Discrete multiscale systems: stability results

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***Abstract*— We introduce discrete time-scale filtering by the way of certain double convolution systems. We prove stability theorems for these systems and make connections with function theory in the poly-disc. We also make connections with the white noise space framework.**

I. INTRODUCTION

In the present work we study a new type of double convolution system, which arises in the theory of multi-scale systems. We use the approach of the second named author presented at the *Mathematical Theory of Networks and Systems* conference in 2006 in Kyoto, see [1].

Consider the group $\mathcal{G}$ of automorphisms of the unit disc $\mathbb{D}$, *i.e.* linear fractional transformations (LFT) of the form

$$\gamma(z) = \frac{\gamma_1 z + \gamma_2}{\gamma_2^* z + \gamma_1^*}, \quad |\gamma_1|^2 - |\gamma_2|^2 = 1. \tag{1}$$

A LFT of this form, with $|\Re(\gamma_1)| > 1$, is said to be of hyperbolic type (see *e.g.* [2]). For $\alpha > 0$, denote by S_α the map

$$s \mapsto S_\alpha(s) = \alpha s$$

corresponding to the shift in scale α, in the right half plane. We also consider the Möbius transformation

$$G_\theta(s) = \frac{e^{i\theta} - s}{e^{-i\theta} + s}, \quad |\theta| < \frac{\pi}{2}$$

which maps conformally the open right half-plane $\mathbb{C}_+$ onto the open unit disc. Then we may verify (see [3]) that any element γ of $\mathcal{G}$ can be written as

$$e^{i\xi_\gamma}\gamma(z) = (G_{\theta_\gamma} \circ S_{\alpha_\gamma} \circ G_{\theta_\gamma}^{-1})(e^{i\xi_\gamma} z) \tag{2}$$

for some $\xi_\gamma, \theta_\gamma$. The parameter $\alpha_\gamma > 0$ is called the multiplier of the hyperbolic LFT γ. For any element γ of $\mathcal{G}$, the map $z \mapsto \gamma(z)$ is therefore conformally equivalent to $s \mapsto S_{\alpha_\gamma}(s)$. Following [1], we consider the unitary map

$$f \mapsto (T_\gamma f)(z) = \frac{1}{\gamma_2^* z + \gamma_1^*} f(\gamma(z)).$$

If f is analytic in $\mathbb{D}$ then $T_\gamma f$ is also analytic there, so let $f(z) = \sum_{n=0}^\infty f_n z^n$ and $(T_\gamma f)(z) = \sum_{n=0}^\infty f_n(\gamma) z^n$ be their Taylor expansions. The map which associates to the sequence $\{f_n\}_{n\in\mathbb{N}}$ the sequence $\{f_n(\gamma)\}_{n\in\mathbb{N}}$ is called the *scaling operation*. This definition was motivated in [1] by the study of the self-similarity property [4]. This property, which appears in many engineering applications, in particular on high quality LAN ethernet network traffic (see [5]), may be seen as a weighted form of stationarity in scale. Now the scale shift does not admit a clear cut definition for discrete-time signals.

Consider now a discrete subgroup Γ of $\mathcal{G}$, which represents the scales we will use to study the signals and systems. One associates to the sequence $\{f_n\}_{n\in\mathbb{N}}$ its *scale transform* $\{f_n(\gamma)\}_{n\in\mathbb{N},\gamma\in\Gamma}$, which is a function of $n \in \mathbb{N}$ and $\gamma \in \Gamma$.

In the present work, we introduce and study discrete time-scale invariant filtering in terms of double convolution input-output linear systems as described by

$$y_n(\gamma) = \sum_{m=0}^{n} \left(\sum_{\varphi\in\Gamma} h_{n-m}(\gamma \circ \varphi^{-1}) u_m(\varphi) \right), \quad \gamma \in \Gamma. \tag{3}$$

In [6] (see also [7], [8]), the first named author together with David Levanony considered another example of double convolution system, when both h_n and u_n are random variables, which belong to the white noise space, or more generally to the Kondratiev space. There are parallel and analogies between the theory of linear stochastic systems presented there and the theory developed here. These parallel and analogies are exploited here as guide and motivation for some of the proofs in the present paper.

Using the Hermite transform, one defines a generalized transfer function, which is a function analytic in z and in a countable number of scale variables (these variables take into account the randomness in [6]). Then we relate properties of the generalized transfer function with that of the system. In particular, we study the notion of BIBO stability and of dissipativity.

Note that our approach to multiscale system is different from that of wavelets. Indeed, the scale transform is the starting point of our approach (initiated in [1]) to multi-scale analysis in discrete time. In opposition to wavelets, we propose a transform which has on the same level both the time and the scale aspect.

