

# Recursive Identification of Continuous-Time Linear Stochastic Systems – An Off-Line Approximation

László Gerencsér\*, Vilmos Prokaj†\*

\*MTA SZTAKI, Kende u. 13-17, Budapest, 1111, Hungary

† Eötvös Loránd University, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary

**Abstract**—We consider multi-variable continuous-time linear stochastic systems given in innovation form, with system matrices depending on an unknown parameter that is locally identifiable. A computable continuous-time recursive maximum likelihood (RML) method with resetting has been proposed in our ECC 09 paper. Resetting takes place if the estimator process hits the boundary of a pre-specified compact domain, or if the rate of change, in a stochastic sense, of the parameter process would hit a fixed threshold. An outline of a proof of convergence almost surely and in  $L_q$  was given, under realistic conditions. In the present paper we show that the RML estimator differs from the off-line estimator by an error of the magnitude of  $\log T/T$  in an appropriate sense. With this result a conjecture formulated back in 1984 has been settled.

## I. INTRODUCTION

Continuous-time stochastic systems have attracted a lot of attention recently due to their wide-spread use in finance. Practically feasible recursive identification of these systems is a basic problem, which has resisted attempts for a rigorous analysis until recently. A convergence theorem for a computable continuous-time recursive maximum likelihood (RML) method for finite dimensional linear stochastic systems, using resetting, has been presented for the first time in [9]. The method itself with a heuristic analysis was given back in 1984 in [7]. The objective of the present paper is to complete the program formulated in [7], and to show that the RML estimator and the *off-line* ML estimator are essentially the same.

Our results extend earlier rigorous results on continuous-time recursive estimation given [14], and later in [3]. In these papers the underlying stochastic systems is essentially an AR-system, for which the recursive maximum-likelihood (RML) estimation reduces to a recursive least squares (RLSQ) estimation. Technically more involved problems have been solved in [2] and [11], but the results of these papers are not satisfactory from a practical point of view.

A continuous-time RML method for general finite dimensional linear stochastic systems have been suggested in [7], along the lines of discrete time recursive maximum-likelihood methods, but the analysis was admittedly incomplete. Recursive estimation in the same generality has been studied in [2] under the condition that the process is stopped if it reaches the boundary of a truncation domain (see Condition b) of Theorem 1 of the cited paper). A rigorous convergence theory for a restricted class of non-linear stochastic systems has been given in [11]. The limitation of the algorithm proposed in [11] is that it requires the

computation of what is called a frozen-parameter process evaluated at the current estimate. Consequently, the proposed algorithm is not computable in a practical sense.

We consider multi-variable Gaussian linear stochastic systems in  $-\infty < t < +\infty$  given as

$$\begin{aligned} dx_t^* &= A(\theta^*)x_t^*dt + K(\theta^*)dw_t \\ dy_t &= C(\theta^*)x_t^*dt + dw_t, \end{aligned}$$

where  $w(\cdot)$  is a vector-valued standard Wiener-process, and  $dw_t$  is the innovation process of  $dy_t$ . The system parameter  $\theta$  is assumed to belong to an open set  $D \subset \mathbb{R}^p$ . For all  $\theta \in D_\theta$  the system is assumed to be stable and inverse stable.  $K(\theta^*)$  is the Kalman-gain. The problem is to estimate the unknown parameter  $\theta^*$  in real time. To compute the negative log-likelihood for any fixed  $\theta$ , we invert the system to get an estimated innovation process  $d\bar{\varepsilon}_t(\theta)$ :

$$d\bar{\varepsilon}_t(\theta) = dy_t - C(\theta)\bar{x}_t(\theta)dt,$$

where  $\bar{x}_t(\theta)$  is the state vector obtained in the course of system inversion. Then the gradient of the negative log-likelihood can be written as

$$V_{\theta t}(\theta) = \int_0^t \dot{\bar{\varepsilon}}_{\theta s}^T(\theta)d\bar{\varepsilon}_s(\theta),$$

where  $\dot{\bar{\varepsilon}}$  denotes time derivative. We assume that the system is locally identifiable, i.e. the Fisher information matrix  $R^*$  is non-singular. The off-line (conditional) maximum-likelihood (ML) estimator  $\bar{\theta}_t$  is given by solving the non-linear algebraic equation

$$V_{\theta t}(\theta) = 0.$$

A rigorous definition of an off-line estimator is given [5].

The estimators of  $\theta^*$  and  $R^*$  will be denoted by  $\theta_t$  and  $R_t$ . The algorithm will be re-initialized if the estimator process hits the boundary of a pre-specified compact domain, or if the rate of change, in a stochastic sense, of the parameter process would hit a fixed threshold. The number of resets up to time  $t$  will be denoted by  $N_t$ . The (RML) method with *resetting* is then defined by

$$\begin{aligned} d\theta_t &= -\frac{1}{t}R_t^{-1}\dot{\bar{\varepsilon}}_{\theta t}d\varepsilon_t + (\theta_0 - \theta_t)dN_t \\ \dot{R}_t &= \frac{1}{t}(\dot{\bar{\varepsilon}}_{\theta t}\dot{\bar{\varepsilon}}_{\theta t}^T - R_t) + (R_0 - R_t)dN_t. \end{aligned}$$

Then, completing the heuristic arguments of [7], that has been made rigorous in [6] for discrete time systems, we get the following result: under reasonable conditions the SDE

with jumps defining the RML method with resetting, has a strong solution in  $[0, \infty)$ , we have  $N_t < \infty$  a.s. for any  $t > 0$ , and

$$\bar{\theta}_t - \theta_t = O_M\left(\frac{\log t}{t}\right) \quad \text{in a limited sense.}$$

The notations on the right hand side will be explained below.

