LMI Conditions of Strictly Bounded Realness On A State-space Realization To Bi-tangential Rational Interpolation

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***Abstract*— We present LMI conditions to characterize the strictly bounded realness of the state-space realization of the solution to the bi-tangential rational interpolation problem with McMillan degree constraint.**

***Index Terms* - Bi-tangential rational interpolation, LMI**

I. Introduction

In this paper, we present the LMI conditions to characterize the strictly bounded realness of the state-space realization of the solution to the bi-tangential rational interpolation problem, i.e., they give the solution to the bi-tangential Nevanlinna-Pick interpolation problem [13], [3]. The bi-tangential Nevanlinna-Pick interpolation problem is the generalization of the classical interpolation problems by Carathéodory, Fejér, Nevanlinna, Pick, see, e.g., [3], [4]. The conventional approach to the interpolation problem is by means of the J-lossless function, which enables one to describe the set of the solutions to the interpolation problem by the linear fractional transformation [3]. Clearly, the rational interpolation problem is independently interesting problem, where the constraint of the bounded realness on the interpolant is removed [1]. A state-space realization is shown in [8], which gives a parameterization of the solutions to the tangential rational interpolation problem.

The approach to the bi-tangential Nevanlinna-Pick interpolation problem in this paper is based on the linear matrix inequality (LMI) [5], [11], [7], [2], [10], [12]. For a rational function, which satisfies the interpolation condition, a convex characterization of the free parameter of the state-space realization of the rational function is given by an LMI. We present the theory for the continuous-time and discrete-time systems. We use the state-space realization, which gives the solutions to the tangential rational interpolation problem without the bounded realness [9], [8]. However, we show that the bi-tangential Nevanlinna-Pick interpolation problem can be equivalently transformed to a tangential Nevanlinna-Pick interpolation interpolation problem under the condition of the strictly bounded realness. Thus, we can use the state-space realization to obtain the solution to the bi-tangential Nevanlinna-Pick interpolation problem.

We discuss the set of strictly bounded real interpolants, which satisfies additional interpolation conditions. The solvability condition of the interpolation problem is given by a matrix inequality, which is not linear. Thus, we give a convex relaxation of the matrix inequality by an LMI, following the scalar case [6].

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Notations

Real numbers are represented by $\mathbb{R}$ and complex numbers are represented by $\mathbb{C}$. $\bar{c}$ denotes the conjugate of $c \in \mathbb{C}$. Denote by $\mathbb{R}^{j\times k}$ $j \times k$ real matrices and by $\mathbb{C}^{j\times k}$ $j \times k$ complex matrices. $I_{m\times m}$ denotes $m\times m$ identity matrix, and $0_{j\times k}$ denotes $j\times k$ zero matrix. They are simply represented by I and 0 if their dimensions are clear in the context.

Denote by $\rho(A)$ the spectrum of a matrix A and by $\bar{\sigma}(A)$ the maximum singular value of a matrix A. We use the notations $A > 0$ to denote that a matrix A is positive definite. Denote by A^* the complex conjugate transpose of a matrix A.

Denote by $\mathbb{C}_+ = \{s \in \mathbb{C} : \text{Re s} > 0\}$ the open right half plane and by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc. The outside of the closed unit disc is denoted by $\bar{\mathbb{D}}^c = \{z \in \mathbb{C} : |z| > 1\}$. The systems are represented by their transfer functions, which are functions of the Laplace transform variable $s \in \mathbb{C}_+$ for continuous-time systems or functions of Z transform variable $z \in \mathbb{D}$ for discrete-time systems.

