Smith-Predictor type Structure for a Class of Infinite-Dimensional Systems: Optimal Control and Performance Limitation Formula

Kenji Kashima

***Abstract*— In this talk we investigate control problems for infinite-dimensional systems whose transfer matrices are expressible in terms of a rational transfer matrix and a scalar (possibly irrational) inner function. This class of systems is capable of describing many practical control problems, when weighting functions are rational and plants have at most a finite number of unstable modes or zeros. In the first half of this talk the concept of Smith-predictors, that was originally used for I/O delay systems, is extended to the aforementioned class of systems. This allows us to reduce the optimal control problems to easily checkable criteria that do not require the solution of operator-valued equations. Furthermore, the obtained (stabilizing or suboptimal) controllers are shown to have the structure of Smith-predictors, or their dual. In the second half of the talk we derive a new expression for the H^2 performance limit, based on state-space representation. The resulting formula, given as a functional of the inner function, helps us to understand how achievable H^2 performance deteriorates due to the plant's non-minimum phase properties or unstable modes. The example of a linear quantum control system suffering from feedback delay is given to illustrate the result.**

I. Introduction

It is known that a large class of standard control problems for infinite-dimensional systems end up with *operator-valued* Riccati equations. Unfortunately, it is not necessarily easy to solve these operator equations due to their infinite-dimensional nature. From the computational point of view, the skew-Toeplitz approach and the so-called AAK (Adamjan-Arov-Krein) theory are more attractive in that they yield a finite rank condition for optimality. The author and his colleagues have generalized these result to a practically useful class of infinite-dimensional systems that maintain the advantage of the finite-dimensionality assumed in the skew-Toeplitz approach [1], [2], [3], [4].

In the talk given by the author at the MTNS2010, these results are introduced focusing on the performance limit formula obtained in [1] and its application to quantum systems [4]. This paper is the outline of the talk. It is not intended to be a complete survey, and not all details could be included here. For further reading see the papers listed in the references section, and their references.

A. H^2 Performance Limit based on State-Space Representation

The analysis of performance limitation, one of the most classical topics in control theory, has been paid renewed attention in recent years, especially in the context of networked control systems (NCS). This research aims to clarify the relationship between the best achievable control performance and plant parameters such as unstable poles/zeros, I/O delay length or weighting functions. These results help us to characterize *easily controllable* plants, or even to *design* plants if these parameters can be chosen by the designer, within some constraints. Thus, for this purpose, it is desirable that the obtained formula should satisfy the following requirements:

- plant parameters remain explicit,
- simple enough to allow intuitive interpretation, and
- capable of dealing with as large a class of plants (or control problems) as possible.

It is, however, not easy to satisfy these conflicting requirements simultaneously.

In this talk, we focus on H^2 control problems. Concerning the third requirement, the standard H^2 control problem can cover a large class of these problems. For this problem, a general solution via matrix Riccati equations is well-known. However, it is obviously difficult to understand the relationship between the achievable performance and plant parameters (in this case, system matrices) because the analytical solution to the corresponding matrix Riccati equations is not obtainable. Hence, most of the existing closed-form expressions of the best achievable performance are based on *transfer function representation*, which is adequate for handling plant parameters directly. Although these results were extended to MIMO cases, from a modern control theoretic point of view, this is not the only possible line of research to pursue.

In the following sections, we derive a closed formula based on *state-space representation*. To this end, we confine ourselves to generalized plants with a special structure represented by a transfer matrix and an inner function. This formulation covers a large class of control problems, and enables us to derive a new analytical expression for the best achievable H^2 control performance via infinite-dimensional control theory. While the obtained formula uses the solutions of a couple of Riccati equations, the plant non-minimum phase factor or unstable modes, both of which are represented by inner function, remains explicit. This result clarifies how these properties degrade the performance limit in the framework of the standard H^2 control problem.

Definition and Notation

For a complex function matrix f, its para-Hermitian conjugate is denoted by $f^{\sim}(s) := \overline{f(-\overline{s})}^{\mathsf{T}}$ where X^{T} is the transpose of X. As usual, H^p (H^p_-) is Hardy p-space on the

K. Kashima is with Graduate School of Information Science and Engineering, Tokyo Institute of Technology, Tokyo, Japan kashima@mei.titech.ac.jp

ISBN 978-963-311-370-7

open right (left) half complex plane, respectively. The H^2-norm of f is denoted by $\|f\|$. For state-space realization of rational transfer matrix,

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := D + C(sI - A)^{-1}B.$$

For transfer matrices of appropriate dimensions, the (lower) linear fractional transform is denoted by $\mathcal{F}_l(\cdot,\cdot)$. The size of matrices is omitted as it is clear from the context.