The main idea is to show, by using the *Itô-Wentzel formula*, that

$$dV_\theta(\theta_t) = \frac{1}{t} b_t dt + \frac{1}{t^{1/2}} s_t dw_t \quad \text{in a limited sense,}$$

where for the processes  $b_t, s_t$  we have  $b_t, s_t = O_M(1)$ . An important technical issue, the estimation of the moments of  $N_{t+1} - N_t$ , the number of jumps in an interval of unit length, will be settled in Section VII. A second technical problem is to show that

$$d\bar{\varepsilon}_t(\theta_t) - d\varepsilon_t = O_M(1/t)dt,$$

in a limited sense, see Section VIII. Similar estimates are needed for the gradient process. This requires in turn the analysis of the exponential stability of homogeneous linear differential equations of the form

$$\dot{\Phi}_t = F(\theta_t)\Phi_t,$$

where  $F(\theta)$  is stable for all  $\theta \in D$ , in our case with  $F(\theta) = A(\theta) - K(\theta)C(\theta)$ . In the analysis we rely heavily on [9], providing a convergence analysis for the RML methods, and on a set of results given on the stability of hybrid stochastic systems given in [10].

## II. THE MODEL

Let us consider a multi-variable Gaussian linear stochastic system given in the state space innovation representation via a linear stochastic differential equation (SDE) and a linear observation:

$$dx_t^* = A(\theta^*)x_t^* dt + K(\theta^*)dw_t \quad (1)$$

$$dy_t = C(\theta^*)x_t^* dt + dw_t, \quad (2)$$

with  $-\infty < t < +\infty$ , where  $x_t^* \in \mathbb{R}^k$  is the state-vector,  $y_t \in \mathbb{R}^l$  is the observation process,  $w(\cdot)$  is an  $l$  dimensional standard Wiener-process over a filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ ,  $A(\theta^*), K(\theta^*), C(\theta^*)$  are matrices of appropriate dimensions, and  $K(\theta^*)$  is the Kalman-gain. The system parameter  $\theta$  is assumed to belong to an open set  $D \subset \mathbb{R}^p$ .

**Condition 1.** For all  $\theta \in D_\theta$  the system is stable and inverse stable, i.e. the matrices  $A(\theta)$  and  $A(\theta) - K(\theta)C(\theta)$  are stable, and  $(A(\theta), K(\theta))$  is controllable. Moreover, the matrices  $A(\cdot), K(\cdot), C(\cdot)$  are in  $C^3$  with respect to  $\theta$ .

To compute the (conditional) log-likelihood of the observation  $\{y(s) : 0 \leq s \leq t\}$  assuming that system parameter takes on the value  $\theta$ , and the initial value for the state is 0 we fix a  $\theta$ , and invert the system to get an estimated innovation process  $d\bar{\varepsilon}_t(\theta)$ :

$$d\bar{x}_t(\theta) = A(\theta)\bar{x}_t(\theta)dt + K(\theta)d\bar{\varepsilon}_t(\theta) \quad (3)$$

$$d\bar{\varepsilon}_t(\theta) = dy_t - C(\theta)\bar{x}_t(\theta)dt, \quad (4)$$

with  $\bar{x}_0(\theta) = 0$ . Then the negative log-likelihood can be written es

$$V_t(\theta) = - \int_0^t (C(\theta)\bar{x}_s(\theta))^T dy_s + \frac{1}{2} \int_0^t |C(\theta)\bar{x}_s(\theta)|^2 ds$$

Formal differentiation with respect to  $\theta$  gives the likelihood equation

$$\frac{\partial}{\partial \theta} V_t(\theta) = V_{\theta t}(\theta) = \int_0^t \dot{\varepsilon}_{\theta s}(\theta) d\bar{\varepsilon}_s(\theta) = 0, \quad (5)$$

where  $\dot{\varepsilon}_{\theta t}$  is obtained from (4) as

$$\dot{\varepsilon}_{\theta t} = - \frac{\partial}{\partial \theta} (C(\theta)\bar{x}_t(\theta)) = -C_{\theta}(\theta)\bar{x}_t(\theta) - C(\theta)\bar{x}_{\theta,t}(\theta).$$

The *asymptotic* negative log-likelihood function is defined, with  $\bar{z}_s(\theta) = C(\theta)\bar{x}_s(\theta)$ , as

$$W(\theta) = L + \lim_{t \rightarrow \infty} \frac{1}{2} \mathbf{E} (\bar{z}_t(\theta) - \bar{z}_t(\theta^*))^T (\bar{z}_t(\theta) - \bar{z}_t(\theta^*)) \quad (6)$$

with some constant  $L$  that does not depend on  $\theta$ . It is easy to see, that the limit above exist. Also it is easy to see that for  $\theta = \theta^*$  we have

$$\left. \frac{\partial}{\partial \theta} W(\theta) \right|_{\theta=\theta^*} = 0. \quad (7)$$

For the elements of the Hessian of  $W(\theta)$  at  $\theta = \theta^*$  we have

$$\left. \frac{\partial^2}{\partial \theta_i \partial \theta_j} W(\theta) \right|_{\theta=\theta^*} = \lim_{s \rightarrow \infty} \mathbf{E} \dot{\varepsilon}_{\theta_i, s}^T(\theta) \dot{\varepsilon}_{\theta_j, s}(\theta).$$

Finally, set

$$R^* = \left. \frac{\partial^2}{\partial \theta^2} W(\theta) \right|_{\theta=\theta^*} = W_{\theta\theta}(\theta)|_{\theta=\theta^*}.$$

It is easily seen directly that  $R^*$  is positive semi-definite.

