

# Mean-Square Minimization in Mathematical Finance with Control and State Constraints <sup>1</sup>

Andrew J. Heunis <sup>2</sup>

## Abstract

We study a problem of optimal stochastic control from mathematical finance. The problem involves both a control constraint (on the portfolio) together with an almost-sure state constraint (on the wealth process), giving a rather challenging combination of constraints. We demonstrate existence of a Lagrange multiplier, show that this is a pair comprising a finitely additive measure (for the state constraint) and an Ito process (for the portfolio constraint), and construct an optimal portfolio in terms of the Lagrange multiplier.

## 1 Introduction

We study a problem of stochastic control drawn from the area of mathematical finance. The goal is to trade in a standard complete market in such a way as to minimize the expected value of a general quadratic loss function of the terminal wealth (that is the wealth close of trade) when there is a convex constraint on the portfolio and a specified almost-sure lower-bound on the terminal wealth. By appropriate choice of the quadratic loss function one can address more specific problems, such as minimizing the mean-square discrepancy between the wealth at close of trade and a specified square-integrable contingent claim ( $L_2$ -hedging), or minimizing the variance of the wealth at close of trade given a specified value for its expectation (least-variance portfolio selection). In the terminology of stochastic optimal control the problem involves both a “control constraint” (on the portfolio) as well as an almost-sure “state constraint” (on the wealth).

The preceding optimal control problem has been addressed by Korn [7] and Bielecki *et. al.* [1] when there is an almost-sure nonnegativity constraint on the wealth process but the portfolio is otherwise unconstrained, and by Labbé *et. al.* [8] when there is a convex constraint on the portfolio but the wealth process is unconstrained. The problem that we address involves the *simultaneous* occurrence of constraints which were addressed *singly* in

the preceding works. There does not appear to be any direct way of extending the approach adopted in either of these works to the case where such constraints occur together, that is when one has both a constraint on the portfolio (i.e. the control) and on the almost-sure value of the wealth (i.e. the state). Indeed, one knows from deterministic optimal control that a combination of control and state constraints presents a particular challenge, and generally leads to “singular” Lagrange multipliers (see e.g. Dubovitskii and Mil’utin [4]); this suggests of course that one will similarly get Lagrange multipliers with some kind of singularity in stochastic control problems with such combined constraints as well. A particularly effective approach for dealing with general convex portfolio constraints in stochastic control problems of finance is due to Cvitanic and Karatzas [3]; the basic idea is to formulate a surrogate or auxiliary complete market with the property that *unconstrained* optimization in the auxiliary market amounts to *constrained* optimization in the given market. Despite the effectiveness of this method we have nevertheless not been able to extend the approach to the case where one has the *joint effect* of an almost sure constraint on the wealth and a convex constraint on the portfolio. In fact, this combination of constraints seems to present its own special challenges and to call for a new approach.

In light of the preceding observations we resort to a duality method of Rockafellar and Moreau (set forth in Rockafellar [10]) which pertains to convex optimization problems of considerable generality. The essence of the approach is to *perturb* the given (i.e. primal) problem, then calculate concave conjugates in terms of the perturbation. This yields (i) an appropriate linear space of dual variables; (ii) a Lagrangian function on the product of the spaces of primal and dual variables, as well as a dual function on the space of dual variables; (iii) existence of a Lagrange multiplier under suitable conditions; and (iv) a set of Kuhn-Tucker optimality relations which characterize the saddle-points of the Lagrangian. For the stochastic optimal control problem of the present work it will be seen that a natural *Slater-type* qualification on the terminal wealth constraint is enough to ensure existence of a Lagrange multiplier which enforces the constraints, the multiplier comprising an Itô process paired with a member of the adjoint (or conjugate) of the space of essentially-bounded random variables  $L_\infty(\Omega, \mathcal{F}_T, P)$ , in

<sup>1</sup>Supported by NSERC of Canada

<sup>2</sup>Department of Electrical and Computer Engineering, Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada. email: heunis@uwaterloo.ca

which  $\mathcal{F}_T$  is the event  $\sigma$ -algebra at close of trade  $t = T$ . The Kuhn-Tucker optimality relations are then used to synthesize an optimal portfolio in terms of the Lagrange multiplier.

## 2 Market Model and Primal Problem

We assume given a complete probability space  $(\Omega, \mathcal{F}, P)$  on which is fixed some  $\mathbb{R}^N$ -valued standard Brownian motion  $\{W(t), t \in [0, T]\}$ , in which the constant  $T \in (0, \infty)$  defines a finite “horizon”  $t \in [0, T]$  over which trading take place. The information available to the investor is assumed to be given by the filtration

$$\mathcal{F}_t := \sigma\{W(\tau), \tau \in [0, t]\} \vee \mathcal{N}(P), \quad t \in [0, T], \quad (2.1)$$

in which  $\mathcal{N}(P)$  denotes the collection of all  $P$ -null events in  $\mathcal{F}$ . The market comprises  $N + 1$  assets traded continuously on the interval  $[0, T]$ , namely a bond with price  $\{S_0(t)\}$  and  $N$  stocks with prices  $\{S_n(t)\}$ ,  $n = 1, 2, \dots, N$ , given by

$$dS_0(t) = r(t)S_0(t) dt, \quad S_0(0) = 1, \quad (2.2)$$

$$dS_n(t) = S_n(t) \left[ b_n(t) dt + \sum_{m=1}^N \sigma_{nm}(t) dW_m(t) \right], \quad (2.3)$$

the initial values  $S_n(0)$  being strictly positive constants. We shall always postulate

**Condition 2.1.** In (2.2) and (2.3) the interest rate  $\{r(t)\}$ , the entries  $\{b_n(t)\}$  of the  $\mathbb{R}^N$ -valued process  $\{b(t)\}$  of appreciation rates on stocks, and the entries  $\{\sigma_{nm}(t)\}$  of the  $N \times N$  matrix-valued volatility process  $\{\sigma(t)\}$  are uniformly bounded and  $\{\mathcal{F}_t\}$ -progressively measurable scalar processes on  $[0, T] \times \Omega$ , and the process  $\{r(t)\}$  is non-negative. There is a constant  $\kappa \in (0, \infty)$  such that  $z' \sigma(t, \omega) \sigma'(t, \omega) z \geq \kappa \|z\|^2$  for all  $z \in \mathbb{R}^N$  and  $(t, \omega) \in [0, T] \times \Omega$  (the prime symbol denotes transposition, and  $\|z\|$  denotes the Pythagorean norm of  $z \in \mathbb{R}^N$ ).

**Condition 2.2.** We are given (i) a closed convex portfolio constraint set  $K \subset \mathbb{R}^N$  with  $0 \in K$ ; (ii) a non-random initial wealth  $x_0 \in (0, \infty)$ ; (iii)  $\mathcal{F}_T$ -measurable random variables  $a, B, c$ , on  $(\Omega, \mathcal{F}, P)$  such that  $0 < \inf_{\omega \in \Omega} a(\omega) \leq \sup_{\omega \in \Omega} a(\omega) < \infty$ , the random variable  $B$  is  $P$ -essentially bounded, and the random variable  $c$  is  $P$ -square integrable.

**Remark 2.3.** Define

$$\theta(t, \omega) := (\sigma(t, \omega))^{-1} [b(t, \omega) - r(t, \omega)\mathbf{1}],$$

for all  $(t, \omega) \in [0, T] \times \Omega$ , in which  $\mathbf{1} \in \mathbb{R}^N$  has all unit entries (this is the *market price of risk*). From Condition 2.1 we see that the  $\mathbb{R}^N$ -valued process  $\{\theta(t)\}$  is uniformly bounded on  $[0, T] \times \Omega$ .