II. DISCRETE TIME-SCALE FILTERING

A. Discrete time-scale signals

For Γ a discrete subgroup of $\mathcal{G}$, we recall that the Γ-scale transforms of a discrete-time sequence $\{x_n\}_{n\in\mathbb{N}}$ yield the double discrete time and scale sequence $\{x_n(\gamma)\}_{n\in\mathbb{N},\gamma\in\Gamma}$. For any fixed $\gamma = \gamma_0$, $\{x_n(\gamma_0)\}_{n\in\mathbb{N}}$ is a discrete-time sequence while $\{x_{n_0}(\gamma)\}_{\gamma\in\Gamma}$ is a discrete-scale sequence for any fixed $n = n_0$.

This is a short version of the paper [3].

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Definition 2.1: The scale-causal projection of of $\{x_m(\varphi)\}, \boldsymbol{\varphi} \in \Gamma$ is given by the restriction of $\{x_m(\varphi)\}$ to the scales $\boldsymbol{\varphi}$ for which the multiplier is strictly less than one: $\alpha_\varphi < 1$, where α_φ is the multiplier of the hyperbolic LFT $\varphi(z)$ as defined in (2).
Now on, we consider only discrete Abelian subgroups of hyperbolic transformations.

Definition 2.2: Given a discrete Abelian subgroup Γ of $\mathcal{G}$ we denote by Γ_+ the set of hyperbolic transformations consisting of the identity and of the scales $\boldsymbol{\varphi}$ for which the multiplier is strictly less than one: $\alpha_\varphi < 1$.
Given γ and φ two elements of Γ, we will say that γ succeeds φ and will note $\varphi \preccurlyeq \gamma$, if $\gamma \circ \varphi^{-1} \in \Gamma_+$ that is:

$$\varphi \preccurlyeq \gamma \quad \iff \alpha_{\gamma\circ\varphi^{-1}} \leqslant 1.$$

Proposition 2.3: The relation $\preccurlyeq$ defines a total order in Γ.

Proof: Since we assume that Γ is Abelian, all the transformations must have the same fixed points. The parameters ξ_γ and θ_γ in (2) are therefore constant. The proof then follows upon noting that the multiplier $\alpha_{\gamma\circ\varphi}$ is given by: $\alpha_{\gamma\circ\varphi} = \alpha_\gamma\alpha_\varphi$. ■

With this order we obtain a bijection

$$\gamma \mapsto \varrho(\gamma) \tag{4}$$

between Γ and $\mathbb{Z}$, and one can identify $\ell_2(\Gamma)$ and $\ell_2(\mathbb{Z})$ and $\ell_2(\Gamma_+)$ and $\ell_2(\mathbb{N})$.

Using the isomorphism we introduce the following definition:

Definition 2.4: The function

$$u \;:\; \Gamma_+ \mapsto \mathbb{C}$$

has finite support if

$$N(u) \stackrel{\triangle}{=} \max\{\varrho(\gamma) \text{ such that } u(\gamma) \neq 0\} < \infty,$$

where ϱ is the bijection defined by (4). The support of the function u is the interval $[0, N(u)] \subset \mathbb{N}$.

Definition 2.5: A stable signal will be a sequence $\{u_n(\cdot)\}_{n\in\mathbb{N}}$ of elements of $\ell_2(\Gamma)$, and such that the condition

$$\sup_{n=0,1,\ldots} \|u_n(\cdot)\|_{\ell_2(\Gamma)} < \infty \tag{5}$$

holds.
A stable and scale-causal signal will be a sequence $\{u_n(\cdot)\}_{n\in\mathbb{N}_0}$ of elements of $\ell_2(\Gamma_+)$, and such that the condition

$$\sup_{n=0,1,\ldots} \|u_n(\cdot)\|_{\ell_2(\Gamma_+)} < \infty \tag{6}$$

holds.

In the sequel, we will impose the following stronger norm constrains on a signal, besides (5) or (6), namely:

$$\sum_{n=0}^{\infty} \|u_n(\cdot)\|^2_{\ell_2(\Gamma)} < \infty, \tag{7}$$

or

$$\sum_{n=0}^{\infty} \|u_n(\cdot)\|_{\ell_2(\Gamma)} < \infty, \tag{8}$$

and similarly for scale-causal signals.

We note the following: a dissipative filter cannot be effective at all scales. At some stage, details cannot be seen. These intuitive facts are made more precise in the following proposition.

Proposition 2.6:

1) Assume that the supports of the u_n are uniformly bounded. Then, (7) is in force.
2) Assume that the support of u_n is infinite for all n. Then, the sum on the left side of (7) diverges.

