A new approach to strong practical stability and stabilization of discrete linear repetitive processes

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Abstract— **Repetitive processes are a distinct class of $2D$ systems of both theoretical and practical interest. The stability theory for these processes originally consisted of two distinct concepts termed asymptotic stability and stability along the pass respectively where the former is a necessary condition for the latter. Recently applications have arisen where asymptotic stability is too weak and stability along the pass is too strong for meaningful progress to be made. This, in turn, has led to the concept of strong practical stability for such cases, where previous work has formulated this property and obtained necessary and sufficient conditions for its existence together with Linear Matrix Inequality (LMI) based tests, which then extend to allow control law design. This paper develops considerably simpler, and hence computationally more efficient, stability tests that also extend to allow control law design.**

I. INTRODUCTION

The unique characteristic of a repetitive, or multipass [7], process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique stabilization problem in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

Physical examples of these processes include long-wall coal cutting and metal rolling operations [7]. Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives, including classes of iterative learning control (ILC) laws [4] and iterative algorithms for solving nonlinear dynamic optimal stabilization problems based on the maximum principle [6]. In this last case, for example, use of the repetitive process setting provides the basis for the development of highly reliable and efficient iterative solution algorithms and in the former it, unlike alternatives, provides a setting where the (potentially) conflicting objective of trial-to-trial (or pass-to-pass) error convergence and performance along the trials can be treated. Recently, highly successful experimental verification of control laws designed using this approach on a gantry robot system executing a pick and place operation under synchronization that emulates many industrial applications, has been reported [4].

Attempts to stabilize these processes using standard (or 1D) systems theory/algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2D systems structure, i.e. information propagation occurs from pass-to-pass and along a given pass and also the initial conditions are reset before the start of each new pass. To remove these deficiencies, a rigorous stability theory has been developed [7] based on an abstract model of the dynamics in a Banach space setting that includes a very large number of processes with linear dynamics and a constant pass length as special cases. Also the results of applying this theory to a range of sub-classes, including the discrete linear repetitive processes considered here, have been reported [7]. This stability theory consists of the distinct concepts of asymptotic stability and stability along the pass respectively where the former is a necessary condition for the latter.

Recognizing the unique control problem, this stability theory is of the bounded input bounded output (BIBO) form, that is, a bounded initial pass profile is required to produce a bounded sequence of pass profiles, where boundedness is defined in terms of the norm on the underlying Banach space. Asymptotic stability guarantees this property over the finite and fixed pass length whereas stability along the pass is stronger in that it requires this property uniformly, that is, for all possible values of the pass length and hence it is not surprising that asymptotic stability is a necessary condition for stability along the pass.

If asymptotic stability holds for discrete linear repetitive processes then the sequence of pass profiles produced will converge in the pass-to-pass direction to a limit profile which is described by a 1D discrete linear systems state-space model. This fact has clear implications for the design of stabilization schemes. Moreover, the condition for asymptotic stability is very easy to test whereas one of the extra for stability along the pass is much more involved. This raises the question of whether or not asymptotic stability alone would be sufficient for at least some practically relevant cases. The answer [3] for at least some applications is no

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but for these it may be acceptable to use strong practical stability as an alternative to stability along the pass. Note also that in the optimal control application [6] asymptotic stability is all that can ever be achieved and see also [7] for discussion centered round the ILC application area where strong practical stability could be the most appropriate way forward.

The basis of strong practical stability was developed in [3] and in subsequent work [2] it was shown that necessary and sufficient conditions for this property could be formulated in LMI terms that immediately give algorithms for the design of a stabilizing control law. In this paper we develop much simpler LMI based results for these problems, which are more computationally effective with the eventual aim of using them in the ILC and other applications.

Throughout this paper, the null and identity matrices of the required dimensions are denoted by 0 and I respectively. Moreover, $M > 0$ (< 0) denotes a real symmetric positive (negative) definite matrix.

II. Background

The state-space model of a discrete linear repetitive process [7] has the following form over $0 \leq p \leq \alpha - 1$, $k \geq 0$,

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p) \end{aligned} \tag{1}$$

where $\alpha < \infty$ is the pass length and on pass k $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the pass profile vector, and $u_k(p) \in \mathbb{R}^r$ is the vector of control inputs. The boundary conditions (i.e. the pass state initial vector sequence and the initial pass profile) are

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \geq 0 \\ y_0(p) &= f(p), \quad 0 \leq p \leq \alpha - 1 \end{aligned} \tag{2}$$

where the $n \times 1$ vector d_{k+1} has known constant entries and $f(p)$ is an $m \times 1$ vector whose entries are known functions of p.

