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The best state space for the SCOLE model

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***Abstract*— It is well-known that the SCOLE model (a beam coupled to a rigid body) is not exactly controllable in the energy state space with L^2 input signals, since the control operator is compact from the input space to state space. In this paper, we derive its exactly controllable space for L^2 input signals and we prove its well-posedness and regularity in this space.**

I. INTRODUCTION

This paper investigates the exact controllability of the SCOLE (NASA Spacecraft Control Laboratory Experiment) model with L^2 input signals and its well-posedness with the state space that makes it exactly controllable. The SCOLE system models a flexible beam with one end clamped and the other end linked to a rigid body. The inputs of the system are the force and the torque acting on the rigid body, while the outputs are the velocity and the angular velocity of the rigid body. The importance of the SCOLE model stems from it being used to model the vibrations of a flexible mast holding an antenna on a spacecraft, see Littman and Markus [5], [6].

Assuming that the beam is uniform and moves only in one plane, the model is

$$\begin{cases} \rho w_{tt}(x,t) + EI w_{xxxx}(x,t) = 0, \\ (x,t) \in (0,l)\times[0,\infty), \\ w(0,t) = 0, \qquad w_x(0,t) = 0, \\ m w_{tt}(l,t) - EI w_{xxx}(l,t) = f(t), \\ J w_{xtt}(l,t) + EI w_{xx}(l,t) = v(t), \end{cases} \tag{1.1}$$

where l is the length of the beam, w is its the transverse displacement, and $EI > 0$ and $\rho > 0$ are its flexural rigidity and mass density. $m > 0$ and $J > 0$ are the mass and the moment of inertia of the rigid body. f and v are the force input and the torque input acting on the rigid body. We define the input and output signals of the model as follows:

$$u_e = \begin{bmatrix} u_{e1} \\ u_{e2} \end{bmatrix} = \begin{bmatrix} f \\ v \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} w_t(l,\cdot) \\ w_{xt}(l,\cdot) \end{bmatrix}. \tag{1.2}$$

The natural state and state space of the SCOLE model are

$$z^c(t) = \begin{bmatrix} w(\cdot,t) & w_t(\cdot,t) & w_t(l,t) & w_{xt}(l,t) \end{bmatrix}^T, \tag{1.3}$$

$$H^c = \mathcal{H}_l^2(0,l) \times L^2[0,l] \times \mathbb{C}^2,$$

where $\mathcal{H}_l^2(0,l) = \{h \in \mathcal{H}^2(0,l) \mid h(0) = 0,\ h_x(0) = 0\}$. The natural norm on H^c is

$$\|z^c(t)\|^2 = EI\|w(\cdot,t)\|^2_{\mathcal{H}_l^2} + \rho\|w_t(\cdot,t)\|^2_{L^2} + m|w_t(l,t)|^2 + J|w_{xt}(l,t)|^2, \tag{1.4}$$

which represents twice the physical energy.

It is well-known that the SCOLE model is not exactly controllable with the natural energy state space H^c using L^2 inputs, since the control operator is bounded from the input space $\mathbb{C}^2$ to H^c, and hence compact. Exact controllability can be achieved either by expanding the input signal space (bringing in distributions) or by shrinking the state space. Here are some results obtained by expanding the input signal space. Using the Hilbert Uniqueness Method, Rao [7] obtained the exact controllability of the uniform SCOLE model with the state space H^c by means of a singular input signal. He considered $f \in L^2[0,T]$ but allowed the torque input v to be in in the dual of $\mathcal{H}^1(0,T)$, where $T > 0$ is an arbitrarily short time. He also proved the exact controllability in arbitrarily short time of the SCOLE model with the state space H^c by singular torque input (and zero force input) if $l < 3$, where all the constants (EI, ρ, m, J) are one. Guo and Ivanov [3] removed this length limitation and they allowed the SCOLE model to be non-uniform.

Smaller state spaces have been investigated in at least three papers. The null-controllability of the SCOLE model with a state space of type $\mathcal{H}^6(0,l) \times \mathcal{H}^4(0,l)$ (with boundary conditions) was proved in Littman and Markus [5] based on the theory of semi-infinite beams. Using a constructive cutoff approach, they proved the existence of smooth torque and force inputs for the finite beam leading to the final state zero. Using the Riesz basis approach, Guo [2] proved that the non-uniform SCOLE model with only torque input is exactly controllable with the state space $\mathcal{D}(A^c)$:

$$\left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in \left(\mathcal{H}^4 \cap \mathcal{H}_l^2\right) \times \mathcal{H}_l^2 \times \mathbb{C}^2 \;\middle|\; \begin{matrix} q_1 = z_2(l) \\ q_2 = z_{2x}(l) \end{matrix} \right\}. \tag{1.5}$$

Here A^c is the generator of the SCOLE system with the state space H^c. We suppressed the interval notation $(0,l)$ for the Sobolev spaces in the above formula. We will do this also in other places due to the length limitation.

Guo and Ivanov [3] have shown that for the non-uniform SCOLE model the space $\mathcal{D}(|A^c|^{\frac{1}{2}})$ is reachable using only force control. The definition of $\mathcal{D}(|A^c|^{\frac{1}{2}})$ will be given in Section 2. So far $\mathcal{D}(|A^c|^{\frac{1}{2}})$ is the largest known reachable space using L^2 inputs. An explicit description of $\mathcal{D}(|A^c|^{\frac{1}{2}})$ like in (1.5) has not been given in [3], nor were there well-posedness results in case we use this space as the state space.

In this paper, using a new approach to the well-posedness and exact controllability of coupled system developed in Weiss and Zhao [12], we show that the SCOLE model described by (1.1) and (1.2) is well-posed, regular and

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exactly controllable in any time $T > 0$ with the state space

$$\mathcal{X} = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in [\mathcal{H}^3 \cap \mathcal{H}_l^2] \times \mathcal{H}_l^1 \times \mathbb{C}^2 \;\Big|\; z_2(l) = q_1 \right\}$$

using both torque and force control in L^2. Here $\mathcal{H}_l^1(0,l) = \{h \in \mathcal{H}^1(0,l) \mid h(0) = 0\}$. The system remains regular with $y = \begin{bmatrix} -EIw_{xxx}(l,\cdot) \\ EIw_{xx}(l,\cdot) \end{bmatrix}$ as an additional output. We suspect that $\mathcal{X} = \mathcal{D}(|A^c|^{\frac{1}{2}})$, but we did not verify this.

II. Background on controllability and coupled systems

For the background on admissible control and observation operators and controllability of infinite-dimensional systems, we refer to Tucsnak and Weiss [11], and for the background on coupled systems that is needed here, we refer to Weiss and Zhao [12]. For easy reference we reproduce below several well-known results which can be found, e.g., in [11].