**Condition 2.** The pair  $(A(\theta^*), K(\theta^*))$  is controllable, and the pair  $(C(\theta^*), A(\theta^*))$  is observable pair. Moreover, the model class (1), (2) is locally identifiable, i.e.  $R^* > 0$ , i.e.  $R^*$  is positive definite.

## III. THE RML METHOD

Let  $\theta_t$  be an on-line, recursive approximation of the off-line estimator to be defined below. We approximate  $\bar{x}_t(\theta_t)$  by a process  $x_t$  which is defined by the time-varying linear stochastic system

$$dx_t = A(\theta_t)x_t dt + K(\theta_t)d\varepsilon_t,$$

Here  $\varepsilon_t$ , is an approximation of  $\bar{\varepsilon}(t, \theta_t)$ , which is obtained in a similar way, see (10) below. Thus we arrive at the following set of state-equations:

$$dx_t = A(\theta_t) x_t dt + K(\theta_t) d\varepsilon_t \quad (8)$$

$$dx_{\theta,t} = A_\theta(\theta_t) x_t dt + A(\theta_t) x_{\theta,t} dt + K_\theta(\theta_t) d\varepsilon_t + K(\theta_t) d\varepsilon_{\theta,t} \quad (9)$$

$$d\varepsilon_t = dy_t - C(\theta_t) x_t dt \quad (10)$$

$$d\varepsilon_{\theta,t} = -C_\theta(\theta_t)x_t dt - C(\theta_t)x_{\theta,t} dt = \dot{\varepsilon}_{\theta,t} dt. \quad (11)$$

The description of the recursive maximum-likelihood (RML) method in continuous time, without resetting, is completed

by defining the dynamics for  $\theta_t$  and  $R_t$ , where the latter is expected to converge to  $R^*$ :

$$d\theta_t = -\frac{1}{t}R_t^{-1}\dot{\varepsilon}_{\theta t}d\varepsilon_t \quad (12)$$

$$\dot{R}_t = \frac{1}{t}(\dot{\varepsilon}_{\theta t}\dot{\varepsilon}_{\theta t}^T - R_t). \quad (13)$$

To ensure regularity at  $t = 0$  we replace the step-sizes  $1/t$  by  $1/(T_0 + t)$  with some  $T_0 > 0$ , and we set  $R_0 = r_0I$  with some  $r_0 > 0$ .

#### IV. RESETTING

To ensure convergence of the RML method we need to impose certain safeguards, in particular we need to introduce a *resetting* mechanism, see [1] or [8]. Let  $D_R$  denote the open set of positive definite matrices, denoted by  $\mathbf{R}^{p \times p+}$ . First, we force estimators to stay inside a compact domain of the form  $D_{0\theta} \times D_{0R} \subset D_\theta \times D_R$  such that  $\theta^* \in \text{int } D_{0\theta}$  and  $R^* \in \text{int } D_{0R}$ . If the estimator process  $(\theta_t, R_t)$  hits the boundary of  $D_0 = D_{0\theta} \times D_{0R}$  then we reset it to  $(\theta_0, R_0)$ .

A second safeguard is to prevent  $(\theta_t, R_t)$  from having excessive rate of change in a stochastic sense, i.e. if the drift or the diffusion in (12), (13) exceeds a fixed threshold. Recall that the dynamics for  $\theta_t$  is given by

$$d\theta_t = -\frac{1}{T_0 + t}R_t^{-1}\dot{\varepsilon}_{\theta t}d\varepsilon_t + (\theta_0 - \theta_{t-})dN_t,$$

with

$$d\varepsilon_t = (C(\theta^*)x_t^*dt + dw_t) - C(\theta_t)x_tdt,$$

and  $\dot{\varepsilon}_{\theta t} = -C_\theta(\theta_t)x_t - C(\theta_t)x_{\theta t}$ . Thus the dynamics of  $\theta_t$  involves terms like  $\dot{\varepsilon}_{\theta t}C(\theta^*)x_t^*$ , so we do not have full control over the rate of  $\theta_t$ . Taking into account (10), (11), an alternative, more convenient way of bounding the rate of change from above is obtained by imposing the condition

$$\frac{1}{t + T_0} \left( |x_t|^2 + \|x_{\theta,t}\|^2 \right) \leq \delta. \quad (14)$$

A resetting takes place if the left hand side equals the threshold  $\delta$ . Thus we enforce that  $|x_t|, |x_{\theta,t}| \leq Ct^{1/2}$ . With this in mind we conclude that  $\theta_t$  satisfies the condition:

**Condition 3.** (*Slow variation*). *The process  $\theta_t$  satisfies*

$$d\theta_t = \beta_t dt + \sigma_t dw_t + q_t |x_t^*| dt + (\theta_0 - \theta_{t-})dN_t \quad (15)$$

with  $|\beta_t| + \|\sigma_t\|^2 \leq \delta$  for all  $t$ , and  $\|q_t\| \leq c$  for any  $c > 0$  if only  $t$  is large enough.