Applying the stability theory of [7] to (1) and (2) gives the necessary and sufficient condition for asymptotic stability as $r(D_0) < 1$, where $r(\cdot)$ denotes the spectral radius of its matrix argument. At first sight, this result is somewhat surprising since it is independent of the plant state dynamics and, in particular, places no constraints the location of the eigenvalues of the matrix A that clearly determine the dynamics produced along any pass. This condition is a result of the finite pass length and its consequences are discussed next.

Suppose that asymptotic stability holds and the input sequence applied $\{u_{k+1}\}$ converges strongly as $k \to \infty$ (i.e. in the sense of the norm on the underlying function space) to u_∞. Then the strong limit $y_\infty := lim_{k\to\infty} y_k$ is termed the limit profile corresponding to this input sequence and its state-space model is

$$\begin{aligned} x_\infty(p+1) &= (A + B_0(I - D_0)^{-1}C)x_\infty(p) \\ &+ (B + B_0(I - D_0)^{-1}D)u_\infty(p) \\ y_\infty(p) &= (I - D_0)^{-1}Cx_\infty(p) \\ &+ (I - D_0)^{-1}Du_\infty(p) \\ x_\infty(0) &= d_\infty \end{aligned} \tag{3}$$

where d_∞ is the strong limit of the sequence $\{d_k\}$. In physical terms, this result states that under asymptotic stability the repetitive dynamics can, after a sufficiently large number of passes have elapsed, be replaced by those of a 1D discrete linear system.

As an example, consider the case when $A = -0.5$, $B = 1$, $B_0 = 0.5 + \beta$, $C = 1$, $D = 0$, $D_0 = 0$, where β is a real scalar. Asymptotic stability holds in this case with resulting limit profile

$$y_\infty(p+1) = \beta y_\infty(p) + u_\infty(p)$$

Hence if $|\beta| \geq 1$, the sequence of pass profiles converge (in the pass-to-pass direction (k)) to an unstable 1D discrete linear system. Note also that this occurs even though the state matrix A is stable in the 1D sense.

The problem here is the finite pass length over which duration even an unstable 1D discrete linear system can only produce a bounded output. If the limit profile is unstable, as a 1D discrete linear system, then clearly this is unacceptable in most applications where tracking a reference signal is required.

Stability along the pass prevents this problem from arising by demanding the BIBO property uniformly with respect to the pass length and can be analyzed mathematically by letting $\alpha \to \infty$. This leads to several sets of necessary and sufficient conditions [7] for this property, such as the following.

Theorem 1: Suppose that the pair $\{A, B_0\}$ is controllable and the pair $\{C, A\}$ is observable. Then a discrete linear repetitive process described by (1) and (2) is stable along the pass if, and only if, $r(D_0) < 1$, $\quad r(A) < 1$ and all eigenvalues of

$$G(z) = C(zI - A)^{-1}B_0 + D_0 \tag{4}$$

have modulus strictly less than unity $\forall\ |z| = 1$

These conditions can be tested by direct application of well known 1D linear systems tests. Application of them to the example given above shows that stability along the pass also places a constraint on the state dynamics of both the current pass ($r(A) < 1$) and, in the single-input single-output case for simplicity, the complete frequency response of the transfer-function describing the contribution of the previous pass profile, and not just on D_0. Also it is easy to see that stability along the pass ensures that the resulting limit profile is stable as a 1D discrete linear system, i.e. $r(A + B_0(I - D_0)^{-1}C) < 1$.

The only difficulty with using Theorem 1 is that testing (4) could be computationally intensive and, despite its Nyquist basis, it has not proved to be a starting point for onward analysis, such as the design of a stabilizing control law. This is in contrast to the case for 1D linear systems, and the most effective way currently available for control law design is based on using an LMI interpretation of stability along the pass.

The price paid for this progress is that the LMI based analysis works on sufficient, as opposed to necessary and sufficient (such as those of Theorem 1), conditions and hence there will be a degree of conservativeness associated with all results and analysis that start from this basis. There is hence great potential benefit in seeking an alternative approach that does not have this undesirable feature.

Another way to consider the stability properties of the discrete linear repetitive processes is to use the links with 2D discrete linear systems described, for example, by the Roesser state-space model, where the weaker concept of practical stability is available through the following result.

