{-\frac{1}{2}\rho^2} \Big|_{\rho_1(\theta)}^{\rho_2(\theta)} d\theta \\ &= \int_{I_\theta} \left[ e^{-\frac{1}{2}\rho_1^2(\theta)} - e^{-\frac{1}{2}\rho_2^2(\theta)} \right] d\theta \end{aligned}$$

- The ellipse  $\Gamma$  contains the origin, and thus we need to integrate on  $[0, 2\pi)$ . Thus, for each  $\theta$ , only one root of (48) is positive. Denoting this root by  $\rho_2(\theta)$ , we can write

$$\begin{aligned} \int_{D_{\xi_1 \xi_2}} e^{-\frac{1}{2}(\xi_1^2 + \xi_2^2)} d\xi_1 d\xi_2 &= \int_{D_{\rho, \theta}} e^{-\frac{1}{2}\rho^2} \rho d\rho d\theta \\ &= - \int_0^{2\pi} e^{-\frac{1}{2}\rho^2} \Big|_0^{\rho_2(\theta)} d\theta \\ &= \int_0^{2\pi} \left[ 1 - e^{-\frac{1}{2}\rho_2^2(\theta)} \right] d\theta \end{aligned}$$

In both cases, this is a simple integral whose value is easily computed numerically. The computation of the integrals in (39) is now straightforward.

## APPENDIX

The computation of bond prices and interest rates becomes definitely simple provided we have a solution to the equations (3). We provide now an explicit solution  $C(t, T)$  to (3) provided we have a solution to two time invariant equations of which the solution can be easily be computed numerically. This will suffice if  $B \equiv 0$ . Otherwise, a fundamental solution of the second equation in (3) is needed. It turns out that also this can be computed explicitly. We follow [1] for this approach.

Since, for each  $T$ , these are ordinary differential equations, to simplify the notation, in what follows we will drop the explicit dependence on this variable as well as the dependence on  $t$  where possible.

*A. A complete solution for matrix  $C$  and vector  $B$*

We have the following general result that leads to a solution of the first equation in (3).

*Theorem 1.1:* Let  $C(t)$  be a solution to

$$\begin{cases} \frac{dC(t)}{dt} + C(t)F(t) + F'(t)C(t) \\ \quad - C(t)GN^{-1}(t)G'C(t) + c(t) = 0 \\ C(t_1) = C_1 \end{cases} \quad (49)$$

where  $C_1$  and  $c(t)$  are symmetric non negative definite and  $N(t)$  is positive definite on the interval  $[t_0, t_1]$ . Then a solution  $C(t)$  to (49) always exists between  $t_0$  and  $t_1$  and it can be expressed as

$$C(t) = Y(t)X(t)^{-1} \quad (50)$$

where  $X$  and  $Y$  satisfy the following linear differential equation:

$$\frac{d}{dt} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} F & -GN^{-1}G' \\ -c & -F' \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \quad (51)$$

$$\begin{bmatrix} X(t_1) \\ Y(t_1) \end{bmatrix} = \begin{bmatrix} I \\ C_1 \end{bmatrix}$$

Moreover, if  $\Phi(t, s)$  denotes the fundamental solution associated with

$$\frac{dx(t)}{dt} = [F(t) - GN^{-1}(t)G'C(t)]x(t) \quad (52)$$

then  $X$  and  $Y$  admit the following interpretation

$$X(t) = \Phi(t, t_1) \quad (53)$$

and (in view of (50))

$$Y(t) = C(t)\Phi(t, t_1)$$

Notice that, if  $F, G, N, c$  are constant, setting  $H = \begin{bmatrix} F & -GN^{-1}G' \\ -c & -F' \end{bmatrix}$  to be the Hamiltonian matrix in (51), we obtain the explicit representation

$$\Phi(t, t_1) = X(t) = [I, 0]e^{H(t-t_1)} \begin{bmatrix} X(t_1) \\ Y(t_1) \end{bmatrix} \quad (54)$$

Concerning the solution of the second equation in (3) we have now (recall that in our case we have  $N = \frac{1}{2}$ )

*Corollary 1.2:* Let  $B(t)$  be the solution to

$$\begin{cases} \frac{dB(t)}{dt} + B(t)[F(t) - GN^{-1}(t)G'C(t)] + b(t) = 0 \\ B(t_1) = B_1 \end{cases} \quad (55)$$

and let  $\Phi(t, s)$  be the fundamental solution for the system (55). Then  $B(t)$  can be written as

$$B(t) = B_1\Phi(t_1, t) + \int_{t_1}^t b(s)\Phi(s, t)ds = B_1X(t)^{-1} + \int_{t_1}^t b(s)X(s)X^{-1}(t)ds \quad (56)$$

Again, using the representation (54), we easily see that, if  $b$  is constant vector (or even a polynomial vector) the integral (56) can be computed explicitly.

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