Proof: Let ϱ be the bijection (4), and let N be such that the support of all the functions $\gamma \mapsto u_n(\gamma)$ is inside $[0, N]$. Then,

$$\begin{aligned}
\sum_{n=0}^{\infty} \|u_n(\cdot)\|^2_{\ell_2(\Gamma)} &= \sum_{n=0}^{\infty}\sum_{\varrho(\gamma)=0}^{N} |u_n(\gamma)|^2 \\
&= \sum_{\varrho(\gamma)=0}^{N}\sum_{n=0}^{\infty} |u_n(\gamma)|^2 \\
&\leq N\|u\|_{\ell_2(\mathbb{N}_0)}.
\end{aligned}$$

This stems from the fact that the maps T_γ are unitary from $\mathbf{H}_2(\mathbb{D})$ onto itself.

The second claim is proved similarly. ■

An example of (u_n) satisfying Condition 1) of Proposition (2.6) has been presented in the paper [1], where the corresponding group Γ is Fuchsian. This was used therein, to define the scale unit-pulse signal. A similar condition was also considered by P. Yuditskii [9] in the description of the direct integral of spaces of character-automorphic functions.

B. Discrete time-scale invariant systems

We define a linear scale-invariant discrete system in much the same way as for the classical linear time-invariant discrete counterpart. For this, it suffices to replace the usual additive group $\mathbb{N}$ by the discrete Abelian group Γ and to consider the convolution product on Γ. Such systems have been introduced in [1] but without further studies of their propoerties. In this paper, we consider both discrete time and scale. We define linear both time and scale invariant systems along with a stability analysis. These systems are described in terms of input-output relation given in (3).

Before we proceed, let us recall the following.

The scaling operators $T_{\boldsymbol{\varphi}}$ form a group of operators from the Hardy space $\mathbf{H}_2(\mathbb{D})$ onto itself. Recall that we have discretized the scale axis and we have restricted $\boldsymbol{\varphi}$ to a discrete subgroup Γ of $\mathcal{G}$. Also we take Γ Abelian (cyclic). Let $\widehat{\Gamma}$ stands for the dual group of $\widehat{\Gamma}$: it is formed by the set of functions $\sigma : \Gamma \to \mathbb{T}$ such that

$$\sigma(\iota) = 1 \text{ and } \forall\ \gamma, \varphi \in \Gamma,\ \sigma(\gamma\circ\varphi) = \sigma(\gamma)\sigma(\varphi),$$

where ι stands for the identity transformation. The elements of $\widehat{\Gamma}$ are called characters of the group Γ (see [10]). We denote by $\widehat{\mu}$ the Haar measure of $\widehat{\Gamma}$, which is compact by the Pontryagin duality [10]. We recall the definition of the Fourier transform on Γ and of its inverse:

$$\widehat{x}(\sigma) = \sum_{\gamma\in\Gamma} x(\gamma)\sigma(\gamma)^*,$$
$$x(\gamma) = \int_{\widehat{\Gamma}} \widehat{x}(\sigma)\sigma(\gamma)d\widehat{\mu}(\sigma).$$

The Haar measure $d\widehat{\mu}$ is normalized so that Plancherel's theorem holds:

$$\|f\|^2_{\ell_2(\Gamma)} \stackrel{\triangle}{=} \sum_{\gamma\in\Gamma}|f(\gamma)|^2 = \int_{\widehat{\Gamma}}|\widehat{f}(\sigma)|^2 d\widehat{\mu}(\sigma) \stackrel{\triangle}{=} \|\widehat{f}\|^2_{\mathbf{L}_2(d\widehat{\mu})}.$$

See [11, Theorem 8.4.2 p. 123].

Definition 2.7: An impulse response (resp. a scale-causal impulse response) will be a sequence $\{h_n(\cdot)\}_{n\in\mathbb{N}}$ of elements of $\ell_2(\Gamma)$ (resp. of $\ell_2(\Gamma_+)$) such that for every $n \in \mathbb{N}$, the multiplication operator

$$\mathcal{M}_{h_n}: \quad u \mapsto h_n \star u, \quad n = 0, 1, \ldots \tag{9}$$

is bounded from $\ell_2(\Gamma)$ into itself (resp. from $\ell_2(\Gamma_+)$ into itself) and such that

$$\sup_{n=0,1,\ldots}\|h_n\|_{\ell_2(\Gamma)} < \infty \quad (resp. \quad \sup_{n=0,1,\ldots}\|h_n\|_{\ell_2(\Gamma_+)} < \infty). \tag{10}$$

We note that we do not require the operator norms of the operators $\mathcal{M}_{h_n}$ to be uniformly bounded in n. There is no direct connections between the norm $\|h_n\|_{\ell_2(\Gamma)}$ and $\mathcal{M}_{h_n}$. Condition (10) is needed for expression (11) below to make sense.