Lemma 1: A discrete linear repetitive process described by (1) and (2) is practically stable if and only if $r(D_0) < 1$ and $r(A) < 1$.

In terms of applying this result to a given example, we simply need to complete two tests for the 1D discrete linear systems stability property.

Consider now an industrial example such as a gantry robot whose task is to collect an object from a location and place it on a moving conveyor belt after a finite time has elapsed, then return to the original location to pick up the next one and so on. This is an obvious application for ILC and hence repetitive process theory, since the time taken to complete the return journey can be used to update the control law using previous pass information to sequentially improve performance. Maximum benefit will arise if this operation can be executed a very large number of times without the need to stop and hence lose throughput. Hence we have the case when α is finite and $k \to \infty$ and for this we clearly need a form of stability where the limit profile (3), i.e. $k \to \infty$, exists with stable along the pass dynamics and, ideally, acceptable tracking of the reference signal can be achieved.

It is already known [7] that linear model based ILC can be described as a linear repetitive process and clearly what is required in situations such as one just described is asymptotic stability plus a stable limit profile, where the latter requirement cannot be guaranteed by practical stability. (The conditions of Lemma 1 do not guarantee a limit profile that is stable in the p direction as the example given earlier in this paper, i.e. $A = -0.5$, $B = 1$, $B_0 = 0.5 + \beta$, β a real scalar, $C = 1$ and $D = D_0 = 0$, demonstrates). Similar situations motivated the development of strong practical stability for discrete linear repetitive processes, for which we next summarize the relevant background.

A. Strong Practical Stability

Strong practical stability as an alternative to the property of Lemma 1 refines stability along the pass by removing the uniform boundedness requirement as both $k \to \infty$ and $\alpha \to \infty$ but still demands this property when (i) both k and α are finite, (ii) the pass index $k \to \infty$ and the pass length α finite, and (iii) the pass index k is finite and the pass length $\alpha \to \infty$. Cases (i) and (ii) have obvious practical motivation and Case (iii) is the mathematical formulation of an example where the process completes a finite number of passes but the pass length is very long and there is also a requirement to control the along the pass dynamics.

Consider the case of $p = 0$ in the process state-space model (1) with $x_{k+1}(0) = 0$, $k \geq 0$, and zero control input vector. Then $y_k(0) = D_0^k y_0(0)$ and hence we require $r(D_0) < 1$. Under this condition, i.e. asymptotic stability, we achieve the limit profile (3) as $k \to \infty$ that is stable when $r(A + B_0(I - D_0)^{-1}C) < 1$.

Consider any finite k. Then clearly, consider the case when there is no previous pass profile contribution, we require that $r(A) < 1$. Also, again with zero state initial vector sequence and zero control input vector, as $p \to \infty$

$$y_{k+1}(\infty) = \left(C(I-A)^{-1}B_0 + D_0\right) y_k(\infty) \tag{5}$$

and hence we require $r(C(I-A)^{-1}B_0 + D_0) < 1$.

Combining the outcomes of the analysis above gives the following conditions for strong practical stability

[a] $r(D_0) < 1$
[b] $r(A) < 1$
[c] $r(A + B_0(I - D_0)^{-1}C) < 1$, and
[d] $r(C(I-A)^{-1}B_0 + D_0) < 1$

and these can, assuming no numerical problems associated with computing the eigenvalues of the matrices involved, be easily checked for a given example.

In order to enforce these conditions for a given example this paper considers application of the control law (see also Section IV)

$$u_{k+1}(p) = K_1 x_{k+1}(p) + K_2 y_k(p) \tag{6}$$

where K_1 and K_2 are matrices to be selected. This control law is a combination of current pass state feedback plus a feedforward term from the previous pass profile and when applied to the process state-space model the matrices A, B_0, C, D_0 are mapped to $A + BK_1$, $B_0 + BK_2$, $C + DK_1$ and $D_0 + BK_2$ respectively. Hence design to satisfy conditions [a] and [b] for the controlled process is simply two applications of the 1D pole placement problem for discrete linear systems. The case for conditions [c] and [d] is far from clear and the only previously reported approach [2] has used results from 1D singular discrete linear systems theory for the state-space model, where the basics of this method is summarized next for the uncontrolled process.