A scalar function $m \in H^\infty$ is said to be *inner* if $|m(s)| = 1$ a.e. on the imaginary axis. Since any inner function satisfies

$$mm^{\sim} = 1 \tag{1}$$

a.e. on the imaginary axis, the domain of m is analytically extended to $\mathbb{C}_-$ by using this equality, i.e., $m = 1/m^{\sim}$. Therefore, zeros (respectively, poles) of m are poles (respectively, zeros) of $m^{\sim}$. For any inner m, mH^2 is right shift invariant subspace in H^2. Let $H(m)$ be the orthogonal complement of mH^2 on H^2. Function space $H(m)$ is left shift invariant and satisfies

$$H(m) = \{x \in H^2 : m^{\sim}x \in H^2_-\}. \tag{2}$$

From (2) and $m^{\sim} = 1/m$, every singularity of any element of $H(m)$ is a pole of m. See also [2] and references therein for other properties of $H(m)$.

For a scalar complex function f, let $\mathcal{M}_f$ be the set of square matrices X such that $m^{\sim}$ is analytic in a neighborhood of every eigenvalue of X. For $X \in \mathcal{M}_f$, matrix function $m^{\sim}(X)$ can be defined as follows:

$$f^{\sim}(X) := \frac{1}{2\pi j}\int_{\Delta} f^{\sim}(s)(sI - X)^{-1}ds, \tag{3}$$

where the closed contour Δ encircles all eigenvalues of X counter-clockwise and $f^{\sim}$ is analytic inside Δ.

The *generalized m-truncation* $\pi^m[\cdot]$ plays a crucial role in this talk ([3]):

Definition 1: Let m be an inner function and $W(s) = C(sI - A)^{-1}B$ with $A \in \mathcal{M}_m$. Define

$$W^{(m)} := \left[\begin{array}{c|c} A & m^{\sim}(A)B \\ \hline C & 0 \end{array}\right] \tag{4}$$

$$\pi^m[W] := W - mW^{(m)} \in H(m). \tag{5}$$

This definition does not depend on the choice of the realization of W. It should be noted that $\pi^m[W]$ is stable even if W is unstable.

Throughout this paper, the internal stability of the interconnected system is defined as the stability of all input-output maps; see e.g., Definition 1 of [3]. Moreover, we impose the following standard assumption on systems matrices in generalized plants:

Assumption 1:

1) (A, B_2) is stabilizable and (A, C_2) is detectable.
2) For any $\omega \in \mathbb{R}$,
$$\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}, \begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$$
are row- and column-full rank, respectively.
3) $D_{12}^{\mathsf{T}}D_{12} = I,\ D_{21}D_{21}^{\mathsf{T}} = I.$
4) $D_{22} = 0.$

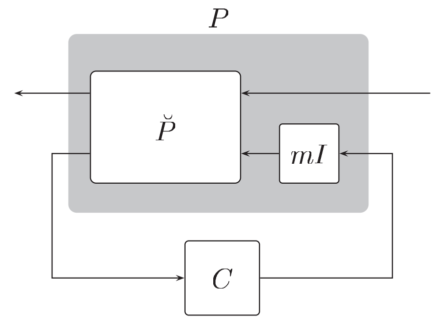

Fig. 1. Problem formulation

II. Performance deterioration due to non-minimum phase properties

Define the generalized plant P by

$$P := \left[\begin{array}{c:c} \breve{P}_{11} & \breve{P}_{12} \\ \hdashline \breve{P}_{21} & \breve{P}_{22} \end{array}\right]\left[\begin{array}{c:c} I & 0 \\ \hdashline 0 & mI \end{array}\right] \tag{6}$$

with rational transfer matrix $\breve{P}$

$$\breve{P} := \left[\begin{array}{c:c} \breve{P}_{11} & \breve{P}_{12} \\ \hdashline \breve{P}_{21} & \breve{P}_{22} \end{array}\right] := \left[\begin{array}{c|c:c} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ \hdashline C_2 & D_{21} & D_{22} \end{array}\right] \tag{7}$$

and an inner function m; see Fig. 1. We denote the set of its internally stabilizing controllers by $\mathcal{C}$. Our goal is to derive an analytical expression of

$$E_{\rm opt} := \inf_{C \in \mathcal{C}} \|\mathcal{F}_l(P, C)\|. \tag{8}$$

To see how practical control problems fall into this problem, let us consider the mixed sensitivity optimization problem for

$$G = m \cdot G_r \cdot G_o, \tag{9}$$

where m is inner, G_r is rational (possibly with unstable poles and/or unstable zeros), and $G_o, {G_o}^{-1} \in H^\infty$. While G_o, G_r are allowed to be matrix-valued m is limited to a scalar function. For rational weighting function W_s, W_t,

$$\breve{P} = \left[\begin{array}{c:c} W_s & W_sG_r \\ 0 & W_tG_r \\ \hdashline 1 & G_r \end{array}\right] \tag{10}$$

yields

$$\begin{bmatrix} W_s(1 - GC)^{-1} \\ W_tGC(1 - GC)^{-1} \end{bmatrix} = \mathcal{F}_l(P, C). \tag{11}$$

Here we ignored G_o since it can be absorbed into the controllers.

In this problem formulation, there are 2 problem data: a rational transfer matrix $\breve{P}$ and an inner function m. As seen in the example above,

- m characterizes (a part of) plant non-minimum phase property of the actual plant, and
- $\breve{P}$ represents other dynamics of the actual plant (non-minimum phase factor which is not included in m, unstable modes and outer part) and weighting functions.