The systems that we consider here are defined by the double convolution (3), that we recall below:

$$y_n(\gamma) = \sum_{m\in\mathbb{Z}}\left(\sum_{\delta\in\Gamma} h_{n-m}(\gamma\circ\delta^{-1})u_m(\delta)\right).$$

In view of (10) the series

$$H(z,\sigma) = \sum_{n=0}^{\infty} z^n \widehat{h}_n(\sigma) \tag{11}$$

converges in the $\mathbf{L}_2(d\widehat{\mu})$ norm for every $z \in \mathbb{D}$. Taking the Fourier transform (with respect to Γ) of both sides of (3) we obtain

$$\widehat{y}_n(\sigma) = \sum_{m=0}^{n} \widehat{h}_{n-m}(\sigma)\widehat{x}_m(\sigma), \tag{12}$$

where the equality is in the $\mathbf{L}_2(d\widehat{\mu})$ sense. Taking now the Z transform we get:

$$Y(z,\sigma) = H(z,\sigma)U(z,\sigma), \tag{13}$$

where $Y(z,\sigma)$ is defined by:

$$Y(z,\sigma) = \sum_{n=0}^{\infty} z^n\widehat{y}_n(\sigma) \quad \text{and} \quad U(z,\sigma) = \sum_{n=0}^{\infty} z^n\widehat{u}_n(\sigma), \tag{14}$$

and where, for every $z \in \mathbb{D}$ the equality in (13) is $\widehat{\mu}$-a.e.

The function $H(z,\sigma)$ can be seen as the transfer function of the discrete-time scale-invariant system. Formula (11) suggests to define and study hierarchies of transfer functions, for which the functions $\widehat{h_n}$ depend on σ in some pre-assigned way (for instance, when they are polynomials in σ), or when the function $H(z,\sigma)$ is a rational function of z or of σ. In the next two sections, under the hypothesis that the subgroup Γ has a finite number, say p, of generators, we will associate to the system (3) an analytic function of $p+1$ variables, which we will call the *generalized transfer function of the system.*

III. Generalized transfer function

A. The case of one generator

In this section, we consider the case of a cyclic group Γ. Any transformation in Γ is thus of the form $\gamma_0^m \stackrel{\triangle}{=} \underbrace{\gamma_0\circ\ldots\circ\gamma_0}_{m \text{ times}}$, $m \in \mathbb{Z}$, where γ_0 is the generator.

Theorem 3.1: There exists a positive measure $d\nu(\theta)$ on $[0, 2\pi)$ such that

$$\int_{\widehat{\Gamma}} \sigma(\gamma_0^m)d\widehat{\mu}(\sigma) = \int_0^{2\pi} e^{im\theta}d\nu(\theta), \quad m \in \mathbb{Z}. \tag{15}$$

Proof: The proof follows from the trigonometric moment problem [12]. See [3] for the details. ■

Remark 3.2: This theorem formalizes the intuitive idea that one can make the "change of variable"

$$\sigma(\gamma_0) = e^{i\theta(\sigma)}.$$

Theorem 3.3: The linear map $\mathbf{I}$ which to $\sigma(\gamma_0^m)$ associates the function ζ^m:

$$\mathbf{I}(\sigma(\gamma_0^m)) = \zeta^m, \quad m \in \mathbb{Z}, \tag{16}$$

is an isomorphism from $\mathbf{L}_2(d\widehat{\mu})$ into $\mathbf{L}_2(d\nu)$.

Proof: For a function f of the form

$$f(\sigma) = \sum_{-N}^{M} c_n\sigma(\gamma_0^n) \quad \text{where} \quad N, M \in \mathbb{N}_0 \quad \text{and} \quad c_n \in \mathbb{C}, \tag{17}$$

we have

$$\begin{aligned}\|f\|^2_{\mathbf{L}_2(d\widehat{\mu})} &= \sum_{n,m=-N,\ldots,M} c_n c_m^* t_{m-n} \\ &= \sum_{n,m=-N,\ldots,M} c_n c_m^* \int_0^{2\pi} e^{-i(m-n)\theta}d\nu(\theta) \\ &= \int_0^{2\pi} |\sum_{n=-N}^{M} c_n e^{in\theta}|^2 d\nu(\theta) \\ &= \|\mathbf{I}(f)\|^2_{\mathbf{L}_2(d\nu)}.\end{aligned}$$

The result follows by continuity since such f are dense in $\mathbf{L}_2(d\widehat{\mu})$. To verify this last claim we note the following: By Plancherel's theorem, the map from $\ell_2(\Gamma)$ onto $\mathbf{L}_2(d\widehat{\mu})$ which to the sequence which consists only of zeros, except the n-th element which is equal to 1, associates the function $\sigma(\gamma_0)^n$, extends to a unitary map. ■

We will be interested in particular in the positive powers of $\boldsymbol{\gamma}_0$, which correspond to zooming (we consider that the multiplier of $\boldsymbol{\gamma}_0$, *i.e.* the associated scale α_{γ_0}, is less than 1).