Consider the 1D singular discrete linear system state-space model

$$Ex(h+1) = \widehat{A}x(h) + \widehat{B}u(h) \tag{7}$$

where E is a singular matrix. Then it follows immediately that condition [c] above for strong practical stability of the discrete linear repetitive processes is equivalent to stability of the 1D singular linear system with state-space model

$$\tilde{E}_1 z(h+1) = \tilde{A}_1 z(h) + \Pi u(h) \tag{8}$$

where $\tilde{E}_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\tilde{A}_1 = \begin{bmatrix} A & B_0 \\ C & D_0 - I \end{bmatrix}$, $\Pi = \begin{bmatrix} B \\ D \end{bmatrix}$. Similarly, condition [d] is equivalent to stability of the 1D singular linear system

$$\tilde{E}_2 z(h+1) = \tilde{A}_2 z(h) + \Pi u(h) \tag{9}$$

where $\tilde{E}_2 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$, $\tilde{A}_2 = \begin{bmatrix} A-I & B_0 \\ C & D_0 \end{bmatrix}$ These facts lead to the following result

Theorem 2: [2] A discrete linear repetitive process described by (1) and (2) is strongly practically stable if, and only if, there exist the appropriately dimensioned matrices

$$\begin{array}{l} W_1 > 0, \quad W_2 > 0, \quad X_{21}^1, \quad X_{21}^2 \\ X_{11}^1 = (X_{11}^1)^T, \quad X_{22}^1 = (X_{22}^1)^T, \quad Y_{11}^2, \quad Y_{22}^1 \\ X_{11}^2 = (X_{11}^2)^T, \quad X_{22}^2 = (X_{22}^2)^T, \quad \widetilde{G_1}, \quad \widetilde{G_2} \end{array}$$

such that the following LMIs are feasible for scalars $\beta_1 > 1$, $\beta_2 > 1$

$$\begin{bmatrix} -W_1 & W_1^T D_0^T \\ D_0 W_1 & -W_1 \end{bmatrix} < 0 \tag{10}$$

$$\begin{bmatrix} -W_2 & W_2^T A^T \\ AW_2 & -W_2 \end{bmatrix} < 0 \tag{11}$$

$$\begin{array}{l} \begin{bmatrix} -X_{11}^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (Y_{22}^1)^T \\ 0 & 0 & X_{11}^1 & (X_{21}^1)^T \\ 0 & Y_{22}^1 & X_{21}^1 & X_{22}^1 \end{bmatrix} \\ +Sym\left\{ \begin{bmatrix} \tilde{A}_1 \widetilde{G_1} \\ -\widetilde{G_1} \end{bmatrix} \begin{bmatrix} I & \beta_1 I \end{bmatrix} \right\} < 0 \end{array} \tag{12}$$

$$\begin{array}{l} \begin{bmatrix} 0 & 0 & (Y_{11}^2)^T & 0 \\ 0 & -X_{22}^2 & 0 & 0 \\ Y_{11}^2 & 0 & X_{11}^2 & (X_{21}^2)^T \\ 0 & 0 & X_{21}^2 & X_{22}^2 \end{bmatrix} \\ +Sym\left\{ \begin{bmatrix} \tilde{A}_2 \widetilde{G_2} \\ -\widetilde{G_2} \end{bmatrix} \begin{bmatrix} U_2 & \beta_2 U_2 \end{bmatrix} \right\} < 0 \end{array} \tag{13}$$

where

$$U_2 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

As detailed in [2], Theorem 2 leads to the computationally tractable stability tests that, in turn, extend to enable control laws of the form (6) to be designed to ensure strong practical stability of the controlled process. This, however, comes at the cost of potentially large dimensioned LMIs of complicated structure in terms of how the block entries are constructed from the plant model and control law matrices. The new analysis in this paper results reformulation of these results to substantially remove these difficulties.

III. New strong practical stability tests

The route to developing simpler computationally efficient tests for the conditions [c] and [d], with a natural extension to control law design, also uses 1D descriptor nonsingular linear systems theory. To begin, first note that the limit profile state-space model (3) can be rewritten in the form

$$\begin{aligned} x_\infty(h+1) - B_0 y_\infty(h) &= Ax_\infty(h) \\ (I - D_0) y_\infty(h) &= Cx_\infty(h) \end{aligned} \tag{14}$$

where $I - D_0$ is a nonsingular matrix. In particular, the condition $r(A + B_0(I-D_0)^{-1}C) < 1$ is equivalent to stability of the 1D descriptor linear system with the state-space model

$$E_1 z(h+1) = A_1 z(h) + \Pi u(h) \tag{15}$$

where

$$z(h) = \begin{bmatrix} x_\infty(h) \\ y_\infty(h-1) \end{bmatrix}, \qquad A_1 = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} I & -B_0 \\ 0 & I - D_0 \end{bmatrix}$$