Note that this condition defines a class of slowly time varying processes which is an extension of what was introduced in [9]. The novelty is the appearance of the term  $q_t |x_t^*|$ . Hence the arguments given in [9] have to be supplemented by an analysis of the effect of this term, see the discussion below Theorem 1.

Thus we get a strictly increasing sequence of stopping times  $\tau_i > 0$  with  $\tau_i = \infty$  allowed, such that for any finite  $\tau_i$  we have  $\theta_{\tau_i} = \theta_0$  and  $R_{\tau_i} = R_0$ . Thus  $(\theta_t, R_t) t \geq 0$ , is an adapted, piecewise continuous càdlàg (right-continuous with left limit) process. Moreover  $(\theta_t, R_t)$  is slowly varying in

$(\tau_i, \tau_{i+1})$  for all finite  $\tau_i$ , in a stochastic sense. An important additional device is that at any time-point  $\tau_i$  the *state* is also reset: we redefine  $(x_t, x_{\theta t})$  to be  $(x_0, x_{\theta 0})$ .

#### V. RML METHOD WITH PARAMETER AND STATE-RESETTING

Letting  $N_t$  denote the counting process associated with events that a resetting has taken place, we get the following final algorithm, with the extended state-process  $(x_t, x_{\theta t}, \varepsilon_t, \varepsilon_{\theta t})$  being defined via (8)-(11):

$$d\theta_t = -\frac{1}{T_0 + t}R_t^{-1}\dot{\varepsilon}_{\theta t}d\varepsilon_t + (\theta_0 - \theta_{t-})dN_t \quad (16)$$

$$dR_t = \frac{1}{T_0 + t}(\dot{\varepsilon}_{\theta t}\dot{\varepsilon}_{\theta t}^T - R_t)dt + (R_0 - R_{t-})dN_t. \quad (17)$$

To ensure convergence we need to impose some conditions on the associated ODE defined by:

$$\dot{\theta}_t = -\frac{1}{T_0 + t}R_t^{-1}W_\theta(\theta_t) \quad (18)$$

$$\dot{R}_t = \frac{1}{T_0 + t}(R(\theta_t) - R_t), \quad (19)$$

with  $R_{ij}(\theta) = \lim_{t \rightarrow \infty} \mathbf{E}(\dot{\varepsilon}_{\theta t}^T(\theta)\dot{\varepsilon}_{\theta j t}(\theta))$ . The solution of this ODE, starting from  $\xi$  at time  $s$  will be denoted by  $y(t, s, \xi)$ . The following condition is taken from [9], in a slightly edited form.

**Condition 4.** *Let the compact truncation domain  $D_{0\theta} \times D_{0R} \subset D_\theta \times D_R$  be such that  $\eta^* = (\theta^*, R^*) \in \text{int } D_{0\theta} \times D_{0R}$ . Moreover assume that: (i) There exists a compact convex set  $D'_{0\theta} \times D'_{0R} \subset D_\theta \times D_R$  such that*

$$y(t, s, \xi) \in D'_{0\theta} \times D'_{0R} \quad \text{for } \xi \in D_{0\theta} \times D_{0R}, \quad (20)$$

$$y(t, s, \xi) \in D_\theta \times D_R \quad \text{for } \xi \in D'_{0\theta} \times D'_{0R} \quad (21)$$

for all  $t \geq s \geq 1$ . Furthermore, for  $\xi \in D'_{0\theta} \times D'_{0R}$

$$\lim_{t \rightarrow \infty} y(t, s, \xi) = \eta^*,$$

and also for  $\xi \in D'_{0\theta} \times D'_{0R}$ ,

$$\left\| \frac{\partial}{\partial \xi} y(t, s, \xi) \right\| \leq C_0(s/t)^\alpha \quad (22)$$

with some  $C_0 \geq 1, \alpha > 0$  for all  $t \geq s \geq 1$ . (ii) We have an initial estimate  $\eta_0 = \xi$  such that for all  $t \geq s \geq 1$  we have  $y(t, s, \xi) \in \text{int}(D_{0\theta} \times D_{0R})$ .

The main point in the above condition is in the characterization of the domains. It can be shown by local analysis that we can take  $\alpha > 1 - c$  for any  $c > 0$ . For the formulation of our results we need the following concept.

**Definition 1.** *Let  $u_t$  be a causal function of the above estimator process. We say that  $u_t = O_M(1)$  in a limited sense, (briefly: i.l.s.), if for all finite  $m \geq 1$  there exists a threshold  $\delta_0$ , controlling the rate of change of the estimator process in (14), such that*

$$M_m(u) = \sup_t \mathbf{E}^{1/m} |u_t|^m < \infty,$$

provided that  $0 < \delta \leq \delta_0$ . Similarly, for  $c_t > 0$ , deterministic, we say that  $u_t = O_M(c_t)$  in a limited sense, if  $u_t/c_t = O_M(1)$  in a limited sense.