In what follows, we attempt to derive a closed formula for $E_{\rm opt}$ in which m remains explicit. In other words, we focus on the following question: *How does non-minimum phase property represented by inner functions m change the H^2 performance limit in the standard control problem framework?* For the simplest example, the obtained formula enables us to understand the performance changing effect caused by unstable zero(s) when m is a Blaschke product. Notice that the inner function m does not need to be rational. Therefore, we take the pure delay $m(s) = e^{-hs}$ with $h > 0$ in order to analyze the effect of I/O delay length. For another example of infinite-dimensional cases, it is known that the transfer function of an Euler-Bernoulli beam equation also has the factorization in (9).

Hereafter $A \in \mathcal{M}_m$ is assumed. This guarantees that no unstable pole-zero cancellation exists between $\breve{P}$ and m. Therefore, under this assumption, the non-minimum phase property m always degrades the performance limit, i.e., $E_{\rm opt} \geq E_{\rm n}$ where

$$E_{\rm n} := \min_{\breve{C} \in \breve{\mathcal{C}}} \left\| \mathcal{F}_l \left(\breve{P}, \breve{C} \right) \right\| \tag{12}$$

with $\breve{\mathcal{C}}$ being the set of internally stabilizing controllers of $\breve{P}$. Moreover its effect can be simply characterized by the following:

Theorem 1 ([1]): Suppose that $\breve{P}$ in (7) satisfies Assumption 1 and that an inner function m satisfies $A \in \mathcal{M}_m$. Let $X, Y \geq 0$ be stabilizing solutions to the following Riccati equations:

$$XA + A^{\mathsf{T}}X + C_1^{\mathsf{T}}C_1 - F^{\mathsf{T}}F = 0, \tag{13}$$
$$AY + YA^{\mathsf{T}} + B_1B_1^{\mathsf{T}} - LL^{\mathsf{T}} = 0, \tag{14}$$

where F, L are given by

$$F := -(B_2^{\mathsf{T}}X + D_{12}^{\mathsf{T}}C_1), \tag{15}$$
$$L := -(YC_2^{\mathsf{T}} + B_1D_{21}^{\mathsf{T}}). \tag{16}$$

Then

$$E(m) := \|\pi^m[\Theta]\|, \ \Theta := \left[\begin{array}{c|c} A & L \\ \hline F & 0 \end{array} \right] \tag{17}$$

with F, L defined in (15), (16) satisfies

$$E_{\rm opt}^2 = E_{\rm n}^2 + E(m)^2, \tag{18}$$

where $E_{\rm n}$ and $E_{\rm opt}$ are given in (12) and (8).

Since the first term $E_{\rm n}$ in (18) is the *nominal* performance limit, i.e., $m(s) = 1$, the second term $E(m)$ represents the performance deterioration due to the non-minimum phase property. In (17), Θ does not depend on m. In this sense, the inner function remains explicit in the expression. It is, in general, impossible to write down the relationship between E and system matrices of $\breve{P}$ because the solutions of the Riccati equations are used. This is a state-space counterpart of the fact that most of transfer function based formulae require the coprime factorization of the plant. However, the fact that Θ and $\breve{P}$ have the common A-matrix enables us to intuitively interpret this result.

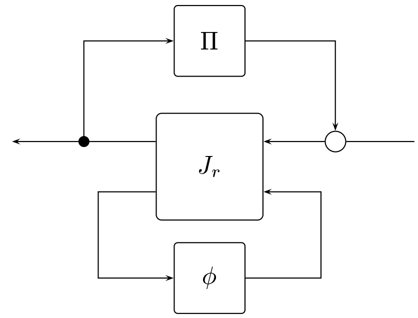

Fig. 2. Smith-predictor type structure

Remark 1: An explicit realization of all required controllers can be obtained as follows: all γ-suboptimal controllers are parameterized in the form depicted in Fig. 2 with

$$\begin{aligned} J_r &:= \left[\begin{array}{c|cc} A_J & -m^{\sim}(A)L & B_2 \\ \hline F & 0 & I \\ -C_2m(A) & I & 0 \end{array} \right] \\ A_J &:= A + B_2F + m^{\sim}(A)LC_2m(A) \\ \Pi &:= \pi^m[P_{22}^*] \in H(m), \quad P_{22}^* := \left[\begin{array}{c|c} A & B_2 \\ \hline -C_2m(A) & 0 \end{array} \right]. \end{aligned}$$

Note that both J_r and Π are independent of γ and the choice of the stable free (but, norm-bounded) parameter ϕ. If we allow ϕ to be any function in H^∞ then Fig. 2 gives a parameterization of all internally stabilizing controllers.

