Block Preconditioned Methods in Solution of Hyperbolic Equations

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***Abstract*— In this article, we compare suitable preconditioners for solving linear systems arising from the class of fourth-order approximations employed for solving hyperbolic equations, $\alpha u_{tt} + \beta u_{xx} = f(x,t,u,u_x,u_t)$ subject to appropriate initial and boundary conditions, where α and β are constants. Numerical results show that the proposed preconditioned methods produces an accurate and oscillation free solution.**

Keywords: Fourth-order approximation, Hyperbolic equations, Krylov subspace methods, Preconditioner.

I. Introduction

The numerical solution of one space second order hyperbolic equations with non-linear first derivative terms in cartesian, cylindrical and spherical coordinates are of great importance in many fields of engineering and sciences. Many computational models give rise to large sparse linear systems. For such systems iterative methods are usually prefered to direct methods which are expensive both in memory and computing requirements. Krylov subspace methods are one of the widely used and successful classes of numerical algorithms for solving large and sparse systems of algebraic equations but the speed of these methods are slow for problems which arise from typical applications. In order to be effective and obtaining faster convergence, these methods should be combined with a suitable preconditioner. The rate of convergence generally depends on the condition number of the corresponding matrix. Since the preconditioner plays a critical role in preconditioned Krylov subspace methods, many preconditioners have been proposed and studied. The ADI method is a preconditioner [12] for non-symmetric systems that can be very effective but this method is not effective for more general block tri-diagonal systems arising from the fourth-order approximations.

In 1997, Bhuruth and Evans [3] proposed BLAGE method as a preconditioner for a class of non-symmetric linear systems. Later, Dehghan and Molavi [5], [12] compared preconditioned techniques for 2-D and 3-D elliptic equations. In this article, we compare different preconditioned methods for solving linear systems arising from the fourth-order approximation of hyperbolic equation

$$\alpha u_{tt} - \beta(x,t) u_{xx} = f(x,t,u,u_x,u_t) \tag{1}$$

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defined in the region $W \times [0 < t < T]$, where

$$W = \{x | 0 < x < 1\}$$

and α is constant. The initial conditions consists of:

$$u(x,0) = g_1(x), \quad u_t(x,0) = g_2(x), \quad 0 \le x \le 1, \tag{2}$$

and boundary conditions consists of:

$$u(0,t) = h_0(t), \quad u(1,t) = h_1(t), \quad t \ge 0, \tag{3}$$

where $u = u(x,t)$. The resulting block tri-diagonal linear system of equations is solved by using Krylov subspace methods. The outline of the paper is as follows:

In Section 2, we briefly introduce some available preconditioners. In Section 3, we consider Krylov subspace methods and in Section 4, we present an example. In Section 5, we report a brief conclusion.

II. Preconditioner

The convergence rate of iterative methods depends on spectral properties of the coefficient matrix. Hence we will attempt to transform the linear system into another equivalent system in the sense that it has the same solution, but has more favorable spectral properties. A preconditioner is a matrix that effects such as a transformation [2]. If the preconditioner be as $M = M_1 M_2$ then the preconditioned system is as

$$M_1^{-1} A M_2^{-1} (M_2 x) = M_1^{-1} b. \tag{4}$$

The matrices M_1 and M_2 are called the left and right preconditioners, respectively. Now, we describe briefly preconditioners that we use for solving linear systems.

A. Preconditioner based on relaxation technique

Let $A = D + L + U$ such that D, L and U are diagonal, lower and upper triangular block matrices, respectively. A splitting of the coefficient matrix is as $A = M - N$ where the stationary iteration for solving a linear system is as

$$x_{k+1} = M^{-1} N x_k + M^{-1} b. \tag{5}$$

In Table 1, we briefly show preconditioners based on relaxation technique. In the above notation, ω is called the relaxation parameter. The optimal value of the parameter ω reduces the number of iterations to a lower order [1]. We have chose M in Jacobi, G-S, SOR as a left preconditioner and in SSOR preconditioner, we have chose $M_1 = \frac{1}{\omega(2-\omega)}(D + \omega L)$ as a left preconditioner and $M_2 = D^{-1}(D + \omega U)$ as a right preconditioner. Also, we take $\omega_{opt} = \frac{2}{1+\sqrt{1-\rho_J^2}}$.