The following theorem was stated as the main result of [9], with an outline of proof, to be discussed after the theorem:

**Theorem 1.** *Assume that Conditions 1, 2, and 4 are satisfied. Then the SDE with jumps defining the RML method with resetting, (16) - (17), has a strong solution in  $[0, \infty)$ , we have  $N_t < \infty$  a.s. for any  $t > 0$ , and for any fixed  $q > 1$*

$$\sup_{s \leq t \leq qs} |\theta_t - \theta^*| = O_M(s^{-1/2}) \quad \text{in a limited sense,}$$

and similarly for  $R_t$ . It follows, that  $(\theta_t, R_t)$  converges to  $(\theta^*, R^*)$  with probability 1, if  $\delta_0$  is sufficiently small.

We should note that the outline of a proof given in [9] did not take into account the effect of  $|x_t^*|$ . The effect of  $|x_t^*|$  show up first in two technical results, which we now restate in a modified form:

**Lemma 1** (see Lemma 1 of [10]). *Let  $V_t > 0$  be an Itô-process satisfying, with  $c > 0$ ,*

$$\begin{aligned} dV_t &= V_t(v_t dt + c|x_t^*| + dM_t) + u_t dt, \\ v_t &\leq -\alpha \quad \text{and} \quad 0 \leq u_t \leq C. \end{aligned}$$

Then, for sufficiently small  $c > 0$ ,

$$\sup_t \mathbf{E}(V_t) < \infty.$$

**Theorem 2** (see Theorem 5 of [10]). *Assume that  $F$  is a stable matrix, and let  $X_t$  satisfy*

$$dX_t = FX_t dt + u_t dt + Gdw_t + (X_0 - X_{t-}) dN_t,$$

with  $u_t \leq c|x_t^*|$  with some  $c > 0$ . Let  $q \geq 0$ , and  $\mathbf{E}|X_0|^q < \infty$ . Then,

$$\sup_t \mathbf{E}(|X_t|^q) < \infty.$$

## VI. ON-LINE VS. OFF-LINE

We have come to the formulation of the main result of the present paper:

**Theorem 3.** *Under the assumptions of Theorem 1 we have, with  $\bar{\theta}_t$  denoting the off-line ML estimation of  $\theta^*$ :*

$$\theta_t - \bar{\theta}_t = O_M\left(\frac{\log t}{t}\right) \quad \text{in a limited sense.} \quad (23)$$

*Outline of proof.* Following the arguments of [7] we consider the dynamics of  $V_\theta(t, \theta_t)$ . Between jumps we can apply the Itô–Wentzel formula, see [12], [13], thus we get

$$\begin{aligned} dV_{\theta,t}(\theta_t) &= dV_{\theta,t}(\vartheta)|_{\vartheta=\theta_t} + V_{\theta\theta,t}(\vartheta)|_{\vartheta=\theta_t} d\theta_t^c + \\ &\quad \frac{1}{2} V_{\theta\theta\theta,t}(\vartheta)|_{\vartheta=\theta_t} d\langle \theta^c \rangle_t + d\langle V_{\theta\theta}(\vartheta), \theta^c \rangle_{\vartheta=\theta_t} + \\ &\quad (V_{\theta,t}(\theta_0) - V_{\theta,t}(\theta_{s-})) dN_t. \end{aligned} \quad (24)$$

We will first show that the effect of the jump term on the r.h.s. is negligible. On the other hand, the first four terms on the right hand side will be denoted as follows

$$\begin{aligned} D_{1,t} &= \dot{\varepsilon}_{\theta,t}(\theta_t) d\bar{\varepsilon}_t(\theta_t) \\ D_{2,t} &= -V_{\theta\theta t}(\theta_t) \frac{1}{T_0 + t} R_t^{-1} \dot{\varepsilon}_{\theta,t} d\varepsilon_t \\ D_{3,t} &= \frac{1}{2} V_{\theta\theta\theta t}(\theta_t) \frac{1}{(T_0 + t)^2} R_t^{-1} \dot{\varepsilon}_{\theta,t} \dot{\varepsilon}_{\theta,t}^T R_t^{-T} dt \\ D_{4,t} &= -\frac{1}{T_0 + t} \dot{\varepsilon}_{\theta\theta,t}(\theta_t) R_t^{-1} \dot{\varepsilon}_{\theta,t} dt. \end{aligned}$$

The key observation is that the first two terms  $D_{1,t}$  and  $D_{2,t}$  approximately cancel each other, while the effect of  $D_{3,t}$  and  $D_{4,t}$  is easily seen to be  $O_M(1/t)dt$  i.l.s. Having proven all these, the proof of Theorem 3 will be completed.

## VII. ESTIMATING $N_{t+1} - N_t$

**Lemma 2.** *Under the conditions of Theorem 1 we have for any  $m \geq 1$*

$$\nu_t = N_{t+1} - N_t = O_M(t^{-m}) \quad \text{i.l.s.}$$

*Proof.* The counting process  $N$  increases at  $s$  exactly if one of the following events takes place:

$$\theta_s \in \partial D_{0\theta} \quad \text{or} \quad |x_s|^2 + \|x_{\theta s}\|^2 \geq s\delta$$

We consider only the case when  $\theta_s \in \partial D$  or, the other case can be handled similarly. Let  $t < \tau_1 < \tau_2 < \dots$  be successive time points when  $\theta_s \in \partial D_{0\theta}$ . The first trick is to consider the non-negative scalar process  $z_t = |\theta_t - \theta_0|^2$ , the dynamics of which can be written as

$$dz_s = \frac{1}{s} \beta_s ds + \frac{1}{s} \sigma_s dw_s - z_{s-} dN_s,$$

where  $\beta_s$  and  $\sigma_s$  are  $M$ -bounded i.l.s. Let us now write

$$\nu_t = N_{t+1} - N_t.$$

The probability of the event that  $\theta_s$  ever enters  $\partial D_{0\theta}$  for  $t \leq s \leq t+1$  is bounded from above as follows: for any  $m \geq 1$

$$P(\nu \geq 1) = O(t^{-m}) \quad \text{i.l.s.}$$

Indeed, this follows directly from by Proposition 2 of [9], stating that  $|\theta_s - \theta^*| = O_M(t^{-1/2})$  i.l.s., combined with a Markov inequality.