**Notation 2.4.** We use  $\mathcal{F}^*$  to denote the  $\mathcal{F}_t$ -progressively measurable  $\sigma$ -algebra on  $[0, T] \times \Omega$ , and write  $\eta \in \mathcal{F}^*$  to indicate that the process  $\eta : [0, T] \times \Omega \rightarrow \mathbb{R}^n$  (the dimension  $n$  being clear from the context) is  $\mathcal{F}^*$ -measurable. The qualifier “a.e.” always refers to the measure  $(\lambda \otimes P)$  on  $\mathcal{B}([0, T]) \otimes \mathcal{F}$ , where  $\mathcal{B}([0, T])$  denotes the Borel  $\sigma$ -algebra of  $[0, T]$  and  $\lambda$  denotes Lebesgue measure on  $\mathcal{B}([0, T])$ .

Given some  $\mathbb{R}^N$ -valued  $\pi \in \mathcal{F}^*$  such that  $\int_0^T \|\pi(t)\|^2 dt < \infty$ , a.s., there exists an a.s. unique scalar-valued, continuous, and  $\{\mathcal{F}_t\}$ -progressively measurable process  $\{X^\pi(t), t \in [0, T]\}$  such that

$$\begin{aligned} dX^\pi(t) &= \{r(t)X^\pi(t) + \pi'(t)\sigma(t)\theta(t)\} dt \\ &\quad + \pi'(t)\sigma(t) dW(t), \quad X^\pi(0) = x_0, \end{aligned} \quad (2.4)$$

Consider a small investor who trades in the market following a *self-funded strategy* from the given initial wealth  $x_0 \in (0, \infty)$ . If  $\pi_n(t)$ , the  $n$ -th entry of the  $\mathbb{R}^N$ -valued vector  $\pi(t)$ , is interpreted as the *dollar amount* invested in the stock with price  $S_n(t)$ ,  $n = 1, 2, \dots, N$ , then  $X^\pi(t)$  is the investor’s wealth at instant  $t \in [0, T]$ . With the preceding in mind we next formulate the problem of quadratic minimization. Recalling Notation 2.4 and Condition 2.2, define the real linear space of  $\mathbb{R}^N$ -valued,  $\mathcal{F}_t$ -progressively-measurable square-integrable processes

$$\begin{aligned} \Pi_2 &:= \{\xi : [0, T] \times \Omega \rightarrow \mathbb{R}^N \mid \xi \in \mathcal{F}^*, \\ &\quad \text{and } E \int_0^T \|\xi(t)\|^2 dt < \infty\}, \end{aligned} \quad (2.5)$$

and define the set of *admissible portfolios*  $\mathcal{A}$ , *loss function*  $J(x, \omega)$ , and *primal value*  $\vartheta$  by

$$\mathcal{A} := \{\pi \in \Pi_2 \mid \pi(t) \in K \text{ a.e.}\}, \quad (2.6)$$

$$J(x, \omega) := \frac{1}{2} [a(\omega)x^2 + 2c(\omega)x], \quad (x, \omega) \in \mathbb{R} \times \Omega, \quad (2.7)$$

$$\vartheta := \inf_{\substack{\pi \in \mathcal{A} \\ X^\pi(T) \geq B}} E[J(X^\pi(T))]. \quad (2.8)$$

The problem of quadratic minimization with constraints is then

$$\begin{aligned} &\text{determine some } \bar{\pi} \in \mathcal{A} \text{ such that } X^{\bar{\pi}}(T) \geq B \text{ a.s.} \\ &\text{and } \vartheta = E[J(X^{\bar{\pi}}(T))], \end{aligned} \quad (2.9)$$

in the sense of constructing an appropriate Lagrange multiplier for the constraints, and synthesizing  $\bar{\pi}$  in terms of the multiplier. For problem (2.9) to make sense we must of course have  $X^\pi(T) \geq B$  a.s. for some  $\pi \in \mathcal{A}$ . In fact we shall need to strengthen this to the following *Slater-type* condition in order to secure existence of Lagrange multipliers:

**Condition 2.5.** There is some  $\bar{\pi} \in \mathcal{A}$  and constant  $\epsilon \in (0, \infty)$  such that  $X^{\bar{\pi}}(T) \geq B + \epsilon$  a.s.

**Remark 2.6.** (i) When the random variable  $B$  is non-negative then it represents a specified “subsistence” or “portfolio insurance”; the case  $B = 0$  amounts to a prohibition on bankruptcy, and negative  $B$  represents a specified limit on debt at close of trade.

(ii) The quadratic criterion given by (2.7) is general enough to include the case of *mean-square hedging*, namely minimizing the mean-square discrepancy  $E[|X^\pi(T) - \gamma|^2]$  of the terminal wealth from a specified claim  $\gamma \in L_2(\Omega, \mathcal{F}_T, P)$ .

(iii) There are many choices of  $B$  for which Condition 2.5 holds. For example, suppose that  $B := \alpha x_0 S_0(T)$  for some constant  $\alpha \in [0, 1)$  (since  $0 \in K$  and  $\{r(t)\}$  is uniformly bounded - by Conditions 2.1 and 2.2 - we can take  $\tilde{\pi} := 0$ ). This is portfolio insurance for an amount corresponding to investing some fraction  $\alpha$  of the initial wealth in a money-market account.

**Remark 2.7.** We next write (2.9) as a primal optimization problem over an appropriate set of Itô processes. To this end, and recalling (2.5), define the real linear spaces

$$\mathbb{S}_{21} := \{v : [0, T] \times \Omega \rightarrow \mathbb{R} \mid v \in \mathcal{F}^* \text{ and } E \left( \int_0^T |v(t)| dt \right)^2 < \infty\}, \quad (2.10)$$

$$\mathbb{B} := \mathbb{R} \times \mathbb{S}_{21} \times \Pi_2, \quad (2.11)$$

and write  $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{B}$  (or  $X \in \mathbb{B}$  for short) to indicate that  $\{X(t), t \in [0, T]\}$  is the  $\mathcal{F}_t$ -Itô process given by

$$X(t) = X_0 + \int_0^t \dot{X}(\tau) d\tau + \int_0^t \Lambda'_X(\tau) dW(\tau), \quad (2.12)$$

for some triplet  $(X_0, \dot{X}, \Lambda_X) \in \mathbb{B}$ , in which the “integrand processes”  $\dot{X} \in \mathbb{S}_{21}$  and  $\Lambda_X \in \Pi_2$  are clearly a.e.-uniquely determined on  $\Omega \times [0, T]$ .

**Remark 2.8.** From (2.12), (2.11), (2.4), and Doob’s maximal  $L^2$ -inequality, together with Condition 2.1 it is easy to verify the following:

- (a)  $E[\max_{t \in [0, T]} |X(t)|^2] < \infty$  for all  $X \in \mathbb{B}$ ;
- (b) one has  $X^\pi \in \mathbb{B}$  if and only if  $\pi \in \Pi_2$ .