Definition 3.4: We denote by $\mathbf{H}_2(d\widehat{\mu})$ the closure in $\mathbf{L}_2(d\widehat{\mu})$ of the functions $\sigma(\gamma_0)^n$, $n = 0, 1, 2, \ldots$. Similarly, we denote by $\mathbf{H}_2(d\nu)$ the closure in $\mathbf{L}_2(d\nu)$ of the functions z^n, $n = 0, 1, 2, \ldots$.

Note that it may happen that $\mathbf{L}_2(d\widehat{\mu}) = \mathbf{H}_2(d\widehat{\mu})$.

Following [6] we introduce the next definition.

Definition 3.5: The map $\mathbf{I}$ will be called the Hermite transform.

Recall that $\widehat{\Gamma}$ is compact and therefore

$$\mathbf{L}_2(d\widehat{\mu}) \subset \mathbf{L}_1(d\widehat{\mu}). \tag{18}$$

In general the product of two elements f and g in $\mathbf{L}_2(d\widehat{\mu})$ does not belong to $\mathbf{L}_2(d\widehat{\mu})$, and one cannot define $\mathbf{I}(fg)$, let alone compare it with the product $\mathbf{I}(f)\mathbf{I}(g)$. On the other hand, we will need in the sequel only the case where at least one of the elements in the product fg defines a bounded multiplication operator from $\mathbf{L}_2(d\widehat{\mu})$ into itself; see Definition 2.7 and the proof of Theorem 5.3 for instance. This is exploited in the next theorem.

Theorem 3.6: Let $f \in \mathbf{L}_2(d\widehat{\mu})$ such that the operator of multiplication by f defines a bounded operator from $\mathbf{L}_2(d\widehat{\mu})$ into itself. Then for every g in $\mathbf{L}_2(d\widehat{\mu})$ it holds that:

$$\mathbf{I}(fg) = \mathbf{I}(f)\mathbf{I}(g). \tag{19}$$

Proof: See [3] ∎

Definition 3.7: The function

$$\mathscr{H}(z,\zeta) = \sum_{n=0}^{\infty} z^n \mathbf{I}(\widehat{h_n})(\zeta) \tag{20}$$

is called the *generalized transfer function* of the system.

Taking the Hermite transform on both sides of (13), or, equivalently, taking the Z transform and the Hermite transform on both sides of (**??**), we obtain

$$\mathscr{Y}(z,\zeta) = \mathscr{H}(z,\zeta)\mathscr{U}(z,\zeta),$$

where $\mathscr{U}(z,\zeta) = \sum_{n=0}^{\infty} z^n \mathbf{I}(\widehat{u_n})(\zeta)$, and similarly for $\mathscr{Y}(z,\zeta)$. The function $\mathscr{H}$ is analytic in a neighborhood of $(0,0) \in \mathbb{C}^2$. It is of interest to relate the properties of $\mathscr{H}$ and of the system. This is done in the remaining of the paper. But before we proceed, we first generalize the preceding result to the case where Γ has a finite number $p > 1$ of generators.

B. The case of p generator

We now assume that the Abelian group Γ has a finite number, say p, of generators, which we will denote by $\gamma_1, \ldots, \gamma_p$. We assume that they are independent in the sense that if

$$\gamma_1^{n_1} \circ \cdots \circ \gamma_p^{n_p} = \iota$$

for some integers $n_1, \ldots n_p \in \mathbb{Z}$, then $n_1 = \cdots = n_p = 0$. In particular, each generator is of the form

$$\gamma_i(z) = \gamma_{\{\alpha_i\}}(z) = (G_\theta \circ S_{\alpha_i} \circ G_\theta^{-1})(z)$$

with θ fixed, and where the set $\{\alpha_i\}_{i=1}^p$ generates a free discrete subgroup of the multiplicative group of positive real numbers. We use in a free way the multi-index notation.

Theorem 3.8: There is a positive measure $d\nu$ on the distinguished boundary of the polydisc such that

$$\int_{\widehat{\Gamma}} \sigma(\gamma_1^{n_1}) \cdots \sigma(\gamma_p^{n_p}) d\widehat{\mu}(\sigma) = \int_{\mathbb{T}^p} e^{in_1\theta_1} \cdots e^{in_p\theta_p} d\nu(\theta_1, \ldots, \theta_p).$$

Proof: The theorem is based on a result of Putinar [13] on the trigonometric moment problem in compact semialgebraic sets. See [3]. ∎