Now consider the case when  $\nu_t \geq 2$ . For each  $\tau_i$  with  $i \geq 2$  we must have  $z_{\tau_i-} > \delta'$  with some  $\delta' > 0$ . The lower bound  $\delta'$  is depends among others on the distance of  $\theta_0$  from the boundary  $\partial D_{0\theta}$ . Then we can write for all  $i = 1, \dots, \nu_t - 1$

$$\int_{\tau_i}^{\tau_{i+1}} \frac{1}{s} \beta_t ds + \frac{1}{s} \sigma_s dw_s \geq \delta'.$$

Adding up these inequalities for  $i = 1, \dots, \nu_t - 1$  gives

$$\int_{\tau_1}^{\tau_{\nu_t}} \frac{1}{s} \beta_t ds + \frac{1}{s} \sigma_s dw_s \geq (\nu_t - 1)\delta'.$$

From here we easily conclude that

$$2 \sup_{t \leq r \leq t+1} \left| \int_t^r \frac{1}{s} \beta_t ds + \frac{1}{s} \sigma_s dw_s \right| \geq (\nu_t - 1)\delta'.$$

But the left hand side is  $O_M(1/t)$  i.l.s. We conclude that

$$(\nu_t - 1)\chi_{(\nu_t - 1 \geq 1)} \leq Y_t/t$$

with some  $Y_t = O_M(1)$  i.l.s. But the l.h.s is at least 1, when  $\chi_{(\nu_t - 1 \geq 1)} = 1$ , and 0 otherwise, hence for any  $m \geq 1$ , we can also write

$$(\nu_t - 1)\chi_{(\nu_t - 1 \geq 1)} \leq (Y_t/t)^m.$$

Thus we conclude that

$$(\nu_t - 1)\chi_{(\nu_t - 1 \geq 1)} = (\nu_t - 1)\chi_{(\nu_t - 1 \geq 0)} = O_M(t^{-m}) \quad \text{i.l.s.}$$

for any  $m \geq 1$ . Now, noting that  $\nu_t \chi_{(\nu_t - 1 \geq 0)} = \nu_t \chi_{(\nu_t \geq 1)} = \nu_t$ , and taking into account that  $\chi_{(\nu_t \geq 1)} = O_M(t^{-m})$  i.l.s., the proposition follows  $\square$

Using the above result the cumulative effect of jumps when integrating (24) can be estimated as follows:

$$\left| \sum_{s \leq t} V_{\theta, s}(\theta_0) - V_{\theta, s}(\theta_{s-}) \right| \leq \sum_{t=0}^{\infty} \sup_{t \leq s \leq t+1, \theta \in D_{0\theta}} |V_{\theta, s}(\theta) - V_{\theta, s}(\theta_0)| (N_{t+1} - N_t).$$

Now it is easy to see that

$$\sup_{t \leq s \leq t+1, \theta \in D_{0\theta}} |V_{\theta, s}(\vartheta) - V_{\theta, s}(\theta_0)| = O_M(t).$$

Taking into account Lemma 2 we get that the cumulative effect of jumps is  $O_M(1)$  i.l.s., thus it is negligible compared to  $O_M(\log(t))$ .

### VIII. THE PROCESS $\bar{x}_t(\theta_t) - x_t$

A key technical step in carrying out the program outlined in Section VI is to find a good upper bound for the differences like  $d\bar{\varepsilon}_t(\theta_t) - d\varepsilon_t$ . This boils down to estimate differences like  $\bar{x}_t(\theta_t) - x_t$ , where  $\bar{x}_t(\theta)$  is a frozen parameter process, and  $x_t$  is its on-line approximation. To put the problem in a general framework, let  $F(\theta)$  be a  $C^2$ -functions of stable matrices defined for  $\theta \in D$ , and let  $\theta_t$  be a parameter process satisfying Condition 3. Let  $\bar{x}_t(\theta)$  be generated by the linear stochastic system, with  $\bar{x}_0(\theta) = O_M(1)$ ,

$$d\bar{x}_t(\theta) = F(\vartheta)\bar{x}_t(\theta)dt + G(\theta)dw_t, \quad (25)$$

**Theorem 4.** Let  $x_t$  be defined by

$$dx_t = F(\theta_t)x_t dt + G(\theta_t)dy_t + (x_0 - x_{t-})dN_t,$$

with  $x_0 = O_M(1)$ . Then

$$\bar{x}_t(\theta_t) - x_t = O_M(1/t) \quad \text{in a limited sense.}$$

*Outline of the proof.* We proceed as above by considering the dynamics of  $\bar{x}_t(\theta_t) - x_t$ . Using the Itô-Wentzel formula we get

$$\begin{aligned} d(\bar{x}_t(\theta_t) - x_t) &= d\bar{x}_t(\vartheta)|_{\vartheta=\theta_t} + \bar{x}_{\theta_t}(\vartheta)|_{\vartheta=\theta_t} d\theta_t^c + \\ &\quad \frac{1}{2} \bar{x}_{\theta\theta t}(\vartheta)|_{\vartheta=\theta_t} d\langle \theta^c \rangle_t + d\langle \bar{x}(\vartheta), \theta^c \rangle_t|_{\vartheta=\theta_t} + \\ &\quad (\bar{x}(\theta_0) - \bar{x}(\theta_{t-}))dN_t - dx_t. \end{aligned}$$