In Fig. 2 the only infinite-dimensional part is the block represented by Π in the feedback path. When $m(s) = e^{-hs}$, this block is a continuous-time FIR system which can be implemented by finite-time integration. This structure in delay compensating controllers is well-known as the modified Smith predictor. As a natural extension, the effect of the general non-minimum phase factor, represented by inner function m, can be compensated for by the *feedback* component $\Pi \in H(m)$. □

Theorem 1 says that the performance degrading effect of non-minimum phase property is given by (17). Since, however, $E(m)$ is defined as the H^2-norm of the infinite-dimensional system $\pi^m[\Theta]$, it is difficult to compute and also to interpret. In view of this, the following formula for $E(m)$ is given:

Corollary 1 ([1]): Under the same assumption and notation as those in Theorem 1, suppose also $A, -A^{\mathsf{T}} \in \mathcal{M}_m$ and that the pair (A, B_1) and $(-A, B_1)$ has no common controllable mode. Then Lyapunov equation

$$AW + WA^{\mathsf{T}} + B_1B_1^{\mathsf{T}} = 0 \tag{19}$$

has a solution W, and

$$E(m)^2 = {\rm trace} \left(F \left(m^{\sim}(A)Zm^{\sim}(A)^{\mathsf{T}} - Z \right) F^{\mathsf{T}} \right) \tag{20}$$

with $Z := Y + W$ independently of the choice of W.

Recall that Z and F are independent of m in (20). Thus Corollary 1 says that $m^{\sim}(A)$ determines the performance degrading effect caused by the non-minimum phase factor represented by inner function m. To be more precise, the effect is given in quadratic form (20) with respect to $m^\sim(A)$. In what follows, we give a detailed interpretation of this formula focusing on the stability of A.

Let us consider the case where A is anti-stable, i.e., $-A^\mathsf{T}$ is stable. In this case, Z is the observability Grammian for stable $\Theta^\sim$, that is, Z is a positive semi-definite matrix such that

$$\|\Theta\|_{H^2_-}^2 = \|\Theta^\sim\|^2 = \operatorname{trace}\left(FZF^\mathsf{T}\right).$$

Hence $E(m)$ becomes arbitrarily large as an unstable zero of m approaches to an eigenvalue of A. This can be viewed as a generalization of the well-known fact that the achievable H^2 performance limit deteriorates when an unstable pole and an unstable zero are located closely. Let us take $m(s) = e^{-hs}$ to analyze the performance degrading effect of I/O delay length h. In this case, $E(m)$ is determined by $m^\sim(A) = e^{Ah}$ and increases exponentially with respect to h.

Conversely, if A is stable, $-Z$ is the controllability Grammian for Θ in (17). Negative semi-definite matrix Z satisfies

$$\|\Theta\|^2 = -\operatorname{trace}\left(FZF^\mathsf{T}\right).$$

In the case of stable A, the performance degrading effect $E(m)$ *saturates*, that is,

$$E(m)^2 \le \|\Theta\|^2 \tag{21}$$

for any inner m (inner function such that $m^\sim(A) = 0$ yields the equality).

Note that $m^\sim(A) - I$ is positive (resp. negative) semi-definite for anti-stable (resp. stable) A. When $A = a$ is a non-zero real scalar we have

$$E(m)^2 = \left|\frac{1}{m(a)^2} - 1\right| \cdot \|\Theta\|_{L^2}^2. \tag{22}$$

III. Duality between Non-Minimum Phase Factor and Unstable Modes

In this section, we analyze the performance degrading effect caused by unstable modes. Problem formulation is the same as that in Section 3 except for the meaning of the inner function. Define the generalized plant P_u by

$$\hat{P} := \left[\begin{array}{c:c} \hat{P}_{11} & \hat{P}_{12} \\ \hdashline \hat{P}_{21} & \hat{P}_{22} \end{array}\right] := \left[\begin{array}{c|c:c} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & 0 & \hat{D}_{12} \\ \hdashline \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right] \tag{23}$$

$$P_\mathrm{u} := \left[\begin{array}{c:c} I & 0 \\ \hdashline 0 & m^\sim I \end{array}\right] \left[\begin{array}{c:c} \hat{P}_{11} & \hat{P}_{12} \\ \hdashline \hat{P}_{21} & \hat{P}_{22} \end{array}\right] \tag{24}$$

with an inner function m. Denote the set of its internally stabilizing controllers by $\mathcal{C}_\mathrm{u}$. In what follows, we derive an analytical expression of

$$E_{\mathrm{u,opt}} := \inf_{C \in \mathcal{C}_\mathrm{u}} \|\mathcal{F}_l(P_\mathrm{u}, C)\|. \tag{25}$$

In comparison to (9), we can formulate control problem for the plant which can be factored as

$$\hat{G} = \hat{G}_r \cdot \hat{G}_o \cdot m^\sim, \tag{26}$$

where m is inner, $\hat{G}_r$ is rational (possibly with unstable poles and/or unstable zeros), and $\hat{G}_o, \hat{G}_o^{-1} \in H^\infty$. Since $m^\sim = 1/m$ unstable zeros of m represents unstable poles of G. Therefore we can analyze the effect on performance limit caused by unstable modes by taking m an appropriate Blaschke product. Again, m is not assumed to be rational. Hence m can deal with even infinitely many unstable modes; see Subsection IV.B.