Definition 3.9: The Hermite transform of the element

$$f(\sigma) = \sum_\alpha h_\alpha \sigma(\gamma^\alpha)$$

is

$$\mathbf{I}(f)(\zeta) = \sum_\alpha h_\alpha \zeta^\alpha.$$

Theorem 3.10: Let $f \in \mathbf{L}_2(d\widehat{\mu})$ be such that the operator of multiplication by f defines a bounded operator from $\mathbf{L}_2(d\widehat{\mu})$ into itself. Then, for every $g \in \mathbf{L}_2(d\widehat{\mu})$:

$$\mathbf{I}(fg) = \mathbf{I}(f)\mathbf{I}(g). \tag{21}$$

The proof is the same as for $p = 1$. As in Definition 3.7, the function of $p+1$ variables

$$\mathscr{H}(z, \zeta_1, \ldots, \zeta_p) = \sum_{n=0}^{\infty} z^n \mathbf{I}(\widehat{h_n})(\zeta_1, \ldots, \zeta_p)$$

is called the *generalized transfer function* of the system.

IV. BIBO STABILITY

The system (3) will be called bounded input bounded output (BIBO) if there is an $M > 0$ such that for every $\{u_n(\gamma)\}$ such that

$$\sup_{n\in\mathbb{N}} \|u_n(\cdot)\|_{\ell_2(\Gamma)} < \infty \tag{22}$$

the output is such that $\{y_n(\gamma)\}_{\gamma\in\Gamma} \in \ell_2(\Gamma)$, $n = 0, 1, \ldots$, and it holds that

$$\sup_{n\in\mathbb{N}} \|y_n(\cdot)\|_{\ell_2(\Gamma)} \leq M \sup_{n\in\mathbb{N}} \|u_n(\cdot)\|_{\ell_2(\Gamma)}. \tag{23}$$

The following theorem gives a characterization of BIBO systems. The proof follows the proof of [6, Theorem 3.2]. We note the following difference between the two theorems: in [6] the multiplication operators, that is the counterparts of the operators $\mathcal{M}_{h_n}$ defined here using the Wick product, are automatically bounded. Here we do not have an analogue of this inequality.

Theorem 4.1: The system (3) is BIBO if and only if the following two conditions hold:

(a) The multiplication operators (9)

$$\mathcal{M}_{h_n}: \quad u \mapsto h_n \star u, \quad n = 0, 1, \ldots$$

are bounded from $\ell_2(\Gamma)$ into itself.

(b) For all $v(\cdot) \in \ell_2(\Gamma_+)$ with $\|v(\cdot)\|_{\ell_2(\Gamma)} = 1$ it holds that

$$\sum_{n=0}^{\infty} \|\mathcal{M}_{h_n}^*(v)\|_{\ell_2(\Gamma)} \leq M. \tag{24}$$

Proof: That the condition (24) is sufficient is readily seen. Indeed, take $v \in \ell_2(\Gamma)$ with $\|v(\cdot)\|_{\ell_2(\Gamma)} = 1$. From (**??**) we have:

$$\langle y_n \, , \, v\rangle_{\ell_2(\Gamma)} = \sum_{m=0}^{n} \langle u_m \, , \, \mathcal{M}^*_{h_{n-m}} v\rangle_{\ell_2(\Gamma)}, \quad n = 0, 1, \ldots, \tag{25}$$

and hence

$$\begin{aligned} |\langle y_n \, , \, v\rangle_{\ell_2(\Gamma)}| &\le \sum_{m=0}^{n} \|u_m(\cdot)\|_{\ell_2(\Gamma)} \|\mathcal{M}^*_{h_{n-m}} v\|_{\ell_2(\Gamma)} \\ &\le \sup_{m=0,\ldots n} \|u_m(\cdot)\|_{\ell_2(\Gamma)} \sum_{m=0}^{n} \|\mathcal{M}^*_{h_m} v\|_{\ell_2(\Gamma)} \\ &\le M \sup_{m \in \mathbb{N}_0} \|u_m(\cdot)\|_{\ell_2(\Gamma)}. \end{aligned}$$

We obtain (24) by taking $v = y_n/\|y_n\|_{\ell_2(\Gamma)}$ when $y_n \neq 0$.

We now show that (24) is necessary. We assume that the system is bounded input and bounded output. We first note that the multiplication operators $\mathcal{M}_{h_n}$ are necessarily bounded. Indeed, assume that (23) is in force and take $u_0 = u \in \ell_2(\Gamma)$ and $u_n = 0$ for $n > 0$. Then,

$$y_n = h_n \star u = \mathcal{M}_{h_n}(u), \quad n = 0, 1, \ldots,$$

and it follows from (23) that $\|\mathcal{M}_{h_n}\| \le M$ for $n = 0, 1, \ldots$

Let us now consider an input sequence (u_n) which satisfies (5). For a given n and v choose

$$u_m = 0 \quad \text{if} \quad \mathcal{M}^*_{h_{n-m}} v = 0,$$

and

$$u_m = \frac{\mathcal{M}^*_{h_{n-m}} v}{\|\mathcal{M}^*_{h_{n-m}} v\|_{\ell_2(\Gamma)}} \quad \text{otherwise}.$$