Similarly, (5) can be rewritten as

$$\begin{aligned} x_{k+1}(\infty) &= Ax_{k+1}(\infty) + B_0 y_k(\infty) \\ y_{k+1}(\infty) &= Cx_{k+1}(\infty) + D_0 y_k(\infty) \end{aligned} \tag{16}$$

or

$$E_2 z(h+1) = A_2 z(h) + \Pi u(h) \tag{17}$$

where

$$z(h) = \begin{bmatrix} x_k(\infty) \\ y_k(\infty) \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & B_0 \\ 0 & D_0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} I - A & 0 \\ -C & I \end{bmatrix}$$

Hence the condition $r(C(I-A)^{-1}B_0 + D_0) < 1$ is equivalent to stability of the 1D descriptor linear system (17).

Lemma 2: [5] A 1D discrete linear system with state matrix of the form $E^{-1}\hat{A}$ is stable if, and only if $\exists$ a matrix $\hat{Q} > 0$ and a nonsingular matrix $\hat{G}$ such that the following LMI is feasible

$$\begin{bmatrix} -\hat{Q} & \hat{A}^T E^{-T} \hat{G}^T \\ \hat{G} E^{-1} \hat{A} & \hat{Q} - \hat{G} - \hat{G}^T \end{bmatrix} < 0 \tag{18}$$

Now we have the following result.

Theorem 3: A discrete linear repetitive process described by (1) and (2) is strongly practically stable if and only if $\exists$ appropriately dimensioned matrices $W_1 > 0$, $W_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, and nonsingular matrices S_1 and S_2 such that the following LMIs are feasible

$$\begin{bmatrix} -W_1 & W_1 D_0^T \\ D_0 W_1 & -W_1 \end{bmatrix} < 0 \tag{19}$$

$$\begin{bmatrix} -W_2 & W_2 A^T \\ AW_2 & -W_2 \end{bmatrix} < 0 \tag{20}$$

$$\begin{bmatrix} -Q_1 & S_1^T A_1^T \\ A_1 S_1 & Q_1 - E_1 S_1 - S_1^T E_1^T \end{bmatrix} < 0 \tag{21}$$

$$\begin{bmatrix} -Q_2 & S_2^T A_2^T \\ A_2 S_2 & Q_2 - E_2 S_2 - S_2^T E_2^T \end{bmatrix} < 0 \tag{22}$$

Proof: The LMIs (19 and 20) are easily seen to be equivalent to $r(D_0) < 1$ and $r(A) < 1$ respectively and hence to conditions [a] and [b] for strong practical stability. The proof that the LMI (21) is equivalent to condition [c] for strong practical stability proceeds by applying Lemma 2 to (15) and introducing the new variables $\hat{S}_1^T = G_1 E_1^{-1}$ to obtain the LMI

$$\begin{bmatrix} -\hat{Q}_1 & A_1^T \hat{S}_1 \\ \hat{S}_1^T A_1 & \hat{Q}_1 - \hat{S}_1^T E_1 - E_1^T \hat{S}_1 \end{bmatrix} < 0 \tag{23}$$

Left- and right multiplying this last condition by $\begin{bmatrix} \hat{S}_1^{-T} & 0 \\ 0 & \hat{S}_1^{-T} \end{bmatrix}$ and its transpose respectively and then introducing $S_1 = \hat{S}_1^{-1}$ and $Q_1 = \hat{S}_1^{-T} \hat{Q}_1 \hat{S}_1^{-1}$ completes this part of the proof.

The equivalence of the LMI (22) to condition [d] follows identical steps to that above for condition [c] and hence the details are omitted here. ■

IV. STABILIZATION IN THE STRONG PRACTICAL SENSE

This section considers the design of the control law (6) to result in a controlled process which is strongly practically stable. The previous pass profile is a measured output and here we assume that it not significantly corrupted by noise etc. Moreover, the current pass state vector in this control law could be replaced by the current pass profile or estimated using an observer if not all entries are available for measurement.