We need some preliminaries. Let A be the generator of a strongly continuous semigroup $\mathbb{T}$ on a Hilbert space X. Then A determines several additional Hilbert spaces: X_1 is $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, X_2 is $\mathcal{D}(A^2)$ with the norm $\|z\|_2 = \|(\beta I - A)^2 z\|$, and X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1} z\|$, where $\beta \in \rho(A)$ is fixed. The spaces X_1, X_2 and X_{-1} are independent of the choice of β, since different values of β lead to equivalent norms on X_1, X_2 and X_{-1}. We have $X_2 \subset X_1 \subset X \subset X_{-1}$, densely and with continuous embeddings. We can continuously extend A to a bounded operator from X to X_{-1}, still denoted by A. The semigroup generated by this extended A is the extension of $\mathbb{T}$ to X_{-1}, still denoted by $\mathbb{T}$. If $X_1^d = \mathcal{D}(A^*)$ with the norm $\|z\|_1^d = \|(\overline{\beta} I - A^*)z\|$, then X_{-1} may be regarded as the dual of X_1^d.

Proposition 2.1: Let H be a Hilbert space and $A_0 : \mathcal{D}(A_0) \to H$ be a strictly positive operator. Denote $H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}})$ with the graph norm. $H_{-\frac{1}{2}}$ is the dual of $H_{\frac{1}{2}}$ with respect to the pivot space H. We define another Hilbert space $X = H_{\frac{1}{2}} \times H$ with the inner product

$$\left\langle \begin{bmatrix} w_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ v_2 \end{bmatrix} \right\rangle_X = \langle A_0^{\frac{1}{2}} w_1, A_0^{\frac{1}{2}} w_2 \rangle + \langle v_1, v_2 \rangle,$$

and another operator A by

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}}).$$

Then A is skew-adjoint on X and $0 \in \rho(A)$. Furthermore

$$X_1 = H_1 \times H_{\frac{1}{2}}, \qquad X_{-1} = H \times H_{-\frac{1}{2}}.$$

If (ϕ_k) ($k \in \Lambda$, Λ countable) is a Riesz basis in the Hilbert space X, we denote by $(\tilde{\phi}_k)$ ($k \in \Lambda$) the biorthogonal sequence to (ϕ_k). Every $z \in X$ can be represented as $z = \sum_{k \in \Lambda} z_k \phi_k$, where $z_k = \langle z, \tilde{\phi}_k \rangle$ and $(z_k) \in l^2(\Lambda)$.

Let $\mathbb{T}$ be a diagonalisable semigroup on X with generator A. This means that there exists a Riesz basis (ϕ_k) ($k \in \Lambda$) in X such that

$$\mathbb{T}_t z = \sum_{k \in \Lambda} e^{\lambda_k t} z_k \phi_k . \qquad (2.1)$$

The generator of $\mathbb{T}$ is given by

$$Az = \sum_{k \in \Lambda} \lambda_k z_k \phi_k ,$$

$$\mathcal{D}(A) = \left\{ z \in X \;\Bigg|\; \sum_{k \in \Lambda} |\lambda_k z_k|^2 < \infty \right\}.$$

For $\alpha \geq 0$ we define

$$|A|^\alpha : \mathcal{D}(|A|^\alpha) \to X$$

by

$$|A|^\alpha z = \sum_{k \in \Lambda} |\lambda_k|^\alpha z_k \phi_k,$$

$$\mathcal{D}(|A|^\alpha) = X_\alpha = \left\{ z \in X \;\Bigg|\; \sum_{k \in \Lambda} |\lambda_k|^{2\alpha} |z_k|^2 < \infty \right\}.$$

The space X_α is a Hilbert space with the norm

$$\|z\|_\alpha = \|(I + |A|)^\alpha z\| . \qquad (2.2)$$

We define $X_{-\alpha}$ as the dual of X_α with respect to the pivot space X. Note that for $\alpha = 1$ we obtain X_1 and X_{-1} as defined earlier and $|A|^\alpha$ commutes with $\mathbb{T}_t$. It is clear that $\mathbb{T}$ can be extended (or restricted) to X_α for any $\alpha \in \mathbb{R}$. The formula (2.1) for $\mathbb{T}$ remains the same, with $(|\lambda_k|^\alpha z_k) \in l^2(\Lambda)$. The generator of $\mathbb{T}$ acting on X_α is an extension (or restriction) of A with $\mathcal{D}(A) = X_{\alpha+1}$ and $\mathcal{D}(A^2) = X_{\alpha+2}$.

In the sequel we recall some results about coupled systems from our paper [12].

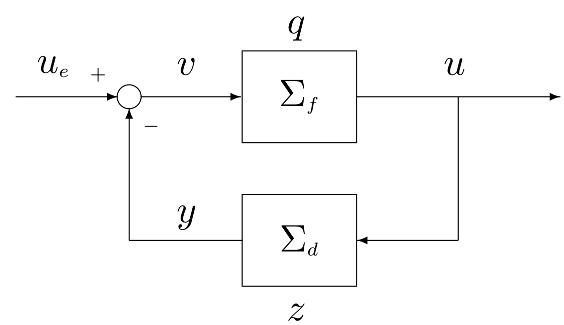

Fig. 1. A coupled system Σ_{cs} consisting of an infinite-dimensional system Σ_d and a finite-dimensional system Σ_f, connected in feedback.

Consider a coupled system Σ_c, in which an infinite-dimensional system Σ_d is connected to a finite-dimensional system Σ_f as shown in Figure 1. The external world interacts with the coupled system Σ_c via the finite-dimensional part Σ_f, which receives the input $v = u_e - y$, where u_e is the input of Σ_c and the signal y comes from Σ_d. The system Σ_f sends out the output u, which is also the output of the coupled system Σ_c. The equations of Σ_f are

$$\begin{cases} \dot{q}(t) = aq(t) + bu_e(t) - by(t), & (2.3) \\ u(t) = cq(t), & (2.4) \end{cases}$$

where $a \in \mathbb{C}^{n \times n}$, $b \in \mathbb{C}^{n \times m}$, $c \in \mathbb{C}^{m \times n}$, and $q(t) \in \mathbb{C}^n$ is the state of the finite-dimensional subsystem at the time t.

Let p be a function defined on some domain in $\mathbb{C}$ that contains a right half-plane, with values in a normed space. We say that p is *strictly proper* if

$$\lim_{\mathrm{Re}\, s \to \infty} \|p(s)\| = 0, \qquad \text{uniformly with respect to } \ \mathrm{Im}\, s.$$

A linear system is called *strictly proper* if its transfer function is strictly proper.

We assume that Σ_d belongs to an abstract class of infinite-dimensional systems called *strictly proper with an integrator* (SPI) systems, introduced in [12].

Definition 2.2: An SPI system Σ_d with input space U, state space X and output space Y (all Hilbert spaces) is determined by three operators A, B, C and a transfer function $\mathbf{G}$, which satisfy the following assumptions:

(a) A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X. The spaces X_1, X_2 and X_{-1} are as introduced at the beginning of this section.

(b) $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$.