Fortunately, we do not need to derive another formula for $E_{\mathrm{u,opt}}$ because this problem can be converted to the computation of E_{opt} by changing problem data appropriately. We again consider the weighted mixed sensitivity optimization to explain the basic idea of this problem conversion. For stable rational weights $\hat{W}_s$ and $\hat{W}_t$, let us find an internally stabilizing controller $\hat{C}$ that minimizes the weighted mixed sensitivity

$$\left\|\begin{bmatrix} \hat{W}_s(1-\hat{G}\hat{C})^{-1} \\ \hat{W}_t(1-\hat{G}\hat{C})^{-1}\hat{G}\hat{C} \end{bmatrix}\right\|.$$

By straightforward computation, it can be verified that $\hat{C}$ internally stabilizes $\hat{G}$ if and only if $C := \hat{C}^{-1}$ internally stabilizes $G := \hat{G}^{-1}$ and

$$\left\|\begin{bmatrix} \hat{W}_s(1-\hat{G}\hat{C})^{-1} \\ \hat{W}_t(1-\hat{G}\hat{C})^{-1}\hat{G}\hat{C} \end{bmatrix}\right\| = \left\|\begin{bmatrix} \hat{W}_t(1-GC)^{-1} \\ \hat{W}_s(1-GC)^{-1}GC \end{bmatrix}\right\|. \tag{27}$$

It readily follows from (26) that G has a factorization of the form of (9). This problem, therefore, falls into the framework in the previous section. We generalize this problem conversion technique under the following:

Assumption 2:

1) $(\hat{A}, \hat{B}_2)$ is stabilizable and $(\hat{A}, \hat{C}_2)$ is detectable.
2) $\hat{D}_{22}$ is nonsingular.
3) $A := \hat{A} - \hat{B}_2\hat{D}_{22}^{-1}\hat{C}_2$ belongs to $\mathcal{M}_m$.
4) $D_{12} = 0$.

Remark 2: As is noted in the previous section, D_{22} in (6) changes neither the stabilizability nor the best achievable performance E in (8). However, this is not necessarily true for $\hat{D}_{22}$ in (24). Roughly speaking, the nonsingularity of $\hat{D}_{22}$ guarantees that $\hat{P}$ and $m^\sim$ cancels no unstable pole/zero *at infinity*. In fact, when $m^\sim$ has a sequence $\{\lambda_i\}$ of unstable poles such that $|\lambda_i| \to \infty$, the right-invertibility is necessary for the existence of an internally stabilizing controller; see [**?**]. □

Proposition 1: Suppose that $\breve{P}_\mathrm{u}$ in (23) satisfies Assumption 2 and define

$$\breve{P} := \left[\begin{array}{c|c:c} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ \hdashline C_2 & D_{21} & D_{22} \end{array}\right] \tag{28}$$

$$:= \left[\begin{array}{c|c:c} \hat{A} - \hat{B}_2\hat{D}_{22}^{-1}\hat{C}_2 & \hat{B}_1 - \hat{B}_2\hat{D}_{22}^{-1}\hat{D}_{21} & -\hat{B}_2\hat{D}_{22}^{-1} \\ \hline \hat{C}_1 & 0 & 0 \\ \hdashline \hat{D}_{22}^{-1}\hat{C}_2 & \hat{D}_{22}^{-1}\hat{D}_{21} & \hat{D}_{22}^{-1} \end{array}\right].$$

Then the following statements hold:

1) (A, B_2) is stabilizable and (A, C_2) is detectable,
2) A belongs to $\mathcal{M}_m$, and

3) $E_{\rm u,opt}$ in (25) is equal to $E_{\rm opt}$ defined by (28), (6) and (8).

This proposition means that the effect of unstable modes can be analyzed according to Theorem 2. We give 2 comments on the interpretation of the result. Firstly, in order to analyze the performance degrading effect caused by unstable poles, we need the solution to the Riccati equations corresponding not to (23) but to (28). Secondly, unstable modes contribute only through

$$m^{\sim}(\hat{A} - \hat{B}_2\hat{D}_{22}^{-1}\hat{C}_2).$$

Since eigenvalues of $\hat{A} - \hat{B}_2\hat{D}_{22}^{-1}\hat{C}_2$ are invariant zeros of $\hat{P}$, we can interpret this result in the same way as in the previous section; the best achievable H^2 performance limit deteriorates when an unstable pole and an unstable zero are located closely.

