2D systems with controls and some their applications

Dorota Bors and Stanisław Walczak

***Abstract*— We consider the 2D continuous counterpart of Marchesini-Fornasini model of the process of gas filtration. The continuous version of the discrete model constitutes the hyperbolic boundary value problem. Our main result is finding sufficient conditions for the existence of an optimal solution for the process of gas filtration minimizing the cost functional.**

I. Introduction

In the theory of automatic control, discrete models of Roesser type and Fornasini-Marchesini type are of fundamental importance. N-dimensional model of Roesser type has the following form

$$\begin{bmatrix} z_1(k_1+1,k_2,...,k_N) \\ z_2(k_1,k_2+1,...,k_N) \\ ... \\ z_N(k_1,k_2,...,k_N+1) \end{bmatrix} \tag{1}$$
$$= \begin{bmatrix} a_{11} & a_{12} & ... & a_{1N} \\ a_{21} & a_{22} & ... & a_{2N} \\ ... & ... & ... & ... \\ a_{N1} & a_{N2} & ... & a_{NN} \end{bmatrix} \begin{bmatrix} z_1(k_1,k_2,...,k_N) \\ z_2(k_1,k_2,...,k_N) \\ ... \\ z_N(k_1,k_2,...,k_N) \end{bmatrix}$$
$$+ \begin{bmatrix} b_{11} & b_{12} & ... & b_{1M} \\ b_{21} & b_{22} & ... & b_{2M} \\ ... & ... & ... & ... \\ b_{N1} & b_{N2} & ... & b_{NM} \end{bmatrix} \begin{bmatrix} u_1(k_1,k_2,...,k_N) \\ u_2(k_1,k_2,...,k_N) \\ ... \\ u_M(k_1,k_2,...,k_N) \end{bmatrix}$$

where

$$z_i(k_1,k_2,...,k_{i-1},0,k_{i+1},...,k_N) = 0$$

for $i = 1, 2, ..., N$,

$$A = [a_{ij}]_{i,j=1,2,...,N}\,,\ B = [b_{ij}]_{i=1,2,...,N}^{j=1,2,...,M}$$

are given matrices,

$$z = [z_1, z_2, ..., z_N]^T$$

is a state vector and

$$u = [u_1, u_2, ..., u_M]^T$$

is a control.
The model (1) was considered in [20], where one can find the proof of the theorem on the existence of solution for this type of system. A continuous equivalent version of the discrete Roesser model can be written in the form of the first order partial differential equations

$$\begin{bmatrix} \frac{\partial z_1}{\partial x_1} \\ \frac{\partial z_2}{\partial x_2} \\ ... \\ \frac{\partial z_N}{\partial x_N} \end{bmatrix}(x) = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1N} \\ a_{21} & a_{22} & ... & a_{2N} \\ ... & ... & ... & ... \\ a_{N1} & a_{N2} & ... & a_{NN} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ ... \\ z_N \end{bmatrix}(x) \tag{2}$$
$$+ \begin{bmatrix} b_{11} & b_{12} & ... & b_{1M} \\ b_{21} & b_{22} & ... & b_{2M} \\ ... & ... & ... & ... \\ b_{N1} & b_{N2} & ... & b_{NM} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ ... \\ u_M \end{bmatrix}(x)$$

where

$$x = (x_1, x_2, ..., x_N)\,,$$

$$z_i(x_1,...,x_{i-1},0,x_{i+1},...,x_N) = 0$$

for $i = 1, 2, ..., N$.
In the literature system (2) is referred to as the system of Dieudonne-Rashevsky type. For continuous controls u, this system was investigated in many papers, see for example [4]. The case of measurable controls were examined in [12].
Furthermore, N-dimensional model of Fornasini-Marchesini type has the following form

$$\begin{aligned} & z(k_1+1,k_2+1,...,k_N+1) \\ & = A_0 z(k_1,k_2,...,k_N) \\ & + A_1 z(k_1+1,k_2,...,k_N) \\ & + A_2 z(k_1,k_2+1,...,k_N) \\ & + ... \\ & + A_N z(k_1,k_2,...,k_N+1) \\ & + Bu(k_1,k_2,...,k_N) \end{aligned} \tag{3}$$

where

$$z = (z_1, z_2, ..., z_N)\,,$$

A_i, $i = 0, 1, ..., N$ are some matrices of $N \times N$ size,
B is a matrix of $N \times M$ size,

$$u = (u_1, u_2, ..., u_M)$$

and

$$z_i(x_1,...,x_{i-1},0,x_{i+1},...,x_N) = 0$$

for $i = 1, 2, ..., N$.
For $N = 2$, system (3) was studied in [7]. Moreover,

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a continuous counterpart of system (3) is given by the following system of partial differential equations

$$\frac{\partial^N z(x)}{\partial x_1 ... \partial x_N} = A_0 z(x) + A_1 \frac{\partial z(x)}{\partial x_1} + ... + A_N \frac{\partial z(x)}{\partial x_N} + Bu(x) \tag{4}$$