(c) $X_2 \subset \mathcal{D}(C) \subset X_1$ and $C : \mathcal{D}(C) \to Y$ is such that its restriction to $\mathcal{D}(A^2)$ is in $\mathcal{L}(X_2, Y)$ and it is an admissible observation operator for $\mathbb{T}$ restricted to X_1.

(d) For some (hence, for every) $s, \beta \in \rho(A)$ we have

$$(sI - A)^{-1}(\beta I - A)^{-1} BU \subset \mathcal{D}(C).$$

(e) We have $\mathbf{G} : \rho(A) \to \mathcal{L}(U, Y)$. For every $s, \beta \in \rho(A)$ we have

$$\mathbf{G}(s) - \mathbf{G}(\beta) = C[(sI - A)^{-1} - (\beta I - A)^{-1}]B.$$

(f) The function $\frac{1}{s}\mathbf{G}(s)$ is strictly proper.

The operators A, B, C are called the *semigroup generator*, the *control operator* and the *observation operator* of Σ_d. $\mathbf{G}$ is called the *transfer function* of Σ_d.

We make some simple comments on SPI systems. The dynamic behavior of Σ_d is assumed to be described similarly as for a system node (as defined in Staffans [9]):

$$\dot{z}(t) = Az(t) + Bu(t), \qquad y(t) = C\&D \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}. \quad (2.5)$$

Here $C\&D$ is defined similarly as for a system node: for some $\beta \in \rho(A)$,

$$C\&D \begin{bmatrix} x \\ u \end{bmatrix} = C[x - (\beta I - A)^{-1} Bu] + \mathbf{G}(\beta) u, \quad (2.6)$$

with the domain $\mathcal{D}(C\&D)$:

$$\left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in X \times U \;\middle|\; x - (\beta I - A)^{-1} Bu \in \mathcal{D}(C) \right\}. \quad (2.7)$$

For a system node, we would have X_1 in place of $\mathcal{D}(C)$ in (2.7), so that for an SPI system, $\mathcal{D}(C\&D)$ is smaller. It is easy to see that $C\&D$ (and its domain) is independent of the choice of β appearing in the formulas. It is also easy to see that we have the following relation between $C\&D$ and $\mathbf{G}$:

$$\mathbf{G}(s) = C\&D \begin{bmatrix} (sI - A)^{-1} B \\ I \end{bmatrix} \qquad \forall\, s \in \rho(A). \quad (2.8)$$

The equations (2.5) have classical solutions if u is of class $\mathcal{H}^2_{\mathrm{loc}}$ and the initial conditions of z and u are compatibe. In this case, y is continuous.

Now consider the situation when Σ_d is an SPI system with input and output space $\mathbb{C}^m$ and state space H^d, semigroup $\mathbb{T}$ and transfer function $\mathbf{G}$. We can consider the coupled system Σ_c as a cascaded system Σ_{casc} (the open loop system in Figure 2) with a feedback. The input of Σ_{casc} is v from Figure 1, and its outputs are u and y. The system Σ_{casc} is described by:

$$\begin{cases} \dot{q}(t) = aq(t) + bv(t), & (2.9) \\ u(t) = cq(t), & (2.10) \\ \dot{z}(t) = Az(t) + Bu(t), & (2.11) \\ y(t) = C\&D \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}. & (2.12) \end{cases}$$

Here $z(t)$ is the state of Σ_d, so that $z(t) \in H^d$.

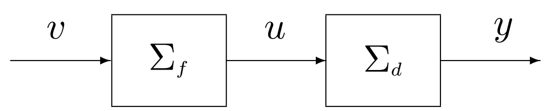

Fig. 2. A cascaded system Σ_{casc} consisting of an infinite-dimensional system Σ_d and a finite-dimensional system $\Sigma_f = (a, b, c)$.

It is easy to show that the above equations (2.9)–(2.11) give rise to a strongly continuous semigroup $\mathcal{S}$ on the state space $H^d \times \mathbb{C}^n$, whose generator $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{bmatrix} A & Bc \\ 0 & a \end{bmatrix},$$

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in H^d \times \mathbb{C}^n \;\middle|\; Az + Bcq \in H^d \right\}. \quad (2.13)$$

In fact, $H^d \times \mathbb{C}^n$ is not a good choice for the state space of Σ_{casc}, because it is too large and the system may not be well-posed with this state space. However, we have shown in [12] that Σ_{casc} (and also Σ_c) is a well-posed system with the state space $\mathcal{X} = \mathcal{D}(\mathcal{A})$, which is a Hilbert space with the graph norm of $\mathcal{A}$.

For the well-posedness and controllability of the coupled system Σ_c, we have the following theorem from [12]:

Theorem 2.3: Let Σ_d be an SPI system described by (2.11)-(2.12), with input space $\mathbb{C}^m$, state space H^d, output space $\mathbb{C}^m$, semigroup generator A, control operator B, observation operator C and transfer function $\mathbf{G}$. Let a, b, c be matrices as in (2.3)-(2.4). Then the coupled system Σ_c from Figure 1 described by (2.3), (2.4), (2.11) and (2.12), with input u_e, state $\left[\begin{smallmatrix} z \\ q \end{smallmatrix}\right]$ and output u, is well-posed with the state space $\mathcal{X} = \mathcal{D}(\mathcal{A})$ from (2.13). The coupled system remains well-posed also with y as an additional output. Moreover, Σ_c is regular, with feedthrough operator zero.

The semigroup of Σ_c, denoted by $\mathcal{S}^c$, is generated by

$$\mathcal{A}^c \begin{bmatrix} z \\ q \end{bmatrix} = \begin{bmatrix} Az + Bcq \\ aq - b[C\&D] \left[\begin{smallmatrix} z \\ cq \end{smallmatrix}\right] \end{bmatrix},$$

$$\mathcal{D}(\mathcal{A}^c) = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in \mathcal{X} \;\middle|\; \mathcal{A}^c \begin{bmatrix} z \\ q \end{bmatrix} \in \mathcal{X} \right\}.$$

If $C \in \mathcal{L}(H_1^d, \mathbb{C}^m)$, then the operators $\mathcal{S}_t^c$ can be extended to form a strongly continuous semigroup on the space $H^d \times \mathbb{C}^n$. The generator of this extension, denoted by $\tilde{\mathcal{A}}^c$, is given by the same formula as $\mathcal{A}^c$ but it has the larger domain $\mathcal{D}(\tilde{\mathcal{A}}^c) = \mathcal{X}$.

Now assume additionally the following:
(i) (A, B) is exactly controllable in time T_0;
(ii) (a, b) is controllable;
(iii) $cb \in \mathbb{C}^{m\times m}$ is invertible;
(iv) Denote $a^\times(\beta) = a + b(cb)^{-1}c(\beta I - a)$. There exists $\beta \in \rho(A)$ such that A^* and $a^\times(\beta)^*$ have no common eigenvalue.

Then Σ_c is exactly controllable in any time $T > T_0$ (on the state space $\mathcal{X}$).