Remark 3: In the dual case investigated in Section 5, since $C\hat{C} = I$ all suboptimal controllers are given by exchanging the input and output signals in Fig. 2. As a result, any suboptimal controller is given in the form of a rational system with a common stable and infinite-dimensional *feedforward* component. This compensates for the effect of unstable modes represented by $m^{\sim}$. □

IV. EXAMPLES

A. Linear Quantum Systems

1) Problem formulation: Consider a quantum system which interacts with a vacuum electromagnetic field through the system operator

$$c = \mathscr{C}x, \tag{29}$$

where $x = [q, p]^\mathsf{T}$ and $\mathscr{C} \in \mathbb{C}^{1\times 2}$. Further, suppose that the system is trapped in a harmonic potential, and that a linear potential is an input to the system. The system Hamiltonian H_t at time t is given by

$$H_t = \frac{1}{2}x^\mathsf{T}Gx - x^\mathsf{T}\Sigma B u_t,\ \Sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{30}$$

where $u_t \in \mathbb{R}$ is the control input at time t, $G = G^\mathsf{T} \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^2$. Then, by using the commutation relation $[q, p] = i$, we obtain the following linear equation for $x_t = [q_t, p_t]^\mathsf{T}$:

$$dx_t = Ax_t dt + Bu_t dt + i\Sigma(\mathscr{C}^\mathsf{T} dB_t^\dagger - \mathscr{C}^\dagger dB_t), \tag{31}$$

where $A := \Sigma[G + {\rm Im}(\mathscr{C}^\dagger\mathscr{C})]$, and the field operators $B_t^\dagger$ and B_t are the creation and annihilation operator processes that satisfy the *quantum Itô rule*. Measurement processes are described as follows. Suppose that the field observable $e^{-i\phi}B_t + e^{i\phi}B_t^\dagger$ is measured by the perfect homodyne detector, where $\phi \in [0, 2\pi)$ denotes the detector parameter that the experimenter can change. Then, the infinitesimal increment of the observable y_t ig given as

$$dy_t = (e^{-i\phi}\mathscr{C} + e^{i\phi}\mathscr{C}^*)x_t dt + e^{-i\phi}dB_t + e^{i\phi}dB_t^\dagger. \tag{32}$$

In the experiment of the feedback control, it is of importance to reduce the time delay by carefully setting up the experimental devices and achieve the best performance possible. However, some quantity of the time delay remains in practice. Those delays are mainly originated from the computational time for a controller and the transition delay of signals. Thus, they should be modelled practically as input-output delays in the feedback loop, i.e., u_t is determined by the past output record $\{y_s\}_{s\le t-h}$.

Let us define the quantum noise vector

$$w_t := \begin{bmatrix} e^{-i\phi}B_t + e^{i\phi}B_t^\dagger \\ -iB_t + iB_t^\dagger \end{bmatrix}. \tag{33}$$

satisfying

$$\langle w_t \rangle = 0, \tag{34}$$

$$dw_t dw_s^\mathsf{T} = \begin{cases} F_\phi dt, & \text{if } s = t \\ 0, & \text{otherwise} \end{cases} \tag{35}$$

where $\langle\cdot\rangle$ denotes the expectation and F_ϕ is the non-negative Hermitian matrix given by

$$F_\phi := \begin{bmatrix} 1 & ie^{-i\phi} \\ -ie^{i\phi} & 1 \end{bmatrix}.$$

Then, (31) and (32) can be rewritten as

$$\begin{aligned} dx_t &= Ax_t dt + B_1 dw_t + Bu_t dt, \\ z_t &= x_t + \begin{bmatrix} 1 & 1 \end{bmatrix}^\mathsf{T} u_t, \\ dy_t &= C_2 x_t dt + \begin{bmatrix} 1 & 0 \end{bmatrix} dw_t. \end{aligned} \tag{36}$$

with

$$\begin{aligned} B_1 &:= \Sigma\ {\rm Im}\left(\mathscr{C}^\dagger \begin{bmatrix} \dfrac{2\exp(-i\phi)}{1+\exp(-i2\phi)} & \dfrac{2i}{1+\exp(-i2\phi)} \end{bmatrix}\right), \\ C_2 &:= e^{-i\phi}\mathscr{C} + e^{i\phi}\mathscr{C}^*. \end{aligned}$$

Here, the additional output signal z_t was introduced to define the cost functional

$$J := \lim_{t\to\infty}\langle z_t^\mathsf{T} z_t\rangle. \tag{37}$$

By introducing $S_\phi := \frac{1}{2}(F_\phi + F_\phi^\mathsf{T})$ and applying the result in the previous section, we can analytically give the infimum $J_{h,\phi}^{\rm opt}$ of the cost functional J over all control laws.

2) Effect of Feedback Delay: In this section, we investigate how the time delay deteriorates the optimal control performance, taking the detector parameter tuning into explicit consideration. It should be noted that the optimal measurement parameter was first discussed by Wiseman and Doherty in 2005 for the delay-free systems, i.e., the optimization of $J_{\phi,0}^{\rm opt}$.

First of all, when A is unstable, $J_{h,\phi}^{\rm opt}$ increases exponentially as $h \to \infty$ independently of ϕ. On the other hand, the remaining two classes, i.e., stable and marginally stable systems, are relatively insensitive to the time delay and it is worth to analyze them in detail.