Next, recalling (2.6) and (2.4), define the set of *admissible* wealth processes

$$\mathbb{A} := \{X^\pi \mid \pi \in \mathcal{A}\} \subset \mathbb{B}, \quad (2.13)$$

(the set-inclusion follows from Remark 2.8(b)). Then  $\mathbb{A}$  is convex, as follows from (2.4) and the fact that  $\mathcal{A}$  is convex (by Condition 2.2(i)). Define the *primal function*  $f : \mathbb{B} \rightarrow (-\infty, \infty]$  as

$$f(X) := \begin{cases} EJ(X(T)), & \text{when } X \in \mathbb{A} \text{ and } X(T) \geq B \text{ a.s.}, \\ +\infty, & \text{otherwise.} \end{cases}$$

$$(2.14)$$

Notice that  $J(X(T))$  is  $P$ -integrable for each  $X \in \mathbb{B}$  (as follows from Remark 2.8(a), (2.7) and Condition 2.2(iii)), and  $f$  is convex on  $\mathbb{B}$ . From (2.14), (2.13) and (2.8), we have

$$\vartheta = \inf_{X \in \mathbb{B}} f(X). \quad (2.15)$$

In the preceding we have re-formulated the problem (2.9) as minimization of the convex function  $f$  given by (2.14) over the linear space  $\mathbb{B}$  (see (2.11)). With this re-formulation in place we are able to apply the general approach of Rockafellar and Moreau, which we summarize next.

### 3 Background

In this section we summarize the approach of Rockafellar and Moreau for convex optimization. A detailed account, with numerous illustrative examples, is given by Rockafellar [10], but the short summary of the present section will be ample for our purposes.

Suppose that  $\mathbb{X}$  is a real linear space, and the *primal problem* is to minimize a convex function  $f : \mathbb{X} \rightarrow [-\infty, \infty]$  on  $\mathbb{X}$ . Note that this does not involve any loss of generality, since the objective function  $f$  can be defined to have the value  $+\infty$  at all points of  $\mathbb{X}$  excluded by possible constraints in the problem (exactly as at (2.14) - indeed, we shall later identify  $\mathbb{X}$  with  $\mathbb{B}$  and will take  $f$  to be the function at (2.14)). The Rockafellar-Moreau approach gives a systematic method for constructing an appropriate Lagrangian and dual function. The first step is to choose a real linear space of *perturbations*  $\mathbb{U}$  and a *perturbation function*  $F : \mathbb{X} \times \mathbb{U} \rightarrow [-\infty, \infty]$  such that

$$F(x, 0) = f(x), \quad x \in \mathbb{X}, \quad (3.16)$$

(cf. (4.1) of [10], p.18). Next, fix another real linear space  $\mathbb{Y}$ , together with a bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathbb{U} \times \mathbb{Y}$ , and define the *Lagrangian*  $K : \mathbb{X} \times \mathbb{Y} \rightarrow [-\infty, \infty]$  by the concave conjugate

$$K(x, y) := \inf_{u \in \mathbb{U}} [\langle u, y \rangle + F(x, u)], \quad (3.17)$$

(cf. (4.2) of [10], p.18). Finally, define the *dual function*  $g : \mathbb{Y} \rightarrow [-\infty, \infty]$  by

$$g(y) := \inf_{x \in \mathbb{X}} K(x, y) = \inf_{(x, u) \in \mathbb{X} \times \mathbb{U}} [\langle u, y \rangle + F(x, u)], \quad (3.18)$$

(cf. (4.6) of [10], p.19). It is immediate from (3.16) to (3.18) that  $g(\cdot)$  is *concave* on  $\mathbb{Y}$  (being the point-wise infimum of a collection of affine functionals on  $\mathbb{Y}$ ), and

$$f(x) \geq K(x, y) \geq g(y), \quad (x, y) \in \mathbb{X} \times \mathbb{Y}. \quad (3.19)$$

A Hausdorff locally convex linear topology  $\mathcal{U}$  on  $\mathbb{U}$  is called *compatible* with the bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathbb{U} \times \mathbb{Y}$  (or  $\langle \mathbb{U}, \mathbb{Y} \rangle$ -compatible for short) when (i) the mapping  $u \rightarrow \langle u, y \rangle : \mathbb{U} \rightarrow \mathbb{R}$  is  $\mathcal{U}$ -continuous for each  $y \in \mathbb{Y}$ , and (ii) each  $\mathcal{U}$ -continuous linear functional  $\phi : \mathbb{U} \rightarrow \mathbb{R}$  is given by  $\phi(u) = \langle u, y \rangle$ ,  $u \in \mathbb{U}$ , for some  $y \in \mathbb{Y}$ . We then have (see Thrms. 17(a) and 18(a) of [10], p.41):

**Theorem 3.1.** *Fix a  $\langle \mathbb{U}, \mathbb{Y} \rangle$ -compatible topology  $\mathcal{U}$  on  $\mathbb{U}$ , and suppose that the perturbation function  $F : \mathbb{X} \times \mathbb{U} \rightarrow [-\infty, \infty]$  is convex on  $\mathbb{X} \times \mathbb{U}$  and subject to the consistency relation (3.16). If there is some  $x_1 \in \mathbb{X}$ , and some  $\mathcal{U}$ -neighborhood  $G$  of the origin  $0 \in \mathbb{U}$ , such that  $\sup_{u \in G} F(x_1, u) < +\infty$ , then  $\inf_{x \in \mathbb{X}} f(x) = \sup_{y \in \mathbb{Y}} g(y) = g(\bar{y})$  for some  $\bar{y} \in \mathbb{Y}$ .*

**Remark 3.2.** Minimization of  $f(\cdot)$  on  $\mathbb{X}$  boils down to constructing a pair  $(\bar{x}, \bar{y}) \in \mathbb{X} \times \mathbb{Y}$  such that  $f(\bar{x}) = g(\bar{y})$ , for then we see from (3.19) that  $\bar{x}$  is a minimizer of  $f(\cdot)$  on  $\mathbb{X}$  (and  $\bar{y}$  is a maximizer of  $g(\cdot)$  on  $\mathbb{Y}$ ). If the conditions of Theorem 3.1 are in force, then we already have at our disposal a solution  $\bar{y} \in \mathbb{Y}$  of the *dual problem* of maximizing  $g(\cdot)$  on  $\mathbb{Y}$ ; in keeping with terminology from *mathematical programming* we refer to the maximizer  $\bar{y}$  as a *Lagrange multiplier*. We can then derive *necessary conditions* resulting from the optimality of  $\bar{y}$  for the dual problem, and try to use these necessary conditions to *construct* an  $\bar{x} \in \mathbb{X}$  in terms of  $\bar{y}$  such that  $f(\bar{x}) = g(\bar{y})$ .

**Remark 3.3.** In the preceding, the vector spaces  $\mathbb{U}$  and  $\mathbb{Y}$ , the function  $F(\cdot, \cdot)$  on  $\mathbb{X} \times \mathbb{U}$ , and the bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathbb{U} \times \mathbb{Y}$  are at our discretion, although subject to (3.16); different choices of these yield different *spaces of dual variables*  $\mathbb{Y}$  as well as different Lagrangian and dual functions. These items must be chosen with the following considerations in mind:

- (a) The conditions of Theorem 3.1 should hold;
- (b) The dual function at (3.18) should have a reasonably tractable form, so that we can obtain useful necessary conditions resulting from the optimality of  $\bar{y}$  given by Theorem 3.1;
- (c) It should be possible to write the condition  $f(x) = g(y)$  (for an *arbitrary* pair  $(x, y) \in \mathbb{X} \times \mathbb{Y}$ ) as reasonably explicit *Kuhn-Tucker* optimality relations, involving in particular transversality conditions, complementary slackness conditions, and feasibility conditions. These relations, together with the necessary conditions from (b), should furthermore be useful for constructing an  $\bar{x} \in \mathbb{X}$  in terms of the maximizer  $\bar{y} \in \mathbb{Y}$  given by Theorem 3.1, such that  $f(\bar{x}) = g(\bar{y})$ .