We obtain from (25) and (23) that

$$\sum_{m=0}^{n} \|\mathcal{M}^*_{h_{n-m}} v\|_{\ell_2(\Gamma)} \le M,$$

from which we get (24). ■

We now make a number of remarks: first, condition (23) is implied by the stronger, but easier to deal with, condition

$$\sum_{n=0}^{\infty} \|\mathcal{M}_{h_n}\| \le M. \tag{26}$$

When Γ is the trivial subgroup of $SU(1,1)$, conditions (24) or (26) reduce to the classical condition

$$\sum_{n=0}^{\infty} |h_n| < \infty.$$

Finally, other versions of this theorem could be given, with non causal systems with respect to the variable n (as in [6]), or with scale-causal signals. We state the last one. The proof is the same as the proof of Theorem 4.1.

Theorem 4.2: The system (3) is scale-causal and BIBO if and only if the following two conditions hold:
(a) The multiplication operators (9)

$$\mathcal{M}_{h_n}: \quad u \mapsto h_n \star u, \quad n = 0, 1, \ldots$$

are bounded from $\ell_2(\Gamma_+)$ into itself.
(b) For all $v(\cdot) \in \ell_2(\Gamma)$ with $\|v(\cdot)\|_{\ell_2(\Gamma_+)} = 1$ it holds that

$$\sum_{n=0}^{\infty} \|\mathcal{M}^*_{h_n}(v)\|_{\ell_2(\Gamma_+)} \le M.$$

V. Dissipative systems

We will call the system (3) *dissipative* if for every input sequence (u_n) such that

$$\sum_{n=0}^{\infty} \|u_n(\cdot)\|^2_{\ell_2(\Gamma)} < \infty$$

it holds that

$$\sum_{n=0}^{\infty} \|y_n(\cdot)\|^2_{\ell_2(\Gamma)} \le \sum_{n=0}^{\infty} \|u_n(\cdot)\|^2_{\ell_2(\Gamma)}. \tag{27}$$

Theorem 5.1: The system is dissipative if and only if the $\mathbf{L}(\ell_2(\Gamma))$-valued function

$$S(z) = \sum_{n=0}^{\infty} z^n \mathcal{M}_{h_n}$$

is analytic and contractive in the open unit disc.

Proof: Equations (27) expresses that the block Toeplitz operator

$$\begin{pmatrix} M_{h_0} & 0 & 0 & \cdots \\ M_{h_1} & M_{h_0} & 0 & \cdots \\ \vdots & \vdots & & \\ & & & \end{pmatrix}$$

is a contraction from $\ell_2(\ell_2(\Gamma))$ into itself, and this is equivalent to the asserted condition on S. ■

We consider the case of scale-causal signals.

Definition 5.2: The system (3) will be called scale-causal dissipative if the following conditions hold:
(1) The operators M_{h_n} are bounded from $\ell_2(\Gamma_+)$ into itself.
(2) Condition (27) holds, with $\ell_2(\Gamma)$ replaced by $\ell_2(\Gamma_+)$.

Recall that we have denoted by $\mathbf{H}_2(d\nu)$ the closure in $\mathbf{L}_2(d\nu)$ of the powers z^α, where all the components of α are greater or equal to 0. Taking the Fourier and Hermite transforms we have:

Theorem 5.3: The system is scale-causal dissipative if and only the function

$$\mathscr{H}(z, \zeta_1, \ldots, \zeta_p) = \sum_{n=0}^{\infty} z^n \mathbf{I}(\widehat{h_n})(\zeta_1, \ldots, \zeta_p) \tag{28}$$

is contractive from $\mathbf{H}_2(\mathbb{D}) \otimes \mathbf{H}_2(d\nu)$ into itself. Furthermore, if the space $\mathbf{H}_2(d\nu)$ is a reproducing kernel Hilbert space, say with reproducing kernel $K(\zeta_1, \ldots, \xi_1, \ldots)$, condition (28) is equivalent to the positivity in $\mathbb{D}^{p+1}$, of the kernel

$$\frac{1 - \mathscr{H}(z, \zeta_1, \ldots)\mathscr{H}(w, \xi_1, \ldots)^*}{1 - zw^*} K(\zeta_1, \ldots, \zeta_p, \xi_1, \ldots, \xi_p) \tag{29}$$

Proof: Since the operators M_{h_n} are assumed bounded, we have

$$h_{n-m} \star u_m \in \ell_2(\Gamma_+), \quad m = 0, \ldots, n,$$

for all entries $u_m \in \ell_2(\Gamma_+)$. Thus

$$\widehat{h_{n-m}}\widehat{u_m} \in \mathbf{H}_2(d\widehat{\mu}),$$

and we may apply Theorem 3.6. We can write:

$$\left\| \sum_{n=0}^{m} h_{n-m} \star u_m \right\|_{\ell_2(\Gamma_+)} = \left\| \sum_{n=0}^{m} \mathbf{I}(\widehat{h_{n-m}})\mathbf{I}(\widehat{u_m}) \right\|_{\mathbf{H}_2(d\nu)}.$$

