The Shortest-Basis Approach to Minimal Realizations of Linear Systems

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***Abstract*— Given a controllable discrete-time linear system $\mathcal{C}$, a shortest basis for $\mathcal{C}$ is a set of linearly independent generators for $\mathcal{C}$ with the least possible lengths. A basis $\mathcal{B}$ is a shortest basis if and only if it has the predictable span property (*i.e.,* has the predictable delay and degree properties, and is non-catastrophic), or alternatively if and only if it has the subsystem basis property (for any interval $\mathcal{J}$, the generators in $\mathcal{B}$ whose span is in $\mathcal{J}$ is a basis for the subsystem $\mathcal{C}_{\mathcal{J}}$). The dimensions of the minimal state spaces and minimal transition spaces of $\mathcal{C}$ are simply the numbers of generators in a shortest basis $\mathcal{B}$ that are active at any given state or symbol time, respectively. A minimal linear realization for $\mathcal{C}$ in controller canonical form follows directly from a shortest basis for $\mathcal{C}$, and a minimal linear realization for $\mathcal{C}$ in observer canonical form follows directly from a shortest basis for the orthogonal system $\mathcal{C}^{\perp}$. This approach seems conceptually simpler than that of classical minimal realization theory.**

The classical theory of minimal realizations of linear systems is based on the “minimal = controllable + observable” paradigm, and involves heavy use of matrix algebra; see *e.g.,* [1]. In this paper, we present an alternative “shortest-basis” approach that seems conceptually simpler.

As in behavioral system theory [9], we characterize a discrete-time linear system over a field $\mathbb{F}$ by the set $\mathcal{C}$ of all of its trajectories on the *symbol time axis* $\mathcal{I} = \mathbb{Z}$, or on a subinterval $\mathcal{I} \subseteq \mathbb{Z}$. The system $\mathcal{C}$ may be time-varying, or defined on a finite time axis $\mathcal{I}$. For simplicity, we consider only systems $\mathcal{C}$ that are *controllable— i.e.,* generated by their finite trajectories [2], [10]— although this restriction can easily be lifted. A *shortest basis* $\mathcal{B}$ for $\mathcal{C}$ is then defined as a set of linearly independent finite generators for $\mathcal{C}$ with the least possible lengths, where the *length* of a nonzero finite trajectory $\mathbf{a} \in \mathcal{C}$ is the size of the shortest interval that contains its support (its *span*).

For a basis $\mathcal{B}$ of a controllable system $\mathcal{C}$, we show that the following properties are equivalent:

1) $\mathcal{B}$ is a shortest basis for $\mathcal{C}$;
2) $\mathcal{B}$ has the *predictable span property*; *i.e.,* if $\mathbf{a} \in \mathcal{C}$ is a linear combination of a subset of generators $\mathcal{S}(\mathbf{a}) \subseteq \mathcal{B}$, then the span of $\mathbf{a}$ is precisely the span of $\mathcal{S}(\mathbf{a})$ (the shortest interval that contains the supports of all generators in $\mathcal{S}(\mathbf{a})$).
3) $\mathcal{B}$ has the has the *subsystem basis property*; *i.e.,* for any subinterval $\mathcal{J}$ of the time axis $\mathcal{I}$, the set of generators in $\mathcal{B}$ whose support is contained in $\mathcal{J}$ is a basis for the subsystem $\mathcal{C}_{\mathcal{J}}$ comprising all trajectories in $\mathcal{C}$ whose support is contained in $\mathcal{J}$.

A journal version of this paper will appear in [6].

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Given a linear system $\mathcal{C}$, it is well known that the *minimal state space* Σ_k at state time k, corresponding to a cut of the symbol time axis $\mathcal{I}$ between symbol time $k-1$ and symbol time k, is the quotient space

$$\frac{\mathcal{C}}{\mathcal{C}_{k^-} \times \mathcal{C}_{k^+}},$$

where $\mathcal{C}_{k^-}$ is the subsystem defined on the *past* $k^- = \{k' \in \mathcal{I} : k' < k\}$ and $\mathcal{C}_{k^+}$ is the subsystem defined on the *future* $k^+ = \{k' \in \mathcal{I} : k' \geq k\}$ [9], [10]. From the subsystem basis property, it follows immediately that the dimension of Σ_k is is the number of active generators at state time k in any shortest basis $\mathcal{B}$ for $\mathcal{C}$, where a generator is said to be *active* at state time k if its support is not contained entirely in either the past k^- or the future k^+.

It is less well known that in a minimal realization of $\mathcal{C}$, the *minimal transition space* $\mathcal{T}_k$ at symbol time $k \in \mathcal{I}$, namely the set of all transitions $(s_k, a_k, s_{k+1}) \in \Sigma_k \times A_k \times \Sigma_{k+1}$ that actually occur (where A_k is the symbol alphabet at time k), is isomorphic to the quotient space

$$\frac{\mathcal{C}}{\mathcal{C}_{k^-} \times \mathcal{C}_{(k+1)^+}}.$$

From the subsystem basis property, it follows immediately that the dimension of $\mathcal{T}_k$ is is the number of active generators at symbol time k in any shortest basis $\mathcal{B}$ for $\mathcal{C}$, where a generator is active at symbol time k if its support is not contained entirely in either the past k^- or the future $(k+1)^+$.