III. Some background on boundary control systems

This section is an introduction to boundary control systems, without any well-posedness assumptions. For proofs and more details we refer to Tucsnak and Weiss [11]. The general theory of such systems started with Fattorini [1] and it was significantly developed by Salamon [8].

Systems described by linear partial differential equations with non-homogeneous boundary conditions often appear in the following, quite different looking form:

$$\dot{z}(t) = Lz(t), \quad Gz(t) = u(t), \quad y(t) = Kz(t). \quad (3.1)$$

Often (but not necessarily) L is a differential operator and G is a boundary trace operator. We assume that U, Z, X and Y are complex Hilbert spaces such that

$$Z \subset X,$$

with continuous embedding. We call U the *input space*, Z the *solution space*, X the *state space* and Y the *output space*.

Definition 3.1: A *boundary control system* on U, Z, X and Y is a triple of operators $\Sigma_b = (L, G, K)$, where

$$L \in \mathcal{L}(Z, X), \qquad G \in \mathcal{L}(Z, U), \qquad K \in \mathcal{L}(Z, Y),$$

if there exists a $\beta \in \mathbb{C}$ such that the following holds:
(i) G is onto,
(ii) Ker G is dense in X,
(iii) $\beta I - L$ restricted to Ker G is onto,
(iv) Ker $(\beta I - L) \cap$ Ker $G = \{0\}$.

Three operators in this definition determine a system via the equations (3.1). With the assumptions of the last definition, we introduce the Hilbert space X_1 and the operator A by

$$X_1 = \text{Ker } G, \qquad A = L|_{X_1}. \quad (3.2)$$

Obviously, X_1 is a closed subspace of Z and $A \in \mathcal{L}(X_1, X)$. Condition (iii) means that $\beta I - A$ is onto. Condition (iv) means that Ker $(\beta I - A) = \{0\}$. Thus, (iii) and (iv) together imply that $\beta \in \rho(A)$, so that

$$(\beta I - A)^{-1} \in \mathcal{L}(X).$$

In fact, $(\beta I - A)^{-1} \in \mathcal{L}(X, X_1)$, so that the norm on X_1 is equivalent to the norm

$$\|z\|_1 = \|(\beta I - A)z\|,$$

which in turn is equivalent to the graph norm of A. We define the Hilbert space X_{-1} as the completion of X with respect to the norm

$$\|z\|_{-1} = \|(\beta I - A)^{-1}z\|.$$

It is easy to see that this space is independent of the choice of $\beta \in \rho(A)$.

Proposition 3.2: Let $\Sigma_b = (L, G, K)$ be a boundary control system on U, Z, X and Y. Let A and X_{-1} be as introduced earlier. Then there exists a unique operator $B \in \mathcal{L}(U, X_{-1})$ such that

$$L = A + BG, \quad (3.3)$$

where A is regarded as an operator from X to X_{-1}. For every $\beta \in \rho(A)$ we have that $(\beta I - A)^{-1}B \in \mathcal{L}(U, Z)$ and

$$G(\beta I - A)^{-1}B = I, \quad (3.4)$$

so that in particular, B is bounded from below.

For the proof see Tucsnak and Weiss [11, Proposition 10.1.2].

Remark 3.3: The following fact is an easy consequence of Proposition 3.2: For every $v \in U$ and every $\beta \in \rho(A)$, the vector $z = (\beta I - A)^{-1}Bv$ is the unique solution of the "abstract elliptic problem"

$$Lz = \beta z, \qquad Gz = v.$$

Definition 3.4: With the notation of Definition 3.1 and Proposition 3.2, we define $C \in \mathcal{L}(X_1, Y)$ as the restriction of K to X_1. Then the *generating triple* of Σ_b is (A, B, C). The *transfer function* of Σ_b is the $\mathcal{L}(U, Y)$-valued function $\mathbf{G}$ defined on $\rho(A)$ by the formula

$$\mathbf{G}(s) = K(sI - A)^{-1}B. \quad (3.5)$$

By the resolvent identity, for any $s, \beta \in \rho(A)$, the difference $(sI - A)^{-1} - (\beta I - A)^{-1}$ maps X_{-1} into X_1, so that (3.5) implies

$$\mathbf{G}(s) - \mathbf{G}(\beta) = C\left[(sI - A)^{-1} - (\beta I - A)^{-1}\right]B. \quad (3.6)$$

It is now clear that if A is the generator of a strongly continuous semigroup on X, then A, B, C and $\mathbf{G}$ determine a system node in the sense of Staffans [9].

Remark 3.5: As a consequence of Proposition 3.2, the first two equations in (3.1) can be rewritten equivalently as a single equation, namely

$$\dot{z}(t) = Az(t) + Bu(t), \quad \text{with} \quad \dot{z}(t) \in X. \quad (3.7)$$

IV. The beam subsystem on the energy state space

To obtain the well-posedness and exact controllability results for the SCOLE model Σ_c described by (1.1) and (1.2), we follow the framework of Theorem 2.3. We decompose Σ_c into an infinite-dimensional system Σ_d (the clamped flexible beam) coupled with a finite-dimensional system Σ_f (the rigid body). We model and analyse the beam subsystem first.

The clamped flexible beam Σ_d that we extract from Σ_c is described by the following Euler-Bernoulli equation with boundary control and boundary observation:

$$\begin{cases} \rho w_{tt}(x,t) + EIw_{xxxx}(x,t) = 0, \\ (x,t) \in (0,l) \times [0,\infty), \\ w(0,t) = 0, \quad w_x(0,t) = 0, \\ w_t(l,t) = u_1(t), \quad w_{xt}(l,t) = u_2(t), \\ y_1(t) = -EIw_{xxx}(l,t), \\ y_2(t) = EIw_{xx}(l,t), \end{cases} \tag{4.1}$$

where $u = \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right]$ is the input of Σ_d (the transverse velocity and angular velocity of the nacelle). $y = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$ is the output of Σ_d (the force and the torque at the top of the tower). The other notation is as in (1.1).

In order to reformulate the system Σ_d as a boundary control system like (3.1), we use the auxiliary functions, which are the first two variables of z^c in (1.3):

$$z_1(x,t) = w(x,t), \qquad z_2(x,t) = w_t(x,t). \tag{4.2}$$

Then (4.1) can be written as:

$$\begin{cases} \dot{z}_1(x,t) = z_2(x,t), \\ \dot{z}_2(x,t) = -\frac{EI}{\rho} z_{1xxxx}(x,t), \\ z_1(0,t) = 0, \qquad z_{1x}(0,t) = 0, \\ z_2(l,t) = u_1(t), \qquad z_{2x}(l,t) = u_2(t), \\ y_1(t) = -EIz_{1xxx}(l,t), \\ y_2(t) = EIz_{1xx}(l,t). \end{cases} \tag{4.3}$$

We denote $z = \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$, and similarly for u and y. The natural state space of Σ_d is

$$X = \mathcal{H}_l^2(0,l) \times L^2[0,l],$$

where $\mathcal{H}_l^2(0,l)$ is defined as after (1.3). We define the norm on X as follows:

$$\|z\|^2 = EI\|z_1\|^2_{\mathcal{H}_l^2} + \rho\|z_2\|^2_{L^2}, \tag{4.4}$$

where

$$\|z_1\|^2_{\mathcal{H}_l^2} = \int_0^l |z_{1xx}|^2 \mathrm{d}x, \quad \|z_2\|^2_{L^2} = \int_0^l |z_2|^2 \mathrm{d}x. \tag{4.5}$$

The physical energy in the system Σ_d is $\frac{1}{2}\|z\|^2$.