where

$$x = (x_1, x_2, ..., x_N) \in P^N = [0,1]^N$$

and

$$z_i(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_N) = 0$$

for $i = 1, 2, ..., N$.
Thus, for $N = 2$, system (4) admits the form

$$\frac{\partial^2 z(x_1,x_2)}{\partial x_1 \partial x_2} = A_0 z(x_1,x_2) + A_1 \frac{\partial z(x_1,x_2)}{\partial x_1} + A_2 \frac{\partial z(x_1,x_2)}{\partial x_2} + Bu(x_1,x_2)$$

where

$$z(x_1, 0) = 0, \;\; z(0, x_2) = 0.$$

In the literature this system is known as the system of Darboux-Goursat type. The nonlinear system of Darboux-Goursat type has the form

$$\frac{\partial^2 z}{\partial x_1 \partial x_2} = F\left(x_1, x_2, z, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, u\right) \tag{5}$$

where u, z and its derivatives depend on x_1 and x_2 variables,

$$F : P^2 \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N,$$

$$z = (z_1, z_2, ..., z_N),$$

$$z(x_1, 0) = 0, \;\; z(0, x_2) = 0.$$

The system (5) was examined in papers [2] and [4] under the continuity assumption imposed on a control u. Whereas, results for measurable controls were presented in [3], [13] and [22].

II. On some applications of continuous 2D systems

In this section we will apply some 2D system to the investigation of a process of gas filtration. Let us consider a system of gas filtration that consist of a container Q and a filter $\mathcal{F}$ being a pipe of length l filled with a substance which absorbs a toxic gas G. The filter $\mathcal{F}$ is connected with the container Q by a valve system which gives us the possibility to control a speed of a gas flow from the container to the filter. Let us suppose that at the moment $t \in [0,T]$ in the container Q there is a certain amount $M(t)$ of a mixture of air and a toxic gas. Supposing that the function M is absolutely continuous, we can write

$$M(t) = M_0 - \int_0^t \omega(\tau)\, d\tau$$

for some $\omega \in L^2([0,T], \mathbb{R})$ such that $\omega(t) = -M'(t)$ where $M_0 = M(0)$. We also assume that $\omega(t) \in [m_0, m_1]$ with fixed $m_1 > m_0 > 0$. By means of the function ω we can control the amount of the mixture being exhausted from the container Q at the moment t. Denote by $u(x,t)$ the concentration of the gas G in the absorbent at the moment t and at the distance x from the inlet of the filter. Let $v(t)$ be a speed at the inlet of filter of the mixture of air and the toxic gas at the moment t. The speed $v(t)$ depends on a control of valve system and on the pressure $p(t)$ in the container Q. Moreover, the pressure $p(t)$ is proportional to the amount $M(t)$ of the mixture in the container Q and therefore we can assume that $v(t)$ is a function of the amount of the mixture in the container Q, i.e.

$$v(t) = \varphi(M(t)) = \varphi\left(M_0 - \int_0^t \omega(\tau)\, d\tau\right) \geq c > 0,$$

where $\varphi \in C^1([0,T], \mathbb{R})$.
If the speed $v(t)$ is sufficiently large (i.e. the constant $c > 0$ is sufficiently large), so that the diffusion does not play an important role in the gas motion, then the process of gas filtration is described by the following hyperbolic equation (for detailed derivation see [21], Chapter 2)

$$u_{xt}(x,t) = -\left(\beta\gamma + \frac{v'(t)}{v(t)}\right) u_x(x,t) - \frac{\beta}{v(t)} u_t(x,t), \quad (x,t) \in P^2 \tag{6}$$

with the boundary conditions

$$u(x,0) = u_0 \exp\left(-\frac{\beta}{v_0} x\right), \;\; x \in [0,l] \qquad u(0,t) = u_0, \;\; t \in [0,T], \tag{7}$$

where $P^2 = [0,l] \times [0,T]$, β, γ are some physical constants; $u_0 = u(0,0)$, $v_0 = v(0)$ and $\omega \in \mathcal{A}$, where by $\mathcal{A}$ we denote the set of all admissible controls such that

$$\mathcal{A} = \left\{\omega \in L^1([0,T], \mathbb{R}); \quad \omega(t) \in [m_0, m_1], \; M_0 - \int_0^T \omega(t)\, dt \geq 0, \; \mathop{\mathrm{V}}_0^T \omega(t) \leq h \right\},$$

where $m_1 > m_0 > 0$, $h \geq 0$, $M_0 = M(0) > 0$ are fixed numbers and $\mathop{\mathrm{V}}_0^T \omega$ is the variation of ω, i.e. the number

$$\mathop{\mathrm{V}}_0^T \omega = \sup_n \sum_{i=0}^{n} |\omega(t_{i+1}) - \omega(t_i)|,$$

for $0 = t_0 < t_1 < ... < t_n < t_{n+1} = T$.
Next let us suppose that the cost of gas filtration is measured by the following functional

$$I(u,\omega) = f(M_0 - M_T) + \int_0^l \int_0^T g(x,t,u(x,t),\omega(t))\, dx dt \tag{8}$$