Stable system – Consider a damped cavity with an on-threshold parametric down converter. The system Hamiltonian and the coupling operator are given by

$$H_t = \frac{\gamma}{2}(qp + pq) - u_t q, \quad c = \delta(q + ip), \tag{38}$$

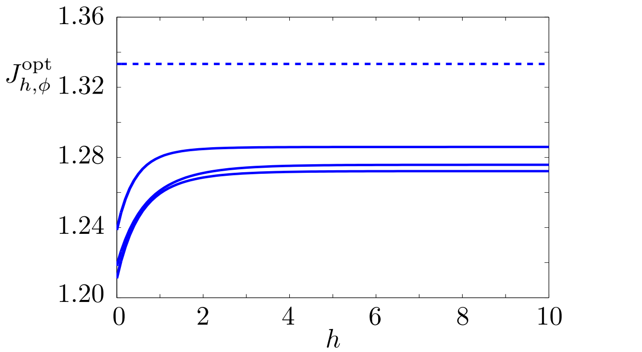

Fig. 3. Optimal performance curves for a damped cavity with an on-threshold parametric down converter. A dashed line depicts the value of J when $u_t \equiv 0$. Three solid lines correspond to the optimal control performances with the detector parameters $\phi = 1.68, 2.28, 1.98$, from the top downwards, respectively, where $\phi = 1.98$ is the optimal detector parameter.

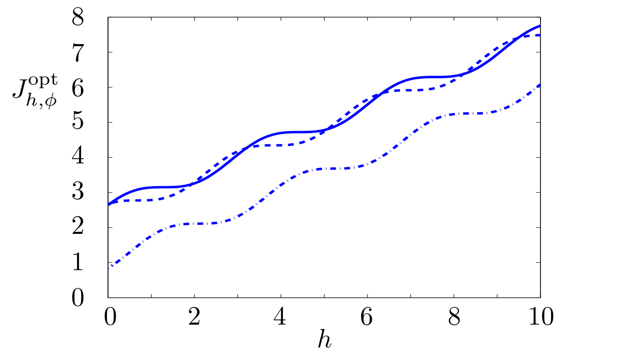

Fig. 4. Optimal performance curves for a harmonic oscillator. A solid line, dashed line and chain line correspond to the optimal performances with $\phi = 0$, $2\pi/18$, $3\pi/18$, respectively.

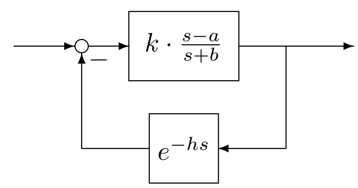

Fig. 5. Block diagram of a system with delayed feedback

where $\gamma > 0$ and $\delta > 0$ are constant parameters. If they satisfy $\gamma < \delta^2$, the system is stable, and consequently $J_{h,\phi}^{\rm opt}$ converges as $h \to \infty$. When we choose the parameter as $\gamma = 1/2$ and $\delta = 1$, the optimal performance curves with the different detector parameters ϕ are given by Fig. 3. The dashed line depicts the value of J when there is no control input field, i.e., $u_t \equiv 0$ for any $t \geq 0$. We can see from the figure that even in the large delay limit, the appropriate choice of ϕ significantly enhances the control performance compared to the uncontrolled case. For the optimal detector parameter, the following relation holds:

Theorem 2: Consider H_t and c defined by (38). Let $\phi_h^{\rm opt}$ denote the detector parameter that minimizes $J_{h,\phi}^{\rm opt}$ for the fixed delay length h. Then,

$$\lim_{h\to\infty} \phi_h^{\rm opt} = \phi_0^{\rm opt}. \tag{39}$$

From the discussion above, we can conclude that the optimal tuning for delay-free case is valid for the stable delay systems in that $\phi_h^{\rm opt} \approx \phi_0^{\rm opt}$ for large h.

Marginally stable system – The next system is a single particle trapped in the harmonic potential and coupled to the probe field via the position operator. The system Hamiltonian and the coupling operator are given by

$$H_t = \frac{1}{2} m\omega^2 q^2 + \frac{1}{2m} p^2 - u_t q, \quad c = q, \tag{40}$$

where m and ω are the mass of the particle and the angular frequency of the harmonic potential, respectively. For this system, the shape of the optimal performance curves is analytically calculated.

Theorem 3: Consider H_t and c defined by (40). Then there exist constants A, B and θ such that the best achievable performance is given by

$$J_{h,\rm opt}^{\rm opt} = J_\phi^{\rm opt} + \mathsf{A}h + \mathsf{B}\sin(\omega h + \theta). \tag{41}$$

Moreover, A and B are independent of the choice of ϕ.

Roughly speaking, the first statement says that $J_{h,\phi}^{\rm opt}$ increases linearly with the oscillation as the delay length h becomes large. This is a natural result of the fact that the matrix A has only pure imaginary eigenvalues. On the other hand, the second statement gives us a nontrivial insight: the growth rate A and the oscillation amplitude B are independent of the detector parameter ϕ. Therefore, we can conclude that if the system suffers from the large delay, the apparatus adjustment is not significant.