We introduce the space $Z \subset X$ by

$$Z = \left[\mathcal{H}^4(0,l) \cap \mathcal{H}_l^2(0,l)\right] \times \mathcal{H}_l^2(0,l). \tag{4.6}$$

We define the operators $L : Z \to X, \quad G, K : Z \to \mathbb{C}^2$ by

$$L = \begin{bmatrix} 0 & I \\ -\frac{EI}{\rho}\frac{d^4}{dx^4} & 0 \end{bmatrix}, \quad G\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_2(l) \\ z_{2x}(l) \end{bmatrix},$$

$$K\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \left[-EIz_{1xxx}(l) EIz_{1xx}(l)\right].$$

With the above notation, (4.3) can be written as follows:

$$\dot{z} = Lz, \quad Gz = u, \quad y = Kz. \tag{4.7}$$

Such equations determine a boundary control system if L, G and K satisfy certain conditions, see Section III. Now we prove that this is indeed the case. Before we do this, we introduce the system operator by $A = L|_{\text{Ker } G}$. It is easy to verify that

$$\mathcal{D}(A) = \text{Ker } G = \left[\mathcal{H}^4(0,l) \cap \mathcal{H}_l^2(0,l)\right] \times \mathcal{H}_0^2(0,l) \tag{4.8}$$

where $\mathcal{H}_0^2(0,l) = \{h \in \mathcal{H}^2(0,l) \mid h(0) = h(l) = 0,\ h_x(0) = h_x(l) = 0\}$. The norm on $\mathcal{H}_0^2(0,l)$ is defined by $\|f\|_{\mathcal{H}_0^2} = \|f''\|_{L^2}$.

Proposition 4.1: The beam system (L, G, K) is a boundary control system.

Proof. It is clear that G is onto. The space $\text{Ker } G$ is dense in X because $\mathcal{H}^4(0,l) \cap \mathcal{H}_l^2(0,l)$ is dense in $\mathcal{H}_l^2(0,l)$, and $\mathcal{H}_0^2(0,l)$ is dense in $L^2[0,l]$. The last two conditions in the definition of a boundary control system are equivalent to the fact that $sI - A$ is invertible for some $s \in \mathbb{C}$. We show that for every $s > 0$, $sI - A$ is invertible, or equivalently, for every $q \in X$, the following equation has a unique solution $z \in \mathcal{D}(A)$:

$$(sI - A)z = q.$$

The above equation is equivalent to

$$\begin{cases} \frac{EI}{\rho} z_{1xxxx} + s^2 z_1 = sq_1 + q_2, \\ z_1(0) = 0, \quad z_{1x}(0) = 0, \\ z_1(l) = \frac{1}{s}q_1(l), \quad z_{1x}(l) = \frac{1}{s}q_{1x}(l), \\ z_2 = sz_1 - q_1. \end{cases} \tag{4.9}$$

Remember that $s > 0$. First we show that the corresponding homogeneous equation, where we replace $sq_1 + q_2$ in the first equation of (4.9) with zero but leave the other equations unchanged, has a unique solution $z_h = \left[\begin{smallmatrix} z_{h1} \\ z_{h2} \end{smallmatrix}\right] \in \mathcal{D}(A)$. Solving this homogeneous equation, we get

$$z_{h1}(x) = c_1 \cosh mx \sin mx - c_1 \sinh mx \cos mx + c_2 \sinh mx \sin mx,$$

and $z_{h2} = sz_{h1} - q_1$, where

$$m = \frac{s}{2}\sqrt{\frac{\rho}{EI}}, \quad c_1 = \frac{d \cdot q_2(l) - bm \cdot q_1(l)}{(ad - bc)ms},$$

$$c_2 = \frac{am \cdot q_1(l) - c \cdot q_2(l)}{(ad - bc)ms}, \quad a = 2\sinh ml \sin ml,$$

$$b = \sinh ml \cos ml + \cosh ml \sin ml,$$

$$c = \cosh ml \sin ml - \sinh ml \cos ml, \quad d = \sinh ml \sin ml.$$

The above solution only makes sense if $ad - bc \neq 0$. Since $ad - bc = \sinh^2(ml) - \sin^2(ml)$ and $\sinh \alpha > |\sin \alpha|$ for any $\alpha > 0$, we obtain that $ad - bc > 0$, so that z_h exists and it is unique.

Similarly it can be shown that the non-homogeneous equation corresponding to (4.9), where we replace $q_1(l)$ and $q_{1x}(l)$ with zero, has a solution $z_n \in \mathcal{D}(A)$. Hence $z = z_h + z_n$ is a solution of (4.9). z is unique because if (4.9) had another solution $\tilde{z}$, then $z - \tilde{z}$ would be a solution of the homogeneous equation with zero boundary conditions (which is 0), hence $z - \tilde{z} = 0$. Therefore $sI - A$ is invertible for $s > 0$. ■

Since $sI - A$ is invertible for $s > 0$, we can introduce the space X_{-1} as the completion of X with respect to the norm

$\|x\|_{-1} = \|(sI-A)^{-1}x\|$. We can extend A to a bounded operator from X to X_{-1}, still denoted by A. We know from Section III that there exists a unique $B : \mathbb{C}^2 \to X_{-1}$ such that $L = A + BG$. According to Remark 3.5, the state trajectories of Σ_d from (4.1) or (4.7) satisfy (3.7).

We decompose the state space X into 2 parts: the null-space of A, X_n, and its orthogonal complement X_r. By a simple computation, we get

$$X_n = \operatorname{Ker} A = \left\{ \begin{bmatrix} ax^3 + bx^2 \\ 0 \end{bmatrix} \,\middle|\, a, b \in \mathbb{C} \right\}. \tag{4.10}$$

Now we determine $X_r = X_n^{\perp}$. Let $z = \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] \in X_r$, then $z_1 \in \mathcal{H}_l^2(0,l)$, $z_2 \in L^2[0,l]$. The condition $\langle q, z\rangle = 0$ for all $q \in X_n$ is equivalent to $\langle q_1, z_1\rangle_{\mathcal{H}_l^2} = 0$ for all q_1 of the form

$$q_1(x) = ax^3 + bx^2, \quad \text{where} \quad a, b \in \mathbb{C}. \tag{4.11}$$