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TABLE I
PRECONDITIONERS BASED ON RELAXATION TECHNIQUE

Preconditioner	M
Jacobi	D
Gauss-Seidel	$(D+L)$
SOR	$\frac{1}{\omega}(D+\omega L)$
SSOR	$\frac{1}{\omega(2-\omega)}(D+\omega L)D^{-1}(D+\omega U)$

B. ADI preconditioner

Peaceman and Rachford [13] in 1955 presented the *ADI* method for solving linear systems. Let $A=H+V+W$ and in the form

$$A=\begin{pmatrix} B_1 & C_1 & & & \\ A_2 & B_2 & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & A_{n-1} & B_{n-1} & C_{n-1} \\ & & & A_n & B_n \end{pmatrix},$$

where $A_i=tridiag(a_{1i},b_{1i},c_{1i})$, $B_i=tridiag(a_{2i},b_{2i},c_{2i})$ and $C_i=tridiag(a_{3i},b_{3i},c_{3i})$ of order $N\times N$ where H and V are bounded and include $H=\{0.5B_i,b_{3i},b_{1i}\}$, $V=\{0.5B_i,a_{1i},c_{1i},a_{3i},c_{3i}\}$. The alternative direction implicit method for solving the linear system $Ax=b$ is in following form:

$$(H+r_1I)u_{i+\frac{1}{2}}=b-(V-r_1I)u_i, \quad (6)$$

$$(V+r_2I)u_{i+1}=b-(H-r_2I)u_{i+\frac{1}{2}}, \quad (7)$$

The *ADI* preconditioner is as $M\equiv(H+r_1I)(V+r_2I)$ that $M_1=(H+r_1I)$ and $M_2=(V+r_2I)$ where Parameters r_1 and r_2 are acceleration parameters. Young and Varga [19], [20] proved that the optimum value for r_1 and r_2 is $\sqrt{\alpha\beta}$ where $\alpha\le\mu_i,\nu_i\le\beta$ and μ_i,ν_i are eigenvalues of matrices H and V respectively.

C. BLAGE preconditioner

The BLAGE method [3], [7] was originally introduced as analogue of the AGE method [6]. The BLAGE uses fractional splitting technique that is applied in two half steps on linear systems with block tri-diagonal matrices of order $N^2\times N^2$ and in the form

$$A=\begin{pmatrix} B_1 & C_1 & & & \\ A_2 & B_2 & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & A_{n-1} & B_{n-1} & C_{n-1} \\ & & & A_n & B_n \end{pmatrix},$$

where A_i, B_i and C_i are tri-diagonal matrices of order $N\times N$. The splitting of matrix A is sum of matrices G_1 and G_2 in which $A=G_1+G_2$ where G_1 and G_2 are of the form

$$G_1=\begin{pmatrix} B'_1 & & & & & & \\ & B'_2 & C_2 & & & & \\ & A_2 & B'_3 & & & & \\ & & & \ddots & & & \\ & & & & B'_{n-1} & C_{n-1} \\ & & & & A_n & B'_n \end{pmatrix}$$

$$G_2=\begin{pmatrix} B'_1 & C_1 & & & & \\ A_2 & B'_2 & & & & \\ & & \ddots & & & \\ & & & B'_{n-2} & C_{n-2} & \\ & & & A_{n-1} & B'_{n-1} & \\ & & & & & B'_n \end{pmatrix},$$

for odd values of n and

$$G_1=\begin{pmatrix} B'_1 & C_1 & & & \\ A_2 & B'_2 & & & \\ & & \ddots & & \\ & & & B'_{n-1} & C_{n-1} \\ & & & A_n & B'_n \end{pmatrix}$$

$$G_2=\begin{pmatrix} B'_1 & & & & & & \\ & B'_2 & C_2 & & & & \\ & A_3 & B'_3 & & & & \\ & & & \ddots & & & \\ & & & & B'_{n-2} & C_{n-2} & \\ & & & & A_{n-1} & B'_{n-1} & \\ & & & & & & B'_n \end{pmatrix},$$