B. Control of Delayed Feedback System

Consider the delayed feedback system in Fig. 5 where $k > 1$, $a > b > 0$ and $h > 0$. Its transfer function $\hat{G}$

$$\hat{G} = \frac{k(s-a)}{s+b+k(s-a)e^{-hs}} \tag{42}$$

can be rewritten in the form of (26) with

$$\hat{G}_r = \frac{s-a}{s+a}, \quad m = \frac{(s+b)+k(s-a)e^{-hs}}{(s-b)e^{-hs}+k(s+a)},$$

$$\hat{G}_o = \frac{k(s+a)}{k(s+a)+(s-b)e^{-hs}}.$$

Here m is an inner function with infinitely many unstable zeros. This means that $\hat{G}$ has infinitely many unstable modes. The performance deterioration caused by these unstable modes is investigated as follows.

Consider the sensitivity optimization for this plant, i.e., the problem of minimizing $\|(1-\hat{G}\hat{C})^{-1}\|_2$ by stabilizing controllers. In view of (27), we evaluate the following complimentary sensitivity optimization limit

$$E^* = \inf_{C\in\mathcal{C}} \left\|(1-GC)^{-1}GC\right\| \tag{43}$$

where $\mathcal{C}$ denotes the set of all internally stabilizing controllers for $G := m \cdot \hat{G}_r^{-1}$. Theorem 4 and Corollary 8 yield the expression

$$E^{*2} = E_{\rm n}(a)^2 + \alpha(a)(m\tilde{}(a)^2 - 1) \tag{44}$$

where functions $E_{\rm n}$ and α are related to the solutions for the corresponding Riccati and Lyapunov equations. To be more precise, $E_{\rm n}$ is the complimentary sensitivity optimization limit for $\hat{G}_r^{-1}$ (i.e., the infimum in (43) with $m = 1$) and $\alpha := F^2 Z$ since $A = a$ is scalar. Hence, feedback delay length h contributes only through

$$|m\tilde{}(a)| = \frac{a-b}{a+b} \cdot e^{-ah} + \frac{2ak}{a+b} \quad (> 1). \tag{45}$$

As is explained after Corollary 1, α takes only positive values due to the antistability of a. This means that the plant with larger $|m\tilde{}(a)|$ is more difficult to control.

Based on this analytical expression we make the following interesting observation: *the delay in the feedback path improves the best achievable sensitivity performance for any fixed* $a > b > 0$ *and* $k > 1$. This shows a clear contrast to I/O delay cases. One reason for this is the fact that m has no direct delay despite delay in the feedback path. Note that $\hat{G}$ has the only (unstable) zero at $s = a$, which is independent of h. Thus, a possible interpretation of the observation stated above is that the distance between the (infinitely many) poles of $\hat{G}$ and $s = a$ increases as h becomes larger. Notice that this performance improvement effect saturates:

$$\lim_{h\to\infty} E^{*2} = E_{\rm n}(a)^2 + \alpha(a)\left(\left(\frac{2ak}{a+b}\right)^2 - 1\right)$$

for any $a > b > 0,\ k > 1$.

We can also conclude that b and k affect the performance limit only through $m\tilde{}(a)$. For example, (44) and (45) imply that E^* monotonically increases with respect to k. For sufficiently large k, the relationship is linear[1]. Therefore, this plant with a larger gain k in the feedforward path is more difficult to control.

V. Conclusion

An analytical expression of the H^2 performance limit for a class of MIMO system is given. Unlike most of existing formula, the main result Theorem1 is based on state-space representation. This is accomplished by assuming a generalized plant with a special structure and by applying several methods in infinite-dimensional control theory. While the obtained formula uses the solutions of a couple of Riccati equations, the performance degrading effect of non-minimum phase factors (represented by inner function) can be analyzed independently of other components. Two equivalent formula are also given in order to help us to compute and interpret Theorem 1. As a dual result, performance degrading effect due to unstable modes can be analyzed in the same manner. The obtained results are applicable to a wide class of infinite-dimensional systems with finite-dimensional input and output channels. As in the example of the delayed feedback plant, the obtained formula is simple (quadratic form of $m\tilde{}(A)$), nevertheless provides a new way of characterizing easily controllable plants.

[1]The growth rate is given by $\frac{2a\sqrt{\alpha(a)}}{a+b}$.

VI. Acknowledgments

The author would like to thank Professor Yutaka Yamamoto, Professor Hitay Özbay, Professor Jun-ichi Imura, and Dr. Kazunori Nishio for the joint works introduced in this talk.

References

[1] K. Kashima, A new expression for the H^2 performance limit based on state-space representation, *Automatica*, vol. 45, 2009, pp 283-290.

[2] K. Kashima and Y. Yamamoto, A new characterization of invariant subspace of H^2 and applications to the optimal sensitivity problem, *Systems and Control Letters*, vol. 54, 2005, pp. 539-545.

[3] K. Kashima and Y. Yamamoto, Finite rank criteria for H^∞ control of infinite-dimensional systems, *IEEE Transactions on Automatic Control*, vol. 53, 2008, pp. 881-893.

[4] K. Nishio, K. Kashima, and J. Imura, Effects of time delay in feedback control of linear quantum systems, *Physical Review A*, vol. 79, 2009, 062105.