#### 4 Perturbation, Lagrangian and Dual Problem

**Notation 4.1.** From now on write  $L_p$  for the spaces  $L_p(\Omega, \mathcal{F}_T, P)$ , for all  $p \in [1, \infty]$  (recall (2.1)). Similarly

write  $L_\infty^*$  for the real linear space  $L_\infty^*(\Omega, \mathcal{F}_T, P)$ , and put  $\mathbb{G} := \{Z \in L_\infty^* \mid Z \leq 0\}$ .

In Section 2 we reduced problem (2.9) to one of minimizing the convex function  $f(\cdot)$  given by (2.14) over the real linear space  $\mathbb{B}$  given by (2.11). We next apply the basic approach of Rockafellar and Moreau, summarized in Section 3, to this problem. To this end fix a linear space of perturbations  $\mathbb{U}$ , and perturbation function  $F : \mathbb{B} \times \mathbb{U} \rightarrow (-\infty, \infty]$ , as follows:

$$\mathbb{U} := L_2 \times L_\infty, \tag{4.20}$$

$$F(X, u) := \begin{cases} \mathbb{E}J(X(T) - u_1), & \text{when } X \in \mathbb{A} \text{ and} \\ & X(T) \geq B + u_2 \text{ a.s.}, \\ +\infty, & \text{otherwise,} \end{cases}$$

for all  $X \in \mathbb{B}$ ,  $u = (u_1, u_2) \in \mathbb{U}$ .

**Remark 4.2.** It is clear that  $F(\cdot, \cdot)$  is convex on  $\mathbb{B} \times \mathbb{U}$ , and (see (2.14)) we have the consistency relation  $F(X, 0) = f(X)$ ,  $X \in \mathbb{B}$  (cf. (3.16)).

Following the plan outlined in Section 3, we next define a duality pairing  $\langle \mathbb{U}, \mathbb{Y} \rangle$  of  $\mathbb{U}$  with another real linear space  $\mathbb{Y}$ . To this end (recalling (2.2), (2.5) and Remark 2.7), put

$$\mathbb{B}_1 := \{Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{B} \mid \dot{Y}(t) = -r(t)Y(t)\}, \tag{4.21}$$

$$\Xi(y, \gamma)(t) := [S_0(t)]^{-1} \left\{ y + \int_0^t S_0(\tau)\gamma'(\tau) \, dW(\tau) \right\}, \tag{4.22}$$

for all  $t \in [0, T]$ ,  $(y, \gamma) \in \mathbb{R} \times \Pi_2$ . Basic properties of the set  $\mathbb{B}_1$  and the mapping  $\Xi(\cdot, \cdot)$  are summarized next. The proof is a straightforward application of Itô's formula and Doob's maximal  $L_2$ -inequality, and is omitted:

**Proposition 4.3.** *Suppose Condition 2.1 and recall (2.5). Then (i)  $\mathbb{B}_1$  is a real linear space;*

*(ii)  $\Xi(y, \gamma) \in \mathbb{B}_1$  for all  $(y, \gamma) \in \mathbb{R} \times \Pi_2$ , and  $\Xi : \mathbb{R} \times \Pi_2 \rightarrow \mathbb{B}_1$  is a linear bijection;*

*(iii) If  $Y := \Xi(y, \gamma)$  for some  $(y, \gamma) \in \mathbb{R} \times \Pi_2$  then  $Y_0 = y$ ,  $\dot{Y} = -rY$ , and  $\Lambda_Y = \gamma$ .*

Next, define the real linear space  $\mathbb{Y}$  and bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathbb{U} \times \mathbb{Y}$  as follows:

$$\mathbb{Y} := \mathbb{B}_1 \times L_\infty^*, \tag{4.23}$$

$$\langle (u_1, u_2), (Y, Z) \rangle := \mathbb{E}[u_1 Y(T)] + Z(u_2), \tag{4.24}$$

for all  $(u_1, u_2) \in \mathbb{U}$ ,  $(Y, Z) \in \mathbb{Y}$ .

**Remark 4.4.** For  $(Y, Z) \in \mathbb{Y}$  we have  $Y \in \mathbb{B}_1$ , thus in particular  $Y(T) \in L_2$  (by Remark 2.8(a)), thus the right side of (4.24) is well-defined and  $\mathbb{R}$ -valued.

Now define the *Lagrangian* as at (3.17), namely

$$K(X, (Y, Z)) := \inf_{(u_1, u_2) \in \mathbb{U}} [((u_1, u_2), (Y, Z)) + F(X, u_1, u_2)], \quad (4.25)$$

for all  $X \in \mathbb{B}$ ,  $(Y, Z) \in \mathbb{Y}$ . Using tools of convex analysis we can evaluate the right side of (4.25) explicitly, as follows: For each  $X \in \mathbb{B}$  and  $(Y, Z) \in \mathbb{Y}$ ,

$$K(X, (Y, Z)) = \begin{cases} \mathbb{E}[X(T)Y(T)] - \mathbb{E}[J^*(Y(T))] \\ + \inf_{\substack{u_2 \in L_\infty \\ u_2 \leq X(T) - B}} Z(u_2), & \text{if } X \in \mathbb{A}_1 \text{ \& } Z \leq 0, \\ -\infty, & \text{if } X \in \mathbb{A}_1 \text{ \& } Z \not\leq 0, \\ +\infty, & \text{if } X \notin \mathbb{A}_1. \end{cases} \quad (4.26)$$

Here we have used the notation

$$\mathbb{A}_1 := \{X \in \mathbb{A} \mid \text{there is some } \alpha \in \mathbb{R} \text{ s.t. } X(T) - B \geq \alpha\}, \quad (4.27)$$

$$J^*(y, \omega) := \sup_{x \in \mathbb{R}} [xy - J(x, \omega)] = \frac{(y - c(\omega))^2}{2a(\omega)}, \quad (4.28)$$

for all  $(y, \omega) \in \mathbb{R} \times \Omega$  (recall (2.7)).

**Remark 4.5.** Note from (4.28) that  $J^*(Y(T))$  is  $P$ -integrable for each  $Y \in \mathbb{B}_1$  (since  $\mathbb{B}_1 \subset \mathbb{B}$  and Remark 2.8(a) ensure that  $Y(T) \in L_2$  for all  $Y \in \mathbb{B}_1$ ).

Following (3.18), we next define the *dual function*. To this end, for each  $(Y, Z) \in \mathbb{Y}$  put

$$g(Y, Z) := \inf_{X \in \mathbb{B}} K(X, (Y, Z)), \quad (4.29)$$

$$\kappa(Y, Z) := \sup_{X \in \mathbb{A}_1} \{-\mathbb{E}[X(T)Y(T)] - \inf_{\substack{u_2 \in L_\infty \\ u_2 \leq X(T) - B}} Z(u_2)\}. \quad (4.30)$$

Then, from (4.29) - (4.30) and (4.26), for each  $(Y, Z) \in \mathbb{Y}$  the dual function is given by

$$g(Y, Z) = \begin{cases} -\kappa(Y, Z) - \mathbb{E}[J^*(Y(T))], & \text{when } Z \leq 0, \\ -\infty, & \text{otherwise.} \end{cases} \quad (4.31)$$

**Remark 4.6.** From (4.29), (4.25), (4.21) and (2.14), we get the inequality (cf. (3.19))

$$f(X) \geq K(X, (Y, Z)) \geq g(Y, Z), \quad X \in \mathbb{B}, \quad (Y, Z) \in \mathbb{Y}. \quad (4.32)$$

We next establish existence of a maximizer for the dual function at (4.31):

**Proposition 4.7.** *Suppose Conditions 2.1, 2.2 and 2.5. Then there exists some  $(\bar{Y}, \bar{Z}) \in \mathbb{Y}$  such that  $\inf_{X \in \mathbb{B}} f(X) = \sup_{(Y, Z) \in \mathbb{Y}} g(Y, Z) = g(\bar{Y}, \bar{Z}) \in \mathbb{R}$  (recall (2.14), (4.23) and (4.31)).*

**Remark 4.8.** Proposition 4.7 is established by using Condition 2.5 to verify the conditions of Theorem 3.1 (see [5] for the proof). We next write the condition  $f(X) = g(Y, Z)$  in the form of *Kuhn-Tucker* relations (see Remark 3.3(c)).