Thus the dissipativity is translated into the contractivity of the block Toeplitz operator

$$\begin{pmatrix} M_{\widehat{h_0}} & 0 & 0 & \cdots \\ M_{\widehat{h_1}} & M_{\widehat{h_0}} & 0 & \cdots \\ \vdots & \vdots & & \\ & & & \end{pmatrix}$$

from $\ell_2(\mathbf{H}_2(d\nu))$ into itself, and hence the claim on $\mathscr{H}$. To prove the second claim, we remark that $\mathbf{H}_2(\mathbb{D}) \otimes \mathbf{H}_2(d\nu)$ is the reproducing kernel Hilbert space with reproducing kernel

$$\frac{1}{1 - zw^*} K(\zeta_1, \cdots, \zeta_p, \xi_1, \cdots, \xi_p).$$

This comes from the fact that the reproducing kernel of a tensor product of reproducing kernel Hilbert spaces is the product of the reproducing kernels.

VI. ℓ_1-ℓ_2 BOUNDED SYSTEMS

The system (3) will be called ℓ_1-ℓ_2 bounded if there is a $M > 0$ such that for all inputs (u_n) satisfying

$$\sum_{n=0}^{\infty} \|u_n(\cdot)\|_{\ell_2(\Gamma)} < \infty,$$

we have

$$\left(\sum_{n=0}^{\infty} \|y_n(\cdot)\|^2_{\ell_2(\Gamma)} \right)^{1/2} \leq M \sum_{n=0}^{\infty} \|u_n(\cdot)\|_{\ell_2(\Gamma)}.$$

Taking the Fourier transform, this condition can be rewritten as:

$$\left(\sum_{n=0}^{\infty} \|\widehat{y_n}\|^2_{\mathbf{L}_2(d\widehat{\mu})} \right)^{1/2} \leq M \sum_{n=0}^{\infty} \|\widehat{u_n}\|_{\mathbf{L}_2(d\widehat{\mu})}, \tag{30}$$

The system (3) will be called *scale-causal ℓ_1-ℓ_2 bounded* if it is moreover scale-causal, that is, if the operators M_{h_n} are bounded from $\ell_2(\Gamma_+)$ into itself. Condition (30) then becomes:

$$\left(\sum_{n=0}^{\infty} \|\widehat{y_n}\|^2_{\mathbf{H}_2(d\widehat{\mu})} \right)^{1/2} \leq M \sum_{n=0}^{\infty} \|\widehat{u_n}\|_{\mathbf{H}_2(d\widehat{\mu})}, \tag{31}$$

from which we obtain, in much the same way as in [6], the following result.

Theorem 6.1: A necessary and sufficient condition for the system (3) to be scalar-causal and ℓ_1-ℓ_2 bounded is that the function

$$H(z, \sigma) = \sum_{n=0}^{\infty} z^n \widehat{h_n}(\sigma) \in \mathbf{H}_2(\mathbb{D}) \otimes \mathbf{H}_2(d\widehat{\mu}), \tag{32}$$

or, equivalently, that the transfer function

$$\mathscr{H}(z, \zeta) = \sum_{n=0}^{\infty} z^n \mathbf{I}(\widehat{h_n})(\zeta) \in \mathbf{H}_2(\mathbb{D}) \otimes \mathbf{H}_2(d\nu). \tag{33}$$

When $\mathbf{H}_2(d\nu) \neq \mathbf{L}_2(d\nu)$ (recall that Γ is finitely generated), (33) can be translated into reproducing kernel conditions. In particular, in the cyclic case, we have:

Theorem 6.2: Assume that $\mathbf{H}_2(d\nu) \neq \mathbf{L}_2(d\nu)$, and let

$$\frac{A(\zeta)A(\xi)^* - B(\zeta)B(\xi)^*}{1 - \zeta\xi^*}$$

be the reproducing kernel of $\mathbf{H}_2(d\nu)$. The system (3) is scale-causal and $\ell_1 - \ell_2$ bounded if and only if there is a $M > 0$ such that the kernel

$$\frac{A(\zeta)A(\xi)^* - B(\zeta)B(\xi)^*}{(1 - zw^*)(1 - \zeta\xi^*)} - M\mathscr{H}(z, \zeta)\mathscr{H}(w, \xi)^*$$

is positive in the bi-disc.

As in the case of equation (29), this comes from the characterization of the reproducing kernel of a tensor product of reproducing kernel Hilbert spaces.

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