The joint contribution of the first and the last terms on the right hand side can be written as

$$\begin{aligned} d\bar{x}_t(\vartheta)|_{\vartheta=\theta_t} - dx_t &= F(\theta_t)\bar{x}(\theta_t)dt + G(\theta_t)dw_t - \\ &\quad (F(\theta_t)x_t dt + G(\theta_t)dw_t + (x_0 - x_{t-})dN_t) = \\ &= F(\theta_t)(\bar{x}(\theta_t) - x_t)dt - (x_0 - x_{t-})dN_t. \end{aligned}$$

Thus we get that  $\bar{x}_t(\theta_t) - x_t$  is defined via a linear, time varying dynamics:

$$d\bar{x}_t(\vartheta)|_{\vartheta=\theta_t} - dx_t = F(\theta_t)(\bar{x}(\theta_t) - x_t)dt + du_t$$

where the input term  $du_t$  is defined by

$$\begin{aligned} du_t &= \bar{x}_{\theta t}(\vartheta)|_{\vartheta=\theta_t} d\theta_t^c + \frac{1}{2} \bar{x}_{\theta\theta t}(\vartheta)|_{\vartheta=\theta_t} d\langle \theta^c \rangle_t + \\ &\quad d\langle \bar{x}(\vartheta), \theta^c \rangle_t|_{\vartheta=\theta_t} + (\bar{x}(\theta_0) - \bar{x}(\theta_{t-}))dN_t - (x_0 - x_{t-})dN_t. \end{aligned}$$

It is easily seen that  $du_t$  can be written as

$$du_t = \frac{1}{t}(b_t dt + s_t dw_t) + v_t dN_t,$$

where the processes  $b, s, v$  are  $M$ -bounded. To complete the proof we have to establish an appropriate form of exponential stability of the underlying homogeneous ODE.

### IX. THE ODE $\dot{\Phi}_t = F(\theta_t)\Phi_t$

Consider now the fundamental matrix of the homogeneous linear ODE:

$$\dot{\Phi}_t = F(\theta_t)\Phi_t, \quad \Phi_0 = I. \quad (26)$$

with  $\theta_t$  satisfying (15).

**Theorem 5.** Assume that  $\theta_t$  satisfies Condition 3. Then for the fundamental matrix  $\Phi_t$  solving (26) we have for any  $0 \leq s \leq t$

$$\|\Phi_t \Phi_s^{-1}\| \leq C_s(\omega) e^{-\alpha(t-s)}$$

with some  $\alpha > 0$ , where  $C_s = O_M(1)$  in a limited sense.

*Outline of proof.* Following the arguments of [10], let  $x_t$  be any column of  $\Phi_t$ , and consider the Lyapunov-function  $V_t = V_t(x_t) = x_t^T P_t x_t$ , where  $P_t$  is smooth solution of the Lyapunov inequality

$$F(\theta_t)^T P_t + P_t F(\theta_t) \leq -\alpha P_t$$

as in Lemma 8 of [10]. Then we get, as in [10], that

$$dV_t \leq V_t(-\alpha dt + dM_t + c|x_t^*| dt).$$

With  $c = 0$  we would apply Theorem 2 of [10], to conclude that  $\mathbf{E}(V_t) = O_M(1)$  in a limited sense. The effect of the term  $c|x_t^*|$  has to be first analyzed in Lemma 1 of [10]. For this purpose the following exponential inequality is useful.

**Lemma 3.** Let  $x_t^*$  be the stationary solution of

$$dx_t^* = Ax_t^* dt + Gdw_t,$$

with  $A$  stable. Then for any  $C > 0$  we have for all  $T \geq 0$

$$\mathbf{E} \left( \exp \left\{ \int_0^T C |x_t^*| dt \right\} \right) \leq K e^{C'T}$$

with some  $C'$  depending on  $C$ .

Taking into account this lemma we get by standard arguments of risk theory the following corollary.

**Corollary 1.** *Let  $x_t^*$  be as above and let  $M > \mathbf{E}(x_t^*)$ . Then*

$$\gamma_s = \sup_{T \geq s} \int_s^T (|x_t^*| - M) dt < \infty \quad a.s.$$

and there exists  $c' > 0$  such that

$$\sup_s \mathbf{E}(\exp\{c'\gamma_s\}) < \infty.$$

#### X. ESTIMATING $D_{1,t} + D_{2,t}$

In order to show that the terms  $D_{1,t}$  and  $D_{2,t}$  approximately cancel each other the key step is to estimate  $V_{\theta\theta t}(\theta_t)R_t^{-1}$ . We proceed as in [6]. First note that by Theorem 1 of [9] we have  $R_t = R^* + O_M(t^{-1/2})$  in a limited sense. Next, we have by Theorem 1.1 and 3.4 of [4] we have

$$\sup_{\vartheta \in D_0} \left| \frac{1}{t} V_{\theta\theta t}(\vartheta) - W_{\theta\theta}(\vartheta) \right| = O_M(t^{-1/2}).$$