for even values of n where $B'_i=\frac{1}{2}B_i$. The BLAGE preconditioner is as $M=(G_1+\omega_1I)(G_2+\omega_2I)$ that $M_1=(G_1+\omega_1I)$ and $M_2=(G_2+\omega_2I)$ where ω_1 and ω_2 are optimal iteration parameters. We have experimentally chosen the relaxation parameter $\omega_1=\sqrt{\alpha_1\beta_2}$ and $\omega_2=\sqrt{\alpha_2\beta_1}$ where $\alpha_1=\lambda_{min}(M_1)$, $\beta_1=\lambda_{max}(M_1)$ and $\alpha_2=\lambda_{min}(M_2)$, $\beta_2=\lambda_{max}(M_2)$. so that we will have the minimum condition number.

III. KRYLOV SUBSPACE METHODS

Let x_0 be an arbitrary initial guess for linear systems given by $Ax=b$, and let $r_0=b-Ax_0$ be the corresponding residual vector. A Krylov subspace of order m that is shown with $K_m(A,r)$ is defined as follows:

$$K_m(A,r_0)=span\{r_0,Ar_0,...,A^{m-1}r_0\}. \quad (8)$$

For non-symmetric matrix A, different methods can be used such as GMRES, GMRES(m), QMR, CGS, BiCG, BiCGSTAB [14]. Now, we briefly describe Krylov subspace methods:

A. Generalized minimal residual(GMRES) method

This method has the property of minimizing the norm of the residual vector over the Krylov subspace method at every step. The major drawback for GMRES method is that the amounts of work and storage required per iteration linearly rises with the iteration number. The usual way for overcome this problem is to restart after m iteration.

[15] Assume that A is a diagonalizable matrix and let $A = XDX^{-1}$ where $D = diag\{\lambda_1, ..., \lambda_n\}$ is the diagonal matrix of eigenvalues. Define,

$$\varepsilon^{(m)} = min_{p\varepsilon P_m, p(0)=1} max_{i=1,...,n} |p(\lambda_i)|.$$

Then, the residual norm achieved by the m-th step of GMRES satisfies the inequality

$$\|r_m\|_2 \leq K(X)\varepsilon^m \|r_0\|_2.$$

where $K(X) = \|X\|_2 \|X^{-1}\|_2$. When A is positive real with symmetric part M, the following error bound can be derived from the proposition,

$$\|r_m\| \leq [1 - \alpha/\beta]^{m/2} \|r_0\|, \tag{9}$$

with $\alpha = (\lambda_{min}(M))^2, \beta = \lambda_{max}(A^T A)$. This proves the convergence of the GMRES(m) for all m when A is positive real [14].

B. Bi-Conjugate Gradient (BiCG) method

Bi-conjugate gradient(BiCG) method was suggested by Fletcher in 1977, is applied to non-symmetric matrices. BiCG method needs matrix-vector products with A and A^T. Also, BiCG method is sensitive to possible breakdowns and numerical instabilities [2].

C. Quasi- Minimal Residual (QMR) method

In 1991, Freund and Nachtigal proposed the quasi-minimal residual (QMR) method for solving non-Hermitian linear systems. Later in 1994, they presented QMR method based on the coupled two-term recurrences instead of three-term recurrences [9]. This method sometimes avoids the break down of BiCG method. Also, QMR method has a regular convergence behavior than other Krylov subspace methods. The residual norm of the approximate solution x_m of QMR method satisfies the relation

$$\|b - Ax_m\| \leq \|V_{m+1}\|_2 |s_1 ... s_m| \|r_0\|_2. \tag{10}$$

The following theorem is well-known, cf. [14].

D. Conjugate gradient squared (CGS) method

In 1989, Sonneveld presented the conjugate gradient squared (CGS) method for nonsymmetric systems [17]. The speed of convergence of this method usually is about twice as fast as BiCG method. Convergence behavior of this method is often quite irregular, which may result loss of accuracy in the updated residual.