It follows that, given any shortest basis $\mathcal{B}$ for $\mathcal{C}$, we can construct an obvious linear state-space realization for $\mathcal{C}$, sometimes called the *controller canonical form* [7], which is evidently minimal. For each generator $\mathbf{g} \in \mathcal{B}$, construct a one-dimensional “atomic” state-space realization, which is active (one-dimensional) when $\mathbf{g}$ is active and dormant (zero-dimensional) otherwise, and which produces a scalar multiple of $\mathbf{g}$, namely $\alpha(\mathbf{g})\mathbf{g}$, where $\alpha(\mathbf{g}) \in \mathbb{F}$, during the active interval. The output of the realization is the sum $\sum_{\mathbf{g} \in \mathcal{B}} \alpha(\mathbf{g})\mathbf{g}$ of the outputs of all of these atomic realizations, which evidently runs through $\mathcal{C}$, the set of all linear combinations of all elements of its basis $\mathcal{B}$. From the results of the two previous paragraphs, the state-space dimension of this controller canonical realization is evidently minimal at all state times, and the transition-space dimension is minimal at all symbol times.

ISBN 978-963-311-370-7

(For multivariable linear time-invariant systems, a construction of a minimal realization in controller canonical form from a shortest ("minimal") basis was given in [3]. In the literature of trellis realizations of block codes, such a construction was first given by Kschischang and Sorokine [8], who introduced the term "atomic.")

An alternative way of defining a linear system $\mathcal{C}$ is via a set of generators for its orthogonal system $\mathcal{C}^\perp$. In linear system theory, such a representation of $\mathcal{C}$ is sometimes called a *kernel representation*, whereas a representation in terms of generators for $\mathcal{C}$ is called an *image representation*. If $\mathcal{B}^\perp$ is a shortest basis for $\mathcal{C}^\perp$, then $\mathcal{C}$ is the set of all trajectories that are orthogonal to all trajectories in $\mathcal{B}^\perp$.

A fundamental duality result is that any minimal state space for $\mathcal{C}^\perp$ has the same dimension as the corresponding minimal state space for $\mathcal{C}$. An equally fundamental result, but less well known, is that the minimal transition spaces of $\mathcal{C}^\perp$ are the orthogonal spaces $(\mathcal{T}_k)^\perp$ to $\mathcal{T}_k$, where orthogonality is defined in an unconventional way, with an inverted sign [6]. It follows that

$$\dim(\mathcal{T}_k)^\perp = \dim \Sigma_k + \dim A_k + \dim \Sigma_{k+1} - \dim \mathcal{T}_k.$$

From these results, we can straightforwardly construct a minimal realization for $\mathcal{C}$ in observer canonical form [7], involving a set of one-dimensional atomic "checkers," one corresponding to each generator $\mathbf{h} \in \mathcal{B}^\perp$.

In a controller canonical realization of a linear time-invariant system $\mathcal{C}$, the lengths of the generators $\mathbf{g} \in \mathcal{B}$ in a shortest basis $\mathcal{B}$ are sometimes called the *controllability indices* of $\mathcal{C}$. Similarly, in a linear time-varying system $\mathcal{C}$, the lengths of the generators of a shortest basis $\mathcal{B}$ may be regarded as generalized controllability indices of $\mathcal{C}$. Dually, in an observer canonical realization of a linear system $\mathcal{C}$, the lengths of the dual generators $\mathbf{h} \in \mathcal{B}^\perp$ of a shortest basis $\mathcal{B}^\perp$ for $\mathcal{C}^\perp$ may be regarded as generalized *observability indices* of $\mathcal{C}$. Thus the generalized observability indices of $\mathcal{C}$ are the generalized controllability indices of $\mathcal{C}^\perp$, and *vice versa*.

As shown in [4], this approach generalizes naturally to discrete-time group systems. As shown in [5], it further generalizes to linear and group systems defined on cycle-free graphs, rather than on a standard discrete time axis, and to some extent to general graphs. Such generality suggests that the shortest-basis approach is rather fundamental.

ACKNOWLEDGMENTS

I have benefitted greatly from interactions with many colleagues over the very many years during which I have been interested in these topics, including particularly F. Fagnani, R. Johannesson, T. Kailath, R. Koetter, F. R. Kschischang, H.-A. Loeliger, J. L. Massey, T. Mittelholzer, S. Mitter, J. Rosenthal, M. D. Trott, A. Vardy, J. C. Willems, and S. Zampieri.

For comments on earlier drafts of this paper, I am grateful to H. Gluesing-Luerssen, F. R. Kschischang, and M. Kuijper.

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