**Proposition 4.9.** *Suppose Conditions 2.1 and 2.2. For each  $(X, (Y, Z)) \in \mathbb{B} \times \mathbb{Y}$ , we have*

$$f(X) = g(Y, Z) \iff \begin{cases} (1) X(T) - B \geq 0, \\ (2) X \in \mathbb{A}_1, \\ (3) Z \leq 0, \\ (4) \inf_{\substack{u_2 \in L_\infty \\ u_2 \leq X(T) - B}} Z(u_2) = 0, \\ (5) \mathbb{E}[X(T)Y(T)] + \kappa(Y, Z) = 0, \\ (6) X(T) = (\partial J^*)(Y(T)). \end{cases} \quad (4.33)$$

**Remark 4.10.** Proposition 4.9 is established on the basis of (4.31), (4.26), and (2.14) (see [5] for the proof). Items (1) - (6) of (4.33) are *Kuhn-Tucker optimality relations* for the problem (2.9). In particular, (4.33)(1)(2) are feasibility conditions on the primal variable  $X$ , (4.33)(3) is a feasibility condition on the dual variable  $Z$ , (4.33)(4)(5) are complementary slackness conditions, and (4.33)(6) is a transversality relation.

## 5 Optimal Portfolio

In this section we shall use the Kuhn-Tucker optimality relations established at Proposition 4.9 to construct an optimal portfolio  $\bar{\pi}$  for the problem (2.9) in terms of the Lagrange multiplier  $(\bar{Y}, \bar{Z}) \in \mathbb{Y}$ , the existence of which is given by Proposition 4.7. To this end we have to establish necessary conditions resulting from the optimality of  $(\bar{Y}, \bar{Z})$  - recall Remark 3.3(b)(c). Observe from Proposition 4.7 that  $g(\bar{Y} + \epsilon Y, \bar{Z}) \leq g(\bar{Y}, \bar{Z}) \in \mathbb{R}$ , for  $\epsilon \in (0, \infty)$  and  $Y \in \mathbb{B}_1$ ; hence from (4.31), (4.30), and  $\mathbb{E}J^*(\bar{Y}(T)) \in \mathbb{R}$ , (see Remark 4.5), we have

$$\kappa(\bar{Y}, \bar{Z}) \in \mathbb{R} \quad \& \quad \bar{Z} \leq 0, \quad (5.34)$$

$$\mathbb{E}[J^*(\bar{Y}(T) + \epsilon Y(T)) - J^*(\bar{Y}(T))] + \kappa(\bar{Y} + \epsilon Y, \bar{Z}) - \kappa(\bar{Y}, \bar{Z}) \geq 0, \quad (5.35)$$

for all  $\epsilon \in (0, \infty)$  and  $Y \in \mathbb{B}_1$ . Moreover, from (4.30) and  $\kappa(\bar{Y}, \bar{Z}) \in \mathbb{R}$ , we have

$$\kappa(\bar{Y} + \epsilon Y, \bar{Z}) - \kappa(\bar{Y}, \bar{Z}) \leq \epsilon \sup_{X \in \mathbb{A}_1} \mathbb{E}[-X(T)Y(T)], \quad (5.36)$$

for all  $\epsilon \in (0, \infty)$ ,  $Y \in \mathbb{B}_1$ . Since  $\mathbb{A}_1 \subset \mathbb{A}$  (recall (4.27)), from (5.36) and (5.35) we obtain

$$\begin{aligned} & \mathbb{E} \left[ \frac{J^*(\bar{Y}(T) + \epsilon Y(T)) - J^*(\bar{Y}(T))}{\epsilon} \right] \\ & + \sup_{X \in \mathbb{A}} \mathbb{E}[-X(T)Y(T)] \geq 0, \end{aligned} \quad (5.37)$$

for all  $\epsilon \in (0, \infty)$  and  $Y \in \mathbb{B}_1$ . From (4.28) and Condition 2.2(iii), it is easy to use dominated convergence to evaluate the limit on the left of (5.37) as  $\epsilon \rightarrow 0$ , to get

$$\sup_{X \in \mathbb{A}} \mathbb{E}[-X(T)Y(T)] + \mathbb{E}[\partial J^*(\bar{Y}(T)) Y(T)] \geq 0, \quad (5.38)$$

for all  $Y \in \mathbb{B}_1$ , where  $\partial J^*(y, \omega) := (y - c(\omega))/a(\omega)$  (recall (4.28)).

**Notation 5.1.** For a real linear space  $\mathbb{X}$  and  $A \subset \mathbb{X}$ , define the *indicator function* of the set  $A$  as the mapping  $x \rightarrow \delta_{\mathbb{X}}(x|A) : \mathbb{X} \rightarrow \{0, \infty\}$  given by  $\delta_{\mathbb{X}}(x|A) = 0$  when  $x \in A$ , and  $\delta_{\mathbb{X}}(x|A) = \infty$  when  $x \notin A$  (this notation is due to Rockafellar ([9], p.28)).

The supremum on left of the inequality at (5.38) defines the *support functional* of the set of admissible wealth processes  $\mathbb{A}$  evaluated at  $-Y$  (recall (2.13), (2.6)). In order to use (5.38) to construct an optimal portfolio for the problem (2.9) we must evaluate this supremum explicitly. The calculation, which uses tools of convex analysis and stochastic integration, is given in [5], and yields

$$\begin{aligned} & \sup_{X \in \mathbb{A}} \mathbb{E}[-X(T)Y(T)] = -x_0 Y_0 \\ & + \mathbb{E} \int_0^T \delta_{\mathbb{R}^N}^*(-\sigma(\tau)[\theta(\tau)Y(\tau) + \Lambda_Y(\tau)]|K) d\tau, \end{aligned} \quad (5.39)$$

for each  $Y \in \mathbb{B}_1$ , in which we have used

$$\delta_{\mathbb{R}^N}^*(x|K) := \sup_{\pi \in K} \pi' x, \quad x \in \mathbb{R}^N, \quad (5.40)$$

for the support functional of the set  $K$  specified by Condition 2.2(i) (following the notation of Rockafellar [9], p.28). Upon combining (5.39) and (5.38) then gives (recall Proposition 4.7, (4.23) and (4.21))

$$\begin{aligned} & \mathbb{E}[\partial J^*(\bar{Y}(T))Y(T)] - x_0 Y_0 \\ & + \mathbb{E} \int_0^T \delta_{\mathbb{R}^N}^*(-\sigma(\tau)[\theta(\tau)Y(\tau) + \Lambda_Y(\tau)]|K) d\tau \geq 0, \end{aligned} \quad (5.41)$$

for each  $Y \in \mathbb{B}_1$ .