For q_1 as above and for every $h \in \mathcal{H}_l^2(0,l)$ we have, using twice integration by parts,

$$\langle q_1, h\rangle_{\mathcal{H}_l^2} = q_{1xx}(l)\cdot \overline{h}_x(l) - \left[q_{1xxx}\cdot \overline{h}\right]_0^l + \int_0^l q_{1xxxx}\cdot \overline{h}\,\mathrm{d}x.$$

Using that $q_{1xxxx} = 0$ and $h(0) = 0$, we get

$$\langle q_1, h\rangle_{\mathcal{H}_l^2} = q_{1xx}(l)\cdot \overline{h}_x(l) - q_{1xxx}(l)\cdot \overline{h}(l).$$

Therefore we have for z_1 in place of h, $q_{1xx}(l)\cdot \overline{z}_{1x}(l) - q_{1xxx}(l)\cdot \overline{z}_1(l) = 0$. Clearly $q_{1xx}(l)$ and $q_{1xxx}(l)$ can be any complex numbers (in fact $q_{1xx}(l) = 6al + 2b$ and $q_{1xxx} = 6a$). Thus $\langle q_1, z_1\rangle = 0$ for all q_1 as in (4.11) is equivalent to

$$z_1(l) = 0, \qquad z_{1x}(l) = 0.$$

Therefore $z_1 \in \mathcal{H}_0^2(0,l)$, where $\mathcal{H}_0^2(0,l)$ is defined as after (4.8). Thus we get

$$X_r = \mathcal{H}_0^2(0,l) \times L^2[0,l].$$

We denote by A_r the restriction of A to X_r. Then

$$\mathcal{D}(A_r) = \left[\mathcal{H}^4(0,l)\cap \mathcal{H}_0^2(0,l)\right] \times \mathcal{H}_0^2(0,l).$$

It is easy to see that X_r is invariant under A, or equivalently, $A_r z \in X_r$, $\forall z \in \mathcal{D}(A_r)$. We can decompose

$$A_r = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \tag{4.12}$$

where

$$A_0 h = \frac{EI}{\rho} h_{xxxx}, \quad \mathcal{D}(A_0) = \mathcal{H}^4(0,l)\cap \mathcal{H}_0^2(0,l). \tag{4.13}$$

Note that A_r corresponds to the equations of a beam clamped at both ends.

Proposition 4.2: A_0 is a strictly positive densely defined operator on $H = L^2[0,l]$, with compact resolvents. We have $\mathcal{D}(A_0^{\frac{1}{2}}) = \mathcal{H}_0^2(0,l)$.

For a proof see, e.g., [11, Example 3.4.13]. This implies that $\sigma(A_0)$ consists of isolated positive eigenvalues, which converge to ∞. Moreover, there exists in H an orthonormal basis consisting of eigenvectors of A_0 (see, e.g., [11, Proposition 3.2.12]).

Proposition 4.3: A_r is skew-adjoint on X_r and A is skew-adjoint on X.

Proof. As $A_0 > 0$, according to Proposition 2.1 A_r is skew-adjoint on X_r and $0 \in \rho(A_r)$. According to the decomposition $X = X_n \oplus X_r$ into $A-$invariant subspaces, it follows that A is skew-adjoint on X. ■

We define $C = K|_{\operatorname{Ker} G}$, so that $C = \left[\begin{smallmatrix} C_1 \\ C_2 \end{smallmatrix}\right]$, where for all $h = \left[\begin{smallmatrix} h_1 \\ h_2 \end{smallmatrix}\right] \in \operatorname{Ker} G$

$$C_1 h = -EI h_{1xxx}(l), \quad C_2 h = EI h_{1xx}(l). \tag{4.14}$$

Proposition 4.4: $B^* = C$.

For the proof, please see the journal version of this paper, Zhao and Weiss [13].

Remark 4.5: If $q_1, q_2 \in \mathbb{C}$, then the vector $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = (I - A)^{-1} B \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ is the unique solution of the "abstract elliptic problem" from Remark 3.3:

$$(I - L)\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = 0, \qquad G\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

As remarked after Definition 3.4, propositions 4.1 and 4.3 imply that the beam system Σ_d is a system node with state space X, skew-adjoint semigroup generator A, control operator B and observation operator $C = B^*$. However this system node is not an SPI system because B is not admissible on the state space X, see [13] for the proof. Thus, in order to use Theorem 2.3, we have to extend the state space.

V. The beam subsystem with state space $X_{-\frac{1}{2}}$

In the sequel, we suppress the notation $(0,l)$ of the standard Sobolev spaces $\mathcal{H}^m(0,l)$ and $\mathcal{H}_0^m(0,l)$. $\mathcal{H}_0^m$ is defined similarly as after (4.8). We need three theorems about interpolation spaces. The following two theorems are taken from Lions and Magenes [4, p. 43, 64]:

Theorem 5.1: Let $s_1 > s_2$, $s_1 > 0$, $0 < \theta < 1$. We have (with equivalent norms)

$$[\mathcal{H}^{s_1}, \mathcal{H}^{s_2}]_\theta = \mathcal{H}^{(1-\theta)s_1 + \theta s_2}.$$

Here, $[\mathcal{H}^{s_1}, \mathcal{H}^{s_2}]_\theta$ denotes the θ-interpolation of $\mathcal{H}^{s_1}$ and $\mathcal{H}^{s_2}$ (see [4] for the definition).

Theorem 5.2: Let $s_1 > s_2 \geq 0$, s_1 and $s_2 \neq$ integer $+\frac{1}{2}$. If $(1-\theta)s_1 + \theta s_2 \neq$ integer $+\frac{1}{2}$, then

$$[\mathcal{H}_0^{s_1}, \mathcal{H}_0^{s_2}]_\theta = \mathcal{H}_0^{(1-\theta)s_1 + \theta s_2}$$

(with equivalent norms).

Actually, in [4] the above results are given for a more general n-dimensional domain. The following theorem is taken from Triebel [10, p. 118].

Theorem 5.3: Let Z_a, Z_b be Banach spaces such that $\{Z_a, Z_b\}$ is an interpolation couple. Let V be a complemented subspace of $Z_a + Z_b$ whose projection P restricted to Z_a is a bounded operator on Z_a, and similarly on Z_b.