where $M_T = M(T)$, $M_0 - M_T = \int_0^T \omega(t)\, dt$ denotes the amount of a purified mixture being the argument of $f : \mathbb{R} \to \mathbb{R}$ function, $g : P^2 \times \mathbb{R} \times [m_0, m_1] \to \mathbb{R}$, $\omega \in \mathcal{A}$ and $u \in AC(P^2, \mathbb{R})$. By $AC(P^2, \mathbb{R})$ we denote the space of all absolutely continuous functions defined on the interval P^2 with the norm

$$\|u\|_{AC(P^2,\mathbb{R})} = \|\alpha\|_{L^1(P^2,\mathbb{R})} + \left\|\alpha^1\right\|_{L^1([0,l],\mathbb{R})} + \left\|\alpha^2\right\|_{L^1([0,T],\mathbb{R})}$$

where functions α, α^1 and α^2 are described in the proposition published in [1], [23].

Proposition 1. *A function $u \in AC\left(P^2, \mathbb{R}\right)$ if and only if there exist functions $\alpha \in L^1\left(P^2, \mathbb{R}\right)$, $\alpha^1 \in L^1\left([0,l], \mathbb{R}\right)$ and $\alpha^2 \in L^1\left([0,T], \mathbb{R}\right)$ such that*

$$u(x,t) = \int_0^x \int_0^t \alpha(y,\tau)\, dy d\tau + \int_0^x \alpha^1(y)\, dy + \int_0^t \alpha^2(\tau)\, d\tau.$$

III. Main results

To show the existence of optimal solution of problem $(6)-(8)$ we need the following conditions:

A1. a function φ is of C^1 class on the interval $[M_T, M_0]$ and satisfies the inequality

$$\varphi(M) \geq c > 0 \text{ for } M \in [M_T, M_0],$$

where $c > 0$ is sufficiently large, so that the diffusion is negligible for the mixture motion.

A2. - a function $f : \mathbb{R} \to \mathbb{R}$ is continuous,
- a function $g : P^2 \times \mathbb{R} \times [m_0, m_1] \to \mathbb{R}$ is measurable with respect to $(x,t) \in P^2$ for any (u,ω), continuous with respect to $(u,\omega) \in \mathbb{R} \times [m_0, m_1]$ for any $(x,t) \in P^2$ and for any bounded set $Z \subset \mathbb{R}$ there exists a function g_z integrable on P^2 such that

$$|g(x,t,u,\omega)| \leq g_z(x,t)$$

for $(x,t) \in P^2$, $z \in Z$ and $\omega \in [m_0, m_1]$.

Then it is possible to prove the theorem on the existence of solution of the control problem $(6)-(7)$.

Theorem 1. *Suppose that the function φ satisfies assumption A1. Then for any admissible control $\omega \in \mathcal{A}$ there exist unique solution $u_\omega \in AC\left(P^2, \mathbb{R}\right)$ satisfying $(6)-(7)$.*

The main result of the paper is the theorem on the existence of an optimal control in the process of gas filtration given by the system $(6)-(8)$.

Theorem 2. *If assumptions A1 and A2 are satisfied, then the control problem $(6)-(7)$ with the cost functional (8) possesses an optimal solution, i.e. there exists admissible control $\omega^* \in \mathcal{A}$ and the trajectory $u^* \in AC\left(P^2, \mathbb{R}\right)$ corresponding to the control ω^* such that $I(u^*, \omega^*) \leq I(u, \omega)$ for any admissible process (u, ω).*

IV. Concluding remarks

The problem considered is the 2D continuous counterpart of Fornasini-Marchesini model of the process of gas filtration, for details see [7]. We have formulated sufficient conditions for the existence of optimal solutions for that problem. Discrete systems of this type were investigated in many papers. One can find numerous results, for example in monograph [15], and papers [5], [7], [8], [9], [10], [11], [13], [16], [17] and [18].
Necessary conditions for the existence of optimal solutions for the system of the form

$$z_{xy}(x,y) = F\left(x, y, z(x,y), z_x(x,y), z_y(x,y), \omega(x,y)\right)$$

where

$$(x,y) \in P^2 \subset \mathbb{R}^2,$$

$$z(x,0) = \varphi(x),\ z(0,y) = \psi(y),$$

$$\omega \in \mathcal{U} = \left\{\omega \in L^\infty\left(P^2, \mathbb{R}^m\right);\ \omega(x,y) \in U\right\}$$

with different cost indices and under various assumptions imposed on functions F, φ, ψ were presented among others in [3], [6] and [22]. Results published therein have the form of maximum principle.
Since the set of admissible controls $\mathcal{A}$ does not satisfy assumptions stated there, necessary conditions published in papers [3], [6] and [22] are not applicable to problem $(6)-(8)$. Thanks to the Ioffe-Tikhomirov extremum principle formulated in [14] or the generalized version of Dubovitski-Milutin method presented in paper [24], it will be possible to formulate necessary conditions for problem $(6)-(8)$, analogous to the maximum principle.

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