Application of (6) to (1) gives the controlled process state-space model

$$\begin{aligned} x_{k+1}(p+1) &= (A+BK_1)x_{k+1}(p) + (B_0+BK_2)y_k(p) \\ y_{k+1}(p) &= (C+DK_1)x_{k+1}(p) + (D_0+DK_2)y_k(p) \end{aligned} \tag{24}$$

and strong practical stability holds if and only if

[e] $r(D_0+DK_2) < 1$

[f] $r[A+BK_1] < 1$

[g] $r[(B_0+BK_2)(I-D_0-DK_2)^{-1}(C+DK_1) + (A+BK_1)] < 1$, and

[h] $r[(C+DK_1)(I-A-BK_1)^{-1}(B_0+BK_2) + (D_0+DK_2)] < 1$

A simpler control law structure would result if $K_2 = 0$ which is equivalent to stabilization using only current pass state feedback but examples are easily constructed where strong practical stability can never be achieved.

The problem of developing a computationally efficient method to design the control such that these conditions hold for the controlled process is more complicated relative to the stability only case because we only have two matrices K_1 and K_2 available for selection. This leads to some conservativeness in the resulting algorithms.

Theorem 4: A controlled discrete linear repetitive process described by (24) is strongly practically stable if ∃ the appropriately dimensioned matrices $Q_1 > 0$, $Q_2 > 0$, a non-singular matrix $S = diag(S_1, S_2)$, and rectangular matrices $\widetilde{N}_1 = \begin{bmatrix} N_1 & 0 \end{bmatrix}$, $\widetilde{N}_2 = \begin{bmatrix} 0 & N_2 \end{bmatrix}$ such that the following LMIs are feasible

$$\begin{bmatrix} -Q_1 & S^T A_1^T + \widetilde{N}_1^T \Pi^T \\ A_1 S + \Pi \widetilde{N}_1 & Q_1 - (E_1 S - \Pi \widetilde{N}_2) - (E_1 S - \Pi \widetilde{N}_2)^T \end{bmatrix} < 0 \tag{25}$$

$$\begin{bmatrix} -Q_2 & S^T A_2^T + \widetilde{N}_2^T \Pi^T \\ A_2 S + \Pi \widetilde{N}_2 & Q_2 - (E_2 S - \Pi \widetilde{N}_1) - (E_2 S - \Pi \widetilde{N}_1)^T \end{bmatrix} < 0 \tag{26}$$

If these hold, stabilizing control law matrices are given by

$$K_1 = N_1 S_1^{-1}, \; K_2 = N_2 S_2^{-1} \tag{27}$$

Proof:

We show that (25) and ((26)) respectively guarantee that [e] and [f] hold. In particular, we apply Theorem 3 with $A_{1new} = A_1 + \Pi \begin{bmatrix} K_1 & 0 \end{bmatrix}$ and $E_{1new} = E_1 - \Pi \begin{bmatrix} 0 & K_2 \end{bmatrix}$ in the case of (25), and $A_{2new} = A_2 + \Pi \begin{bmatrix} 0 & K_2 \end{bmatrix}$ and $E_{2new} = E_2 - \Pi \begin{bmatrix} K_1 & 0 \end{bmatrix}$ in the case of (26). Introducing the additional variables $N_1 = K_1 S_1$ and $N_2 = K_2 S_2$ completes this part of the proof.

The next task is to show that the LMIs (25) and (26) guarantee that [e] and [f] hold respectively, where by Lemma 2 these are equivalent to the following LMIs

$$\begin{bmatrix} -W_1 & S_2^T D_0^T + N_2^T D^T \\ D_0 S_2 + D N_2 & W_1 - S_2 - S_2^T \end{bmatrix} < 0 \tag{28}$$

$$\begin{bmatrix} -W_2 & S_1^T A^T + N_1^T B^T \\ A S_1 + B N_1 & W_2 - S_1 - S_1^T \end{bmatrix} < 0 \tag{29}$$