Let $0 < \theta < 1$. Then $\{Z_a \cap V, Z_b \cap V\}$ is also an interpolation couple, and

$$[Z_a \cap V, Z_b \cap V]_\theta = [Z_a, Z_b]_\theta \cap V.$$

Recall from (4.13) that $A_0 : \mathcal{D}(A_0) \to H$ is a strictly positive operator on $H = L^2[0,l]$. Denote $H_\alpha = \mathcal{D}(A_0^\alpha)$ ($\alpha \geq 0$) with the graph norm. $H_{-\alpha}$ is the dual of H_α with respect to the pivot space H. From (4.12) and Propositions 4.2 and 2.1 we know that

$$H_1 = \mathcal{H}^4 \cap \mathcal{H}_0^2, \qquad H_{\frac{1}{2}} = \mathcal{H}_0^2, \qquad X_r = H_{\frac{1}{2}} \times H,$$

$$(X_r)_1 = \mathcal{D}(A_r) = \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}}) = H_1 \times H_{\frac{1}{2}}, \quad (5.1)$$

$$(X_r)_{-1} = H \times H_{-\frac{1}{2}}. \quad (5.2)$$

According to Theorem 5.2 with $s_1 = 2$, $s_2 = 0$, $\theta = \frac{1}{2}$, we have

$$H_{\frac{1}{4}} = [H, H_{\frac{1}{2}}]_{\frac{1}{2}} = [L_2, \mathcal{H}_0^2]_{\frac{1}{2}} = \mathcal{H}_0^1. \quad (5.3)$$

Note that (by definition) the dual of $\mathcal{H}_0^1$ with respect to L^2 is $\mathcal{H}^{-1}$, i.e.

$$H_{-\frac{1}{4}} = \mathcal{H}^{-1}. \quad (5.4)$$

Recall from Section IV that $X = \mathcal{H}_l^2 \times L^2 = X_n \oplus X_r$, where $\dim X_n = 2$, the spaces $X_\alpha = \mathcal{D}(|A|^\alpha)$ (for $\alpha > 0$) were introduced before (2.2), $X_{-\alpha}$ is the dual of X_α with respect to the pivot space X, and $\mathcal{H}_l^1$ is defined as at the end of Section I.

Proposition 5.4: $X_{-\frac{1}{2}} = \mathcal{H}_l^1 \times \mathcal{H}^{-1}$.

Proof. Let ϕ_k be the eigenvectors of A_r, then

$$\begin{bmatrix} A_0^{\frac{1}{2}} & 0 \\ 0 & A_0^{\frac{1}{2}} \end{bmatrix} \phi_k = \lambda_k^{\frac{1}{2}} \phi_k = |\mu_k| \phi_k = |A_r| \phi_k \quad \forall k \in \mathbb{Z}^*.$$

Thus we get

$$|A_r| = \begin{bmatrix} A_0^{\frac{1}{2}} & 0 \\ 0 & A_0^{\frac{1}{2}} \end{bmatrix}, \quad \text{hence} \quad |A_r|^{\frac{1}{2}} = \begin{bmatrix} A_0^{\frac{1}{4}} & 0 \\ 0 & A_0^{\frac{1}{4}} \end{bmatrix}.$$

Therefore

$$(X_r)_{\frac{1}{2}} = \left\{ z \in X_r \,\middle|\, |A_r|^{\frac{1}{2}} z \in X_r \right\} = H_{\frac{3}{4}} \times H_{\frac{1}{4}}. \quad (5.5)$$

By definition $(X_r)_{-\frac{1}{2}}$ is the dual of $(X_r)_{\frac{1}{2}} = H_{\frac{3}{4}} \times H_{\frac{1}{4}}$ with respect to $X_r = H_{\frac{1}{2}} \times H$. Combining this fact with equations (5.3) and (5.4), we have

$$(X_r)_{-\frac{1}{2}} = H_{\frac{1}{4}} \times H_{-\frac{1}{4}} = \mathcal{H}_0^1 \times \mathcal{H}^{-1}.$$

Therefore we have

$$\begin{aligned} X_{-\frac{1}{2}} &= X_n \oplus (X_r)_{-\frac{1}{2}} \\ &= \left\{ \begin{bmatrix} ax^3 + bx^2 \\ 0 \end{bmatrix} \,\middle|\, a, b \in \mathbb{C} \right\} \oplus \left(\mathcal{H}_0^1 \times \mathcal{H}^{-1} \right) \\ &= \mathcal{H}_l^1 \times \mathcal{H}^{-1}. \end{aligned}$$

■

Proposition 5.5: $X_{\frac{1}{2}} = (\mathcal{H}^3 \cap \mathcal{H}_l^2) \times \mathcal{H}_0^1$.

Proof. From Theorem 5.1 with $s_1 = 4$, $s_2 = 2$ and $\theta = \frac{1}{2}$ we know that

$$[\mathcal{H}^4, \mathcal{H}^2]_{\frac{1}{2}} = \mathcal{H}^3.$$

From this and Theorem 5.3 (with $Z_a = \mathcal{H}^4$, $Z_b = \mathcal{H}^2$ and $V = \mathcal{H}_0^2$), we get

$$[\mathcal{H}^4 \cap \mathcal{H}_0^2, \mathcal{H}_0^2]_{\frac{1}{2}} = [\mathcal{H}^4, \mathcal{H}^2]_{\frac{1}{2}} \cap \mathcal{H}_0^2 = \mathcal{H}^3 \cap \mathcal{H}_0^2.$$

Therefore

$$H_{\frac{3}{4}} = [H_1, H_{\frac{1}{2}}]_{\frac{1}{2}} = [\mathcal{H}^4 \cap \mathcal{H}_0^2, \mathcal{H}_0^2]_{\frac{1}{2}} = \mathcal{H}^3 \cap \mathcal{H}_0^2. \quad (5.6)$$

Substituting (5.6) and (5.3) into (5.5), we get

$$(X_r)_{\frac{1}{2}} = (\mathcal{H}^3 \cap \mathcal{H}_0^2) \times \mathcal{H}_0^1.$$

Therefore we have

$$\begin{aligned} X_{\frac{1}{2}} &= X_n \oplus (X_r)_{\frac{1}{2}} \\ &= \left\{ \begin{bmatrix} ax^3 + bx^2 \\ 0 \end{bmatrix} \,\middle|\, a, b \in \mathbb{C} \right\} \oplus \left((\mathcal{H}^3 \cap \mathcal{H}_0^2) \times \mathcal{H}_0^1 \right). \end{aligned}$$

A simple reasoning shows that by adding functions of the form $ax^3 + bx^2$ to $\mathcal{H}^3 \cap \mathcal{H}_0^2$, we get $\mathcal{H}^3 \cap \mathcal{H}_l^2$. From here, the proposition follows. ■

Proposition 5.6: Let $\mathbb{T}$ be the semigroup generated by A on X, as introduced in Section IV. If we extend $\mathbb{T}$ to $X_{-\frac{1}{2}}$, then its generator is an extension of A (still denoted by A) with $\mathcal{D}(A) = X_{\frac{1}{2}}$ and $\mathcal{D}(A^2) = X_{\frac{3}{2}}$.

Indeed, this follows from what we said after (2.2). If we take $H^d = X_{-\frac{1}{2}}$ as the state space, we get the following results (see [13] for the proof):

Proposition 5.7: B is admissible for $\mathbb{T}$ on the state space $X_{-\frac{1}{2}}$ and (using this state space) and the pair (A, B) is exactly controllable in any time $T_0 > 0$.

Proposition 5.8: The beam subsystem is an SPI system with state space $H^d = X_{-\frac{1}{2}}$.