It follows that

$$\frac{1}{t} V_{\theta\theta t}(\theta_t) = R^* + O_M(t^{-1/2}) \quad \text{in a limited sense.}$$

Thus we conclude that

$$\frac{1}{t} V_{\theta\theta t}(\theta_t)R_t^{-1} - I = O_M(t^{-1/2}) \quad \text{in a limited sense.}$$

Now between jumps we have  $d\varepsilon_t = d\bar{\varepsilon}_t(\theta_t) + O_M(1/t)dt$  and  $\dot{\varepsilon}_{\theta_t} = \dot{\bar{\varepsilon}}_{\theta_t}(\theta_t) + O_M(1/t)$ , both i.l.s., thus

$$\begin{aligned} D_{2,t} &= V_{\theta\theta t}(\theta_t)d\theta_t = -\frac{1}{t} V_{\theta\theta t}R_t^{-1}\dot{\varepsilon}_{\theta_t}d\varepsilon_t = \\ &= -(I + O_M(t^{-1/2}))(\dot{\bar{\varepsilon}}_{\theta_t}(\theta_t)d\bar{\varepsilon}_t(\theta_t) + O_M(t^{-1})dt) \end{aligned}$$

in a limited sense. Expanding the right hand side, and replacing  $\theta_t$  by  $\theta^*$ , except in the first, dominant term we get:

$$V_{\theta\theta t}(\theta_t)d\theta_t = -\dot{\bar{\varepsilon}}_{\theta_t}(\theta_t)d\bar{\varepsilon}_t(\theta_t) + \delta_t\dot{\bar{\varepsilon}}_{\theta_t}(\theta_t)dw_t + O_M(t^{-1})dt$$

where  $\delta_t = O_M(t^{-1/2})$ .

Note that the dominant term exactly cancels  $D_{1,t} = dV_{\theta t}(\theta_t) = \dot{\bar{\varepsilon}}_{\theta_t}(\theta_t)d\bar{\varepsilon}_t(\theta_t)$ . Integrating  $D_{1,t} + D_{2,t}$  over  $[0, T]$ , we get, by Burkholder inequality, that the cumulative contribution is  $O_M(\log T)$  in a limited sense. With this the proof of Theorem 3 is completed.

#### REFERENCES

- [1] B. Delyon, *Stochastic approximation with decreasing gain: Convergence and asymptotic theory*. Technical report, Université de Rennes, 2000.
- [2] D. DeWolf and D. Wiberg, "An ordinary differential equation technique for continuous-time parameter estimation," *IEEE Trans. Autom. Control*, vol. 38, pp. 514–528, 1993.
- [3] T. E. Duncan and B. Pasik-Duncan, "Adaptive control of continuous time stochastic systems," *Internat. J. Adapt. Control Signal Process.*, vol. 16, no. 5, pp. 327–340, 2002.
- [4] L. Gerencsér, "On a class of mixing processes," *Stochastics*, vol. 26, pp. 165–191, 1989.
- [5] —, "On the martingale approximation of the estimation error of ARMA parameters," *Systems & Control Letters*, vol. 15, pp. 417–423, 1990.
- [6] —, "Strong approximation of the recursive prediction error estimator of the parameters of an ARMA process," *Systems & Control Letters*, vol. 21, pp. 347–351, 1993.
- [7] L. Gerencsér, I. Gyöngy, and G. Michaletzky, "Continuous-time recursive maximum-likelihood method. A new approach to Ljung's scheme," in *A Bridge Between Control Science and Technology. Proc. of the 9th Triennial World Congress of IFAC, Budapest*, L. Ljung and K. Åström, Eds. Pergamon Press, Oxford, 1984, pp. 75–77.
- [8] L. Gerencsér and Z. Mátyás, "Almost sure and  $L_q$ -convergence of the re-initialized BMP scheme," in *Proc. 46th IEEE Conference on Decision and Control, New Orleans, LA, USA, 13-15 December 2007*, D. Castanon and J. Spall, Eds. Omnipress, 2007, pp. 965–974.
- [9] L. Gerencsér and V. Prokaj, "Recursive identification of continuous-time linear stochastic systems - convergence w.p.1 and in  $L_q$ ," in *European Control Conference 2009 ECC'09 23-26 August 2009, Budapest, Hungary*, 2009, pp. 1209–1214, cD-ROM MoC1.5.
- [10] —, "Stability of a class of hybrid linear stochastic systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 5, pp. 1233–1238, 2010.
- [11] D. Levanony, A. Shwartz, and O. Zeitouni, "Recursive identification in continuous-time stochastic processes," *Stochastic Process. Appl.*, vol. 49, no. 2, pp. 245–275, 1994.
- [12] B. L. Rozovskii, "The Itô-Ventcel' formula," *Vestnik Moskov. Univ. Ser. I Mat. Meh.*, vol. 28, no. 1, pp. 26–32, 1973.
- [13] A.-S. Sznitman, "Martingales dépendant d'un paramètre: une formule d'Itô," *Z. Wahrsch. Verw. Gebiete*, vol. 60, no. 1, pp. 41–70, 1982.
- [14] J. H. van Schuppen, "Convergence results for continuous-time adaptive stochastic filtering algorithms," *J. Math. Anal. Appl.*, vol. 96, no. 1, pp. 209–225, 1983.