Also (25) can be rewritten in extended form as

$$\begin{bmatrix} -Q_{11} & -Q_{12} & S_1^T A^T + N_1^T B^T & S_1^T C^T + N_1^T D^T \\ -Q_{12}^T & -Q_{22} & 0 & 0 \\ A S_1 + B N_1 & 0 & Q_{11} - S_1 - S_1^T & Q_{12} + B_0 S_2 + B N_2 \\ C S_1 + D N_1 & 0 & Q_{12}^T + S_2^T B_0^T + N_2^T B^T & \Xi \end{bmatrix} < 0 \tag{30}$$

where

$$\Xi = Q_{22} - S_2 - S_2^T + D_0 S_2 + D N_2 + S_2^T D_0^T + N_2^T D^T$$

A real symmetric matrix $F(x) \in \mathscr{R}^{n \times n}$ is positive or negative definite if and only if, all of its principal minors are positive or negative definite respectively. Hence for (30) to be feasible

$$\left[\begin{array}{cc|c} -Q_{11} & -Q_{12} & S_1^T A^T + N_1^T B^T \\ -Q_{12}^T & -Q_{22} & 0 \\ \hline A S_1 + B N_1 & 0 & Q_{11} - S_1 - S_1^T \end{array}\right] < 0 \tag{31}$$

must hold (as the underlying matrix is a principal minor of the matrix of (30)). Next left- and right- multiply this last inequality by $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ to obtain

$$\left[\begin{array}{cc|c} Q_{11} - S_1 - S_1^T & A S_1 + B N_1 & 0 \\ S_1^T A^T + N_1^T B^T & -Q_{11} & -Q_{12} \\ \hline 0 & -Q_{12}^T & -Q_{22} \end{array}\right] < 0 \tag{32}$$

As all principal minors of the underlying matrix of (32) must be negative definite, the following must hold

$$\begin{bmatrix} Q_{11} - S_1 - S_1^T & A S_1 + B N_1 \\ S_1^T A^T + N_1^T B^T & -Q_{11} \end{bmatrix} < 0 \tag{33}$$

Left- and right- multiplying this last condition by $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ and the LMI (29) is obtained for $Q_{11} = W_1$.

In the case of (26) first rewrite this LMI in extended form as

$$\begin{bmatrix} -Q_{211} & -Q_{212} & 0 & 0 \\ -Q_{212}^T & -Q_{222} & S_2^T B_0^T + N_2^T B^T & S_2^T D_0^T + N_2^T D^T \\ 0 & B_0 S_2 + B N_2 & \Upsilon & Q_{212} + S_1^T C^T + N_1^T D^T \\ 0 & D_0 S_2 + D N_2 & Q_{212}^T + C S_1 + D N_1 & Q_{222} - S_2 - S_2^T \end{bmatrix} < 0 \quad (34)$$

where

$$\Upsilon = Q_{211} - S_1 - S_1^T + A S_1 + B N_1 + S_1^T A^T + B^T N_1^T$$

Also for (34) to hold

$$\left[\begin{array}{cc|c} -Q_{222} & S_2^T B_0^T + N_2^T B^T & S_2^T D_0^T + N_2^T D^T \\ B_0 S_2 + B N_2 & \Upsilon & Q_{212} + S_1^T C^T + N_1^T D^T \\ \hline D_0 S_2 + D N_2 & Q_{212}^T + C S_1 + D N_1 & Q_{222} - S_2 - S_2^T \end{array}\right] < 0 \quad (35)$$

Left- and right multiplying this last result by $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ gives

$$\left[\begin{array}{c|cc} Q_{222} - S_2 - S_2^T & D_0 S_2 + D N_2 & Q_{212}^T + C S_1 + D N_1 \\ \hline (D_0 S_2 + D N_2)^T & -Q_{222} & S_2^T B_0^T + N_2^T B^T \\ (Q_{212}^T + C S_1 + D N_1)^T & B_0 S_2 + D N_2 & \Upsilon \end{array}\right] < 0 \quad (36)$$

and for (36) to hold

$$\begin{bmatrix} Q_{222} - S_2 - S_2^T & D_0 S_2 + D N_2 \\ (D_0 S_2 + D N_2)^T & -Q_{222} \end{bmatrix} < 0 \quad (37)$$

Finally, left- and right- multiply this last inequality by $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ to obtain (28) when $Q_{222} = W_2$, and the proof is complete. ■

In the proof of this last result the block diagonal structure of the matrix S is required to avoid additional strong links between the control law matrices K_1 and K_2.

Theorem 4 is much simpler than the corresponding one in [2] and hence more computationally tractable and less conservative. Also, the resulting LMIs are not parameterized and hence there is no parameter which must to be tuned to achieve stabilized process dynamics. Moreover, the number of LMIs has been reduced by a factor of two.