VI. WELL-POSEDNESS, REGULARITY AND EXACT CONTROLLABILITY OF THE SCOLE MODEL

The rigid body system Σ_f that we extract from the SCOLE model Σ_c (see (1.1) and (1.2)) is described by the following Newton-Euler equations with control and observation:

$$\left\{ \begin{array}{l} \dot{q}_1 = -\frac{1}{m} y_1 + \frac{1}{m} f, \\ \dot{q}_2 = -\frac{1}{J} y_2 + \frac{1}{J} v, \\ u_1 = q_1, \qquad u_2 = q_2. \end{array} \right. \quad (6.1)$$

For this system, the state is $q = \left[\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} w_t(l,t) \\ w_{xt}(l,t) \end{smallmatrix}\right]$, which is the last two components of z^c in (1.3). The inputs are $f - y_1$ and $v - y_2$. $u = \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right]$ is the output of both Σ_f and Σ_c. It is easy to see that this system is a particular case of the finite-dimensional subsystem in Theorem 2.3 with $a = 0$, $b = \left[\begin{smallmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{smallmatrix}\right]$ (using both torque and force control) and $c = I$. It is clear that (a, b) is controllable.

Theorem 6.1: The SCOLE model Σ_c described by (1.1) and (1.2) is well-posed, regular, and exactly controllable in any time $T > 0$ with the state space

$$\mathcal{X} = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in \left[\mathcal{H}^3 \cap \mathcal{H}_l^2 \right] \times \mathcal{H}_l^1 \times \mathbb{C}^2 \;\middle|\; z_2(l) = q_1 \right\}$$

when using both torque and force control in L^2. It remains regular with $y = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$ from (4.1) as an additional output.

Proof. From Proposition 5.8 we know that the beam subsystem Σ_d is an SPI system with state space $H^d = X_{-\frac{1}{2}}$. From the descriptions of Σ_c, Σ_d and Σ_f, it is clear that they fit into the framework of Theorem 2.3. Therefore, by

Theorem 2.3, Σ_c (with input u_e and output $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$) is well-posed and regular with the state space

$$\mathcal{X} = \mathcal{D}(\mathcal{A}) = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in X_{-\frac{1}{2}} \times \mathbb{C}^2 \;\middle|\; Az + Bcq \in X_{-\frac{1}{2}} \right\},$$

where $\mathcal{A}$ is the generator of the cascaded system in the state space $X_{-\frac{1}{2}} \times \mathbb{C}^2$.

From Proposition 5.7 we also know that Σ_d is exactly controllable in any time $T_0 > 0$ with the state space $H^d = X_{-\frac{1}{2}}$ using both velocity and angular velocity control. Thus assumption (i) of Theorem 2.3 is satisfied. From the beginning of this section, we know that (a, b) is controllable, so that assumption (ii) of Theorem 2.3 is satisfied. Since $cb = \left[\begin{smallmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{smallmatrix}\right]$ is invertible, assumption (iii) is also satisfied. As

$$a^\times(\beta) = \beta I, \qquad \beta \in \rho(A),$$

we know that $a^\times(\beta)^*$ and A^* have no common eigenvalues, which is assumption (iv). So far all the assumptions of Theorem 2.3 are satisfied. Thus the coupled system Σ_c is exactly controllable in any time $T > 0$ with the state space $\mathcal{X}$.

Now we determine $\mathcal{X}$. Recall that $c = I$. Take $z \in X_{-\frac{1}{2}}$ and $q \in \mathbb{C}^2$. The fact that $Az + Bq \in X_{-\frac{1}{2}}$ is equivalent to $(A - I)z + Bq \in X_{-\frac{1}{2}}$, which is equivalent to

$$z - (I - A)^{-1} Bq \in X_{\frac{1}{2}}.$$

Thus $\mathcal{X}$ is

$$\left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in X_{-\frac{1}{2}} \times \mathbb{C}^2 \;\middle|\; z - (I - A)^{-1} Bq \in X_{\frac{1}{2}} \right\}. \quad (6.2)$$

Take $\begin{bmatrix} z \\ q \end{bmatrix} \in \mathcal{X}$. Let $\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \in X_{\frac{1}{2}}$ be such that

$$z = \gamma + (I - A)^{-1} Bq.$$

Define $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = (I - A)^{-1} B \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$. It is clear that $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in X$, which means that $g_1(0) = 0$ and $g_{1x}(0) = 0$. According to Remark 4.5, g_1, g_2 are the solution of

$$\begin{cases} g_1 - g_2 = 0, & (6.3) \\ g_{1xxxx} + g_2 = 0, & (6.4) \\ g_2(l) = q_1, \qquad g_{2x}(l) = q_2, & (6.5) \end{cases}$$

which is equivalent to $g_2 = g_1$ and $g_{1xxxx} + g_1 = 0$ subject to $g_1(l) = q_1$, $g_{1x}(l) = q_2$, $g_1(0) = 0$ and $g_{1x}(0) = 0$. Thus, g_1 is the solution of a fourth order ODE with four boundary conditions. It is easy to see that

$$g_1 = g_2 \in C^\infty[0, l] \subset \mathcal{H}^3.$$

Combing this fact, the boundary conditions of g_1 and Proposition 5.5, we get that

$$z \in \left[\mathcal{H}^3(0, l) \cap \mathcal{H}_l^2(0, l)\right] \times \mathcal{H}_l^1.$$

From Proposition 5.5 we know that $\gamma_2(l) = 0$. Hence, from equation (6.5) $z_2(l) = g_2(l) = q_1$. Thus, we have proved that

$$\mathcal{X} \subset \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in [\mathcal{H}^3 \cap \mathcal{H}_l^2] \times \mathcal{H}_l^1 \times \mathbb{C}^2 \;\middle|\; z_2(l) = q_1 \right\}. \quad (6.6)$$

Now we prove the reversed inclusion. Take

$$\begin{bmatrix} z \\ q \end{bmatrix} \in \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in [\mathcal{H}^3 \cap \mathcal{H}_l^2] \times \mathcal{H}_l^1 \times \mathbb{C}^2 \;\middle|\; z_2(l) = q_1 \right\}.$$

Consider $z - (I - A)^{-1} Bq = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$, where $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ is the solution of (6.3)-(6.5). So $g_1 = g_2 \in C^\infty[0, l] \subset \mathcal{H}^3$. We also know that $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in [\mathcal{H}^3(0, l) \cap \mathcal{H}_l^2(0, l)] \times \mathcal{H}_l^1(0, l)$ and $z_2(l) = q_1$. Combing these facts with equation (6.5), we get $\left(z - (I - A)^{-1} Bq\right) \in [\mathcal{H}^3(0, l) \cap \mathcal{H}_l^2(0, l)] \times \mathcal{H}_0^1(0, l) = X_{\frac{1}{2}}$. We know that $[\mathcal{H}^3(0, l) \cap \mathcal{H}_l^2(0, l)] \times \mathcal{H}_l^1(0, l) \subset X_{-\frac{1}{2}}$. From (6.2) it is now clear that $\begin{bmatrix} z \\ q \end{bmatrix} \in \mathcal{X}$, i.e, the reversed inclusion of (6.6) holds. ■

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