**Remark 5.2.** The inequality (5.41) is a *necessary condition* resulting from the optimality of  $(\bar{Y}, \bar{Z})$  at Proposition 4.7. In accordance with Remark 3.3(c) our goal is to construct an  $\bar{X} \in \mathbb{B}$  in terms of the Lagrange multiplier  $(\bar{Y}, \bar{Z}) \in \mathbb{Y}$  such that the pair  $(\bar{X}, (\bar{Y}, \bar{Z})) \in \mathbb{B} \times \mathbb{Y}$  satisfies items (1)-(6) of the Kuhn-Tucker optimality relations (4.33). To this end define the usual *state price density*  $H$

in terms of the market price of risk  $\theta$  and bond price  $S_0$  (recall Remark 2.3 and (2.2)), namely

$$\begin{aligned} H(t) & := [S_0(t)]^{-1} \exp\left\{-\int_0^t \theta'(\tau) dW(\tau) \right. \\ & \left. - (1/2) \int_0^t \|\theta(\tau)\|^2 d\tau\right\}, \end{aligned} \quad (5.42)$$

for all  $t \in [0, T]$ . From Itô's formula and Remark 5.9 of ([8], p.87), we have

(a)  $\mathbb{E}[\max_{t \in [0, T]} |H(t)|^p] < \infty$ , for all  $p \in \mathbb{R}$ ;

(b)  $H \in \mathbb{B}_1$  with  $H_0 = 1$  and  $\Lambda_H = -H\theta$  (recall (4.21) and (2.11)).

We shall also need the following technical result, which follows directly from Lemma 5.1 and Lemma 5.2 of [8]:

**Proposition 5.3.** Fix some  $\xi \in L_2$  (recall Notation 4.1), define the  $\mathbb{R}$ -valued process  $\{X(t)\}$  by  $X(t) := H^{-1}(t)E[\xi H(T) | \mathcal{F}_t]$ ,  $t \in [0, T]$ , and put

$$X(t)H(t) = X(0) + \int_0^t \psi'(\tau) dW(\tau), \quad t \in [0, T],$$

for some  $\mathbb{R}^N$ -valued and a.e. unique  $\psi \in \mathcal{F}^*$  (recall Notation 2.4) such that  $\int_0^T \|\psi(t)\|^2 dt < \infty$  a.s. (existence and uniqueness of  $\psi$  follows from the martingale representation theorem and (2.1)). Then, for the  $\mathbb{R}^N$ -valued process  $\pi \in \mathcal{F}^*$  defined by  $\pi(t) := [\sigma'(t)]^{-1} \{H^{-1}(t)\psi(t) + X(t)\theta(t)\}$ ,  $t \in [0, T]$ , one has that  $X \in \mathbb{B}$  and  $\pi \in \Pi_2$  (recall (2.10)-(2.12) and (2.5)). Moreover,  $X$  and  $\pi$  are related by

$$dX(t) = \{r(t)X(t) + \pi'(t)\sigma(t)\theta(t)\} dt + \pi'(t)\sigma(t) dW(t).$$

Now define the  $\mathbb{R}$ -valued continuous process  $\bar{X}$  and the  $\mathbb{R}^N$ -valued processes  $\bar{\psi} \in \mathcal{F}^*$ ,  $\bar{\pi} \in \mathcal{F}^*$ , in accordance with Proposition 5.3 with  $\xi := \partial J^*(\bar{Y}(T)) \in L_2$  (recall from Remark 4.5 that  $\bar{Y}(T) \in L_2$ ); that is

$$\bar{X}(t) := H^{-1}(t)E[\partial J^*(\bar{Y}(T))H(T) | \mathcal{F}_t], \quad (5.43)$$

$$\bar{X}(t)H(t) = \bar{X}(0) + \int_0^t \bar{\psi}'(\tau) dW(\tau), \quad (5.44)$$

$$\bar{\pi}(t) := [\sigma'(t)]^{-1} \{H^{-1}(t)\bar{\psi}(t) + \bar{X}(t)\theta(t)\}, \quad (5.45)$$

and  $\int_0^T \|\bar{\psi}(t)\|^2 dt < \infty$  a.s. From Proposition 5.3 with (5.43), (5.45), we get

$$d\bar{X}(t) = \{r(t)\bar{X}(t) + \bar{\pi}'(t)\sigma(t)\theta(t)\} dt + \bar{\pi}'(t)\sigma(t) dW(t), \quad (5.46)$$

$$\bar{X} \in \mathbb{B}, \quad \bar{\pi} \in \Pi_2. \quad (5.47)$$

We shall use (5.41) to establish that

$$\bar{X}(0) = x_0, \quad \bar{\pi} \in \mathcal{A}, \quad \text{and} \quad \bar{X} = X^{\bar{\pi}} \in \mathbb{A}, \quad (5.48)$$

(recall (2.13), (2.6) and (2.4)). From (5.46), (4.21) and (2.12), along with Proposition I-1 of Bismut ([2], p.387) it is easily seen that  $\mathbb{M}(\bar{X}, Y)(t) = \bar{X}(t)Y(t) - \bar{X}(0)Y_0 - \int_0^t \bar{\pi}'(\tau)\sigma(\tau)[Y(\tau)\theta(\tau) + \Lambda_Y(\tau)]d\tau$  is a  $\mathcal{F}_t$ -martingale for each  $Y \in \mathbb{B}_1$ , thus, in particular,  $\mathbb{E}\mathbb{M}(\bar{X}, Y)(T) = 0$ , that is

$$\begin{aligned} \mathbb{E}[\bar{X}(T)Y(T)] &= \bar{X}(0)Y_0 \\ + \mathbb{E} \int_0^T \bar{\pi}'(\tau)\sigma(\tau)[Y(\tau)\theta(\tau) + \Lambda_Y(\tau)]d\tau, \end{aligned} \quad (5.49)$$

for each  $Y \in \mathbb{B}_1$ . From (5.43) we have  $\bar{X}(T) = \partial J^*(\bar{Y}(T))$ , and combining this with (5.49) and (5.41) gives

$$\begin{aligned} (\bar{X}(0) - x_0)Y_0 & \\ + \mathbb{E} \int_0^T \{ \bar{\pi}'(\tau)\sigma(\tau)[Y(\tau)\theta(\tau) + \Lambda_Y(\tau)] \\ + \delta_{\mathbb{R}^N}^*(-\sigma(\tau)[Y(\tau)\theta(\tau) + \Lambda_Y(\tau)]|K) \} d\tau &\geq 0, \end{aligned} \quad (5.50)$$

for each  $Y \in \mathbb{B}_1$ . In order to use (5.50) to establish (5.48) we shall need the following result on integral equations:

**Lemma 5.4.** *Suppose Condition 2.1 and recall Remark 2.3. For each  $\rho \in \Pi_2$  (recall (2.5)) there is a unique  $\xi \in \Pi_2$  such that  $\rho(t) = \xi(t) + \theta(t) \int_0^t \xi'(\tau) dW(\tau)$  a.e.*

**Remark 5.5.** The proof of Lemma 5.4 is by a standard Picard iterations and is omitted. Now fix some  $y \in \mathbb{R}$ . Using Lemma 5.4 we are going to construct some  $Y \in \mathbb{B}_1$  such that  $Y_0 = y$  and  $Y\theta + \Lambda_Y = 0$  a.e. Then, from (5.50) and the fact that  $\delta_{\mathbb{R}^N}^*(0|K) = 0$  (see (5.40)), we have  $(\bar{X}(0) - x_0)y \geq 0$ . But  $y \in \mathbb{R}$  is arbitrary, thus  $\bar{X}(0) = x_0$ . We construct  $Y \in \mathbb{B}_1$  as follows:

For  $y \in \mathbb{R}$  set  $\rho := -y\theta \in \Pi_2$  (the set-membership in  $\Pi_2$  following from Remark 2.3). From Lemma 5.4 there is some  $\xi \in \Pi_2$  such that  $\rho(t) = \xi(t) + \theta(t) \int_0^t \xi'(\tau) dW(\tau)$  a.e.. Now put  $\gamma(t) := [S_0(t)]^{-1}\xi(t)$ ,  $t \in [0, T]$ . Then  $0 = y\theta(t) + \rho(t) = y\theta(t) + S_0(t)\gamma(t) + \theta(t) \int_0^t S_0(\tau)\gamma'(\tau) dW(\tau)$ , thus, multiplying by  $[S_0(t)]^{-1}$ , gives  $0 = y[\theta(t)]^{-1} + \gamma(t) + [S_0(t)]^{-1}\theta(t) \int_0^t S_0(\tau)\gamma'(\tau) dW(\tau)$ . From this, with  $\gamma \in \Pi_2$  (recall Condition 2.1) and (4.22), we have  $0 = \gamma(t) + \theta(t)\Xi(y, \gamma)$  a.e., and thus, from Proposition 4.3, we get  $0 = \Lambda_Y(t) + \theta(t)Y(t)$  a.e., for  $Y := \Xi(y, \gamma) \in \mathbb{B}_1$ . Since  $Y_0 = y$ , from Remark 5.5 we obtain  $\bar{X}(0) = x_0$ , and from this with (5.46) and (2.4), we get  $\bar{X} = X^{\bar{\pi}}$ .

To complete the proof of (5.48) we next establish  $\bar{\pi} \in \mathcal{A}$ . Put  $A := \{(t, \omega) \in [0, T] \times \Omega \mid \bar{\pi}(t, \omega) \in K\}$ . From ([6], Lemma 5.4.2, p.207), corresponding to  $\bar{\pi} \in \mathcal{F}^*$  is some  $\mathbb{R}^N$ -valued  $\bar{\nu} \in \mathcal{F}^*$  such that

$$\begin{cases} \|\bar{\nu}(t)\| \leq 1, & |\delta_{\mathbb{R}^N}^*(-\bar{\nu}(t)|K)| \leq 1, & \text{a.e. on } [0, T] \times \Omega, \\ \bar{\pi}'(t)\bar{\nu}(t) + \delta_{\mathbb{R}^N}^*(-\bar{\nu}(t)|K) = 0, & & \text{a.e. on } A, \\ \bar{\pi}'(t)\bar{\nu}(t) + \delta_{\mathbb{R}^N}^*(-\bar{\nu}(t)|K) < 0, & & \text{a.e. on } A^c. \end{cases} \quad (5.51)$$

**Remark 5.6.** We shall next use Lemma 5.4 to construct some  $Y \in \mathbb{B}_1$  such that  $Y_0 = 0$  and  $\bar{\nu}(t) = \sigma(t)[Y(t)\theta(t) + \Lambda_Y(t)]$  a.e. Then (5.50) gives  $\mathbb{E} \int_0^T \{ \bar{\pi}'(\tau)\bar{\nu}(\tau) + \delta_{\mathbb{R}^N}^*(-\bar{\nu}(\tau)|K) \} d\tau \geq 0$ ; from this and (5.51) we obtain  $\lambda \otimes P(A^c) = 0$ , i.e.  $\bar{\pi}(t) \in K$  a.e., whence  $\bar{\pi} \in \mathcal{A}$  (recall (5.47) and (2.6)). The construction of  $Y \in \mathbb{B}_1$  is as follows:

Put  $\rho(t) := [S_0(t)]\sigma^{-1}(t)\bar{\nu}(t)$ ,  $t \in [0, T]$ . Then  $\rho \in \Pi_2$  (as follows from (5.51), and Condition 2.1), thus  $\rho(t) = \xi(t) + \theta(t) \int_0^t \xi'(\tau) dW(\tau)$  a.e. for some  $\xi \in \Pi_2$  (as follows from Lemma 5.4). Now put  $\gamma(t) := [S_0(t)]^{-1}\xi(t)$ ,  $t \in [0, T]$ ; then  $\gamma \in \Pi_2$  (since  $\xi \in \Pi_2$ ). Upon combining the preceding we obtain  $\bar{\nu}(t) = [S_0(t)]^{-1}\sigma(t)\{\xi(t) + \theta(t) \int_0^t \xi'(\tau) dW(\tau)\} = \sigma(t)\{\gamma(t) + \theta(t)\Xi(0, \gamma)(t)\}$  (recall (4.22)). Now put  $Y := \Xi(0, \gamma)$ ; then, from Proposition 4.3, we have  $Y \in \mathbb{B}_1$  with  $Y_0 = 0$  and  $\Lambda_Y = \gamma$ , thus  $\bar{\nu}(t) = \sigma(t)[\Lambda_Y(t) + \theta(t)Y(t)]$  a.e. In view of this and Remark 5.6 we obtain  $\bar{\pi} \in \mathcal{A}$ , and since  $\bar{X} = X^{\bar{\pi}}$  (shown just prior to (5.51)), from (2.13) we have  $\bar{X} \in \mathbb{A}$ , as required to obtain (5.48).

**Remark 5.7.** Having established (5.48), we are going to show next that

$$\bar{X}(T) \geq B, \quad \text{a.s.} \quad (5.52)$$

From this, with (5.48), we get  $X^{\bar{\pi}}(T) \geq B$ , so that the terminal wealth constraint at (2.9) holds. To establish (5.52) observe that, for  $\zeta \in L_2$ , the mapping  $u \rightarrow \mathbb{E}[u\zeta] : L_\infty \rightarrow \mathbb{R}$  defines an element of  $L_\infty^*$ ; we shall denote this functional by  $\zeta$ , and put  $\zeta(u) := \mathbb{E}[u\zeta]$ ,  $u \in L_\infty$ . Now fix some  $\zeta_1 \in L_2$ ; from the martingale representation theorem it is easily seen that  $Y_1(T) = \zeta_1$  for some  $Y_1 \in \mathbb{B}_1$ .

Now Proposition 4.7 gives  $g(\bar{Y} + \epsilon Y_1, \bar{Z} - \epsilon \zeta_1) \leq g(\bar{Y}, \bar{Z})$  for all  $\epsilon \in (0, \infty)$ , thus

$$\begin{aligned} \kappa(\bar{Y} + \epsilon Y_1, \bar{Z} - \epsilon \zeta_1) + \mathbb{E}J^*(\bar{Y}(T) + \epsilon Y_1(T)) \\ + \delta_{L_\infty^*}(\bar{Z} - \epsilon \zeta_1|\mathbb{G}) \\ \geq \kappa(\bar{Y}, \bar{Z}) + \mathbb{E}J^*(\bar{Y}(T)) + \delta_{L_\infty^*}(\bar{Z}|\mathbb{G}), \end{aligned} \quad (5.53)$$

for each  $\epsilon \in (0, \infty)$  - recall (4.31), Notation 4.1, and Notation 5.1. From (4.30), (5.53), and  $Y_1(T) = \zeta_1$  (see Remark 5.7), it is easily shown that

$$\begin{aligned} \sup_{\substack{X \in \mathbb{A}_1, u_2 \in L_\infty \\ u_2 \leq X(T) - B}} \mathbb{E}[\zeta_1(u_2 - X(T))] + \delta_{L_\infty^*}(-\zeta_1|\mathbb{G}) \\ + \mathbb{E} \left[ \frac{J^*(\bar{Y}(T) + \epsilon \zeta_1) - J^*(\bar{Y}(T))}{\epsilon} \right] \geq 0, \end{aligned} \quad (5.54)$$

for all  $\epsilon \in (0, \infty)$ . Since  $\bar{X}(T) = \partial J^*(\bar{Y}(T))$  (see (5.43)), it follows easily from (4.28), Condition 2.2(iii) and dominated convergence, that the third term on the left of (5.54) converges to  $\mathbb{E}[\bar{X}(T)\zeta_1]$  as  $\epsilon \rightarrow 0$ . Now (5.54) has

been established for an arbitrarily chosen  $\zeta_1 \in L_2$  (see Remark 5.7), so we obtain

$$\sup_{\substack{X \in \mathbb{A}_1, u_2 \in L_\infty \\ u_2 \leq \bar{X}(T) - B}} E[\zeta(u_2 - (X(T) - B))] + E[(\bar{X}(T) - B)\zeta] \geq 0, \tag{5.55}$$

for all non-negative  $\zeta \in L_2$ .