E. Bi-Conjugate Gradient Stabilized (BiCGSTAB) method

This method is applied for non-symmetric systems. Bi-conjugate gradient stabilized method is an alternative for CGS method that avoids the irregular convergence behavior of CGS method while maintaining about the same speed of convergence [20]. Algorithm of BiCGSTAB method that applied to the preconditioned system (2.1) is presented in [2].

TABLE II

NUMBER OF ITERATIONS WITH GMRES METHOD

N	no pre	Jacobi	SOR	SSOR	ADI	BLAGE
19	147	131	72	53	33	63
39	350	335	170	144	48	183
59	564	530	273	318	220	294
79	770	724	Nun	Nun	405	455

TABLE III

NUMBER OF ITERATIONS WITH BICG METHOD

N	no pre	Jacobi	SOR	SSOR	ADI	BLAGE
19	154	148	103	61	44	82
39	553	608	486	405	91	309
59	1126	Nun	Nun	Nun	Nun	803
79	1922	Nun	Nun	Nun	Nun	1504

IV. NUMERICAL EXPERIMENT

In this section, we present a numerical example to show the computational efficiency of the preconditioning methods introduced in section 2. Our initial guess is the zero vector and the iterations are stopped when the error is less of 10^{-6}. We show the iteration number without using of preconditioner by ”no pre”. The computations have been done on a P.C. with Corw 2 Pue 2.0 Ghz and 1024 MB RAM.
Test: We consider hyperbolic differential equation

$$u_{tt} = u_{xx} + u_x + u_t, \tag{11}$$

with Dirichlet boundary conditions, where

$$u(x,t) = \exp(2x + 3t). \tag{12}$$

We discretized Eq.(4.1) by using forth- order approximation. The coefficient matrix is block tri-diagonal and the diagonal elements are tri-diagonal matrices. We show in Tables $2-6$ the number of iterations with using different Krylov subspace methods. When h is be decreased, we encounter break down by using direct preconditioners while ADI and BLAGE preconditioners work quite well.
We found that the QMR, BiCG, CGS and BiCGSTAB methods in composition preconditioners are not suitable but

TABLE IV

NUMBER OF ITERATIONS WITH CGS METHOD

N	no pre	Jacobi	SOR	SSOR	ADI	BLAGE
19	162	155	82	57	52	79
39	621	878	417	637	114	369
59	1232	Nun	Nun	Nun	Nun	912
79	2003	Nun	Nun	Nun	Nun	1430

TABLE V

NUMBER OF ITERATIONS WITH QMR METHOD

N	no pre	Jacobi	SOR	SSOR	ADI	BLAGE
19	154	143	102	60	45	81
39	553	605	407	409	93	309
59	1113	Nun	Nun	Nun	Nun	781
79	1930	Nun	Nun	Nun	Nun	1356

TABLE VI
NUMBER OF ITERATIONS WITH BICGSTAB METHOD

N	no pre	Jacobi	SOR	SSOR	ADI	BLAGE
19	369	291	96	67	54	83
39	1067	1265	584	917	111	499
59	2169	Nun	Nun	Nun	Nun	1381
79	3503	Nun	Nun	Nun	Nun	2132

with using GMRES method in composition preconditioners, we get less iteration number than other preconditioned krylov subspace methods. Also, preconditioned GMRES method has regular convergence behavior.
Also, we observe that we obtain less iteration number with using ADI and SSOR preconditioners but SSOR preconditioner needs more time consumption than other preconditioners. Also, we saw that using ADI preconditioner we save in time consumption.

V. CONCLUSIONS

Here, we compared the different preconditioners in non-symmetric systems for hyperbolic equation. It is seen when the condition number is high ADI and BLAGE preconditioner aren't work very well but in well-conditioned problems the iteration number of the BLAGE(2), SSOR preconditioners is less and the iteration number of the Jacobi and SOR preconditioners is more but the time consumption of BLAGE(2) preconditioner is less and the time consumption of SSOR preconditioner is more than other. So we propose using BLAGE(2) preconditioner because this preconditioner needs to less computing time and have the less iteration numbers than other. We can use the parallel machines for better comparison of preconditioners because the BLAGE and ADI preconditioners can be employed in parallel environment where the preconditioning operations can be divided into several subproblems which can be run in parallel [3].

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