V. Numerical Example

As an example to illustrate the new results in this paper, consider a simplified model of a metal rolling process, see [1] for the details, where $\alpha = 100$ and

$$\left[\begin{array}{c|c|c} A & B_0 & B \\ \hline C & D_0 & D \end{array}\right] = 10^{-3} \times \left[\begin{array}{cc|c|c} 972.0 & 97.2 & 7.94 & -0.278 \\ -278.0 & 972.0 & 79.4 & -2.78 \\ \hline 972.0 & 97.2 & 722.0 & -0.278 \end{array}\right] \quad (38)$$

with boundary conditions

$$\begin{aligned} x_{k+1}(0) &= [0,\ 0]^T, \quad k \geq 0 \\ y_0(p) &= 5, \quad 0 \leq p \leq \alpha - 1 \end{aligned}$$

This process is practically stable (i.e. $r(D_0) < 1$ and $r(A) < 1$) and Figure 1 shows the pass profile sequence generated with zero control input, which is not acceptable in this application are due to the oscillations present in the along the pass dynamics. One option would be to attempt control law design to ensure practical stability without these oscillations but, as noted in the main body of the paper, there will be cases when this property is simply not strong enough. Here we give representative results from control law design for strong practical stability where acceptable along the pass dynamics result.

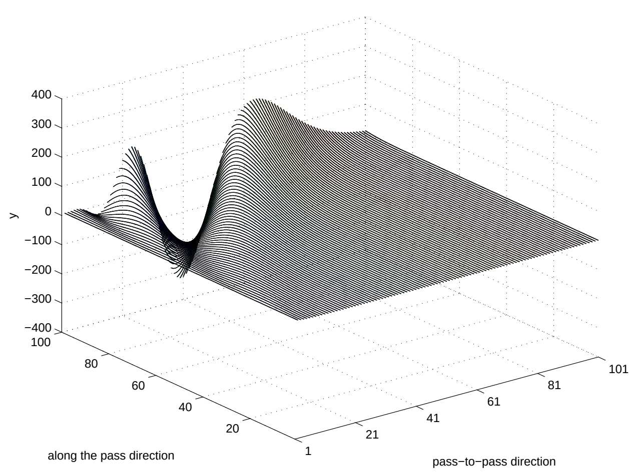

Fig. 1. Uncontrolled output progression.

Solving the LMIs of Theorem 4 gives the control law matrices

$$K_1 = \begin{bmatrix} 749.97 & 291.73 \end{bmatrix}, \qquad K_2 = 22.89.$$

and the sequence of pass profiles dynamics generated by the controlled process is given in Figure 2. These simulations confirm that the controlled process is converging in the pass-to-pass direction and will eventually reach the limit profile described by a stable 1D discrete linear system. Moreover, unacceptable oscillations do not appear in the along the dynamics.

VI. Conclusions

This paper has developed new results on strong practical stability and stabilization of discrete linear repetitive processes, starting from results in descriptor, but nonsingular, 1D

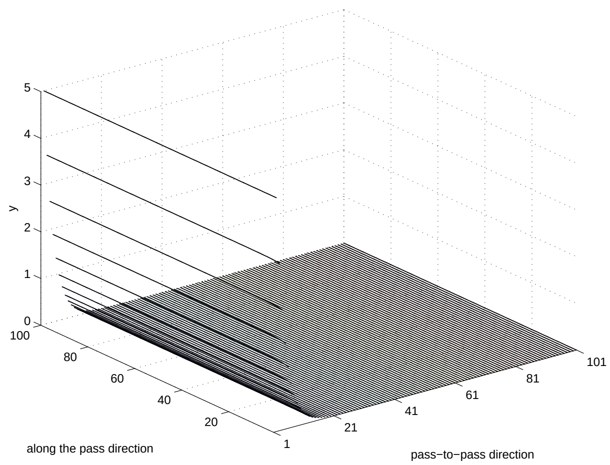

Fig. 2. Closed loop output progression.

linear systems approach. The resulting design algorithms are LMI based and are considerably simpler that those currently available. Of particular note is the reduced dimensions of the LMIs which, in turn, should limit the possibility of computational difficulties in design examples. Future work will include the extension of these results to processes with uncertainty in their state-space models and/or disturbances on, for example, the pass profile measurement that cannot be ignored at even the initial design stage.

The control law considered in this paper is based on feedback of the current state vector and if some its entries are not available for measurement then an observer will be required for implementation. An alternative is to extend the results here to a control law where the current pass state vector is replaced by the current pass profile. In the longer term, it is planned to extend this work to the ILC application area with experimental verification.

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