**Remark 5.8.** The inequality (5.55) is another necessary condition resulting from the optimality of  $(\bar{Y}, \bar{Z})$  given by Proposition 4.7. In order to use (5.55) to establish (5.52) take  $\zeta := I\{\bar{X}(T) < B\}$  (that is,  $\zeta = 1$  on  $\{\bar{X}(T) < B\}$ , and  $\zeta = 0$  otherwise); since the first term on the left of (5.55) is non-positive we immediately get (5.52).

**Remark 5.9.** To summarize what has been established so far, put  $(X, (Y, Z)) := (\bar{X}, (\bar{Y}, \bar{Z}))$  (for  $(\bar{Y}, \bar{Z})$  given by Proposition 4.7 and  $\bar{X}$  defined by (5.43)). Then (5.52) verifies (4.33)(1); from this, with  $\bar{X} \in \mathbb{A}$  (see (5.48)) and (4.27), we get  $\bar{X} \in \mathbb{A}_1$ , which verifies (4.33)(2). Moreover, (4.33)(3) and (4.33)(6) are immediate from (5.34) and (5.43) respectively. It therefore remains only to verify that  $(X, (Y, Z)) := (\bar{X}, (\bar{Y}, \bar{Z}))$  satisfies the complementary slackness conditions (4.33)(4)(5), as follows:

Proposition 4.7 and (5.34) give  $g(\bar{Y} - \epsilon\bar{Y}, \bar{Z} - \epsilon\bar{Z}) \leq g(\bar{Y}, \bar{Z})$  and  $\bar{Z} - \epsilon\bar{Z} \leq 0$ , for all  $\epsilon \in [0, 1)$ , thus, from (4.31)

$$\begin{aligned} &\kappa(\bar{Y} - \epsilon\bar{Y}, \bar{Z} - \epsilon\bar{Z}) + EJ^*(\bar{Y}(T) - \epsilon\bar{Y}(T)) \\ &\geq \kappa(\bar{Y}, \bar{Z}) + EJ^*(\bar{Y}(T)), \end{aligned} \tag{5.56}$$

for all  $\epsilon \in (0, \infty)$ . From (4.30) one sees that  $\kappa(\bar{Y} - \epsilon\bar{Y}, \bar{Z} - \epsilon\bar{Z}) = (1 - \epsilon)\kappa(\bar{Y}, \bar{Z})$ , for all  $\epsilon \in [0, 1)$ . Using this in (5.56), with  $\kappa(\bar{Y}, \bar{Z}) \in \mathbb{R}$  (see (5.34)), then gives that  $-\kappa(\bar{Y}, \bar{Z}) + (1/\epsilon)E[J^*((1 - \epsilon)\bar{Y}(T)) - J^*(\bar{Y}(T))] \geq 0$ , for all  $\epsilon \in (0, 1)$ . Take  $\epsilon \rightarrow 0$  and use  $\bar{X}(T) = \partial J^*(\bar{Y}(T))$  (recall (5.43)), together with (4.28), Condition 2.2(iii), and dominated convergence, to get (i)  $\kappa(\bar{Y}, \bar{Z}) + E[\bar{X}(T)\bar{Y}(T)] \leq 0$ . Moreover, from (5.52), we have (ii)  $\inf_{\{u_2 \in L_\infty, u_2 \leq \bar{X}(T) - B\}} \bar{Z}(u_2) \leq 0$ . From (i) and (ii) we get

$$\kappa(\bar{Y}, \bar{Z}) + E[\bar{X}(T)\bar{Y}(T)] + \inf_{\substack{u_2 \in L_\infty \\ u_2 \leq \bar{X}(T) - B}} \bar{Z}(u_2) \leq 0. \tag{5.57}$$

But (5.57) also holds with  $\geq$  in place of  $\leq$ , as is immediate from (4.30) and the fact that  $\bar{X} \in \mathbb{A}_1$  (recall Remark 5.9), so that (5.57) in fact holds with  $=$  in place of  $\leq$ ; from this, together with (i) and (ii) (see text prior to (5.57)), we obtain

$$\kappa(\bar{Y}, \bar{Z}) + E[\bar{X}(T)\bar{Y}(T)] = - \inf_{\substack{u_2 \in L_\infty \\ u_2 \leq \bar{X}(T) - B}} \bar{Z}(u_2) = 0. \tag{5.58}$$

**Remark 5.10.** For  $(X, (Y, Z)) := (\bar{X}, (\bar{Y}, \bar{Z}))$  we see that (4.33)(4)(5) are immediate from (5.58). In view of this and Remark 5.9, we have verified the Kuhn-Tucker relations (4.33)(1)-(6), so we obtain  $f(\bar{X}) = g(\bar{Y}, \bar{Z})$ . This, together with (4.32), shows that  $\bar{X}$  defined at (5.43) is the optimal wealth process with corresponding optimal portfolio  $\bar{\pi}$  defined by (5.45).

### References

- [1] T.R. BIELECKI, H. JIN, S.R. PLISKA AND X.Y. ZHOU, Continuous-time mean-variance portfolio selection with bankruptcy prohibition, *Math. Finance*, pp. 213–244, v.15, (2005).
- [2] J.M. BISMUT, Conjugate convex functions in optimal stochastic control, *J. Math. Analysis Appl.*, pp. 384–404, v.44, (1973).
- [3] J. CVITANIC AND I. KARATZAS, Conex duality in constrained portfolio optimization, *Ann. Appl. Probab.*, pp. 767–818, v. 2, (1992).
- [4] A. YA. DUBOVITSKII AND A. A. MIL’YUTIN, Necessary conditions for a weak extremum in problems of optimal control with mixed inequality constraints, *Zhur. Vychislitel. Mat. i Mat. Fys.*, pp. 725–779, v.8 (1968) (*USSR Comp. Math. and Math. Phys.*, v.8, pp. 24–98).
- [5] A.J. HEUNIS, Quadratic minimization with portfolio and terminal wealth constraints, Technical Report, University of Waterloo (2009),
- [6] I. KARATZAS AND S.E. SHREVE, *Methods of Mathematical Finance*, Springer-Verlag, New York, (1998).
- [7] R. KORN, Some applications of  $L_2$ -hedging with a non-negative wealth process, *Appl. Math. Finance*, pp. 65–79, v.4, (1997).
- [8] C. LABBÉ AND A.J. HEUNIS, Convex duality in constrained mean-variance portfolio optimization, *Adv. Appl. Probab.*, v.39, 77–104, (2007).
- [9] R.T. ROCKAFELLAR, *Convex Analysis*, Princeton Univ. Press, Princeton, N.J., (1970).
- [10] R.T. ROCKAFELLAR, *Conjugate Duality and Optimization*, SIAM (CBMS-NSF Series No. 16), Philadelphia, (1974).