The state-space realization of a transfer function $G(s)$ with k inputs, j outputs and n states is denoted by

$$G(s) = C(sI - A)^{-1}B + D,$$

where the matrices are $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{j\times n}$ and $D \in \mathbb{C}^{j\times k}$. We also use a notation

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

to denote $G(s)$. For the continuous-time system, the conjugate is defined by

$$\begin{aligned} G(s)^{\sim} &:= G(-\bar{s})^* \\ &= \left[\begin{array}{c|c} -A^* & C^* \\ \hline -B^* & D^* \end{array}\right]. \end{aligned}$$

For the discrete-time system

$$G(z) = C(zI - A)^{-1}B + D,$$

the conjugate is defined by

$$\begin{aligned} G(z)^{\sim} &:= G(\bar{z}^{-1})^* \\ &= \left[\begin{array}{c|c} A^{-*} & -A^{-*}C^* \\ \hline B^*A^{-*} & D^* - B^*A^{-*}C^* \end{array}\right]. \end{aligned}$$

Let $H_\infty(\mathbb{D})$ be the Hardy space of bounded analytic functions in $\mathbb{D}$ [3]. Then, $H_\infty^{m\times n}(\mathbb{D})$ are $m\times n$ matrix-valued

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functions whose entries belong to $H_\infty(\mathbb{D})$. The H_∞ norm of $g(z)$ is defined by

$$||g||_\infty := \text{ess} \sup_{\theta\in[0,2\pi]} \bar{\sigma}(g(e^{i\theta})).$$

The strictly bounded real function is the function $g \in H_\infty^{m\times p}(\mathbb{D})$ such that

$$||g||_\infty < \gamma$$

for $0 < \gamma \leq 1$.

The Hardy space of bounded analytic functions in $\mathbb{C}_+$ is translated from $H_\infty(\mathbb{D})$ by the conformal map between $\mathbb{C}_+$ and $\mathbb{D}$

$$z = \frac{s-1}{s+1}.$$

II. Preliminary

A. A State-space Realization for Right Tangential Interpolant

The problem to be first considered here is the right tangential interpolation problem [3]. For complex numbers p_r, $r = 1,\dots,n_1$, an $m\times p$ matrix D, p-dimensional vectors u_r, $r = 1,\dots,n_1$ and m-dimensional vectors v_r, $r = 1,\dots,n_1$, we want to find an $m\times p$ rational function $Q(z)$, which satisfies the conditions below:

- Right tangential interpolation conditions:

$$Q(p_r)u_r = v_r, \quad r = 1,\dots,n_1$$

$$Q(\infty) = D$$

- $Q(z)$ is rational of McMillan degree n_1.
- $Q(z)$ is analytic in $\rho(A_1)$, where

$$A_1 := \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & p_{n_1} \end{bmatrix}. \quad (1)$$

Let us define

$$\begin{aligned} U &:= \begin{bmatrix} u_1 & u_2 & \dots & u_{n_1} \end{bmatrix} \\ V &:= \begin{bmatrix} v_1 & v_2 & \dots & v_{n_1} \end{bmatrix}. \end{aligned} \quad (2)$$

A solution to the right tangential interpolation problem is given in [8].

Theorem 2.1: The rational function $Q(z)$ is a solution to the right tangential interpolation problem if and only if it has the realization

$$Q(z) = \left[\begin{array}{c|c} A_1 + BU & B \\ \hline DU - V & D \end{array}\right] \quad (3)$$

for some matrix B. Moreover, the rational function $Q(z)$ of the realization (3) is also given by the right coprime factorization

$$Q(z) = G_r(z)F_r(z)^{-1}, \quad (4)$$

where

$$F_r(z) = \left[\begin{array}{c|c} A_1 & B \\ \hline -U & I \end{array}\right]$$

$$G_r(z) = \left[\begin{array}{c|c} A_1 & B \\ \hline -V & D \end{array}\right].$$

B. A State-space Realization for Left Tangential Interpolant

Similar to $Theorem$ 2.1, we derive a state-space realization of the rational function $Q(z)$ for the solution to the left tangential interpolation problem. For complex numbers z_ℓ, $\ell = 1,\dots,n_2$, an $m\times p$ matrix D, m-dimensional vectors x_ℓ, $\ell = 1,\dots,n_2$ and p-dimensional vectors y_ℓ, $\ell = 1,\dots,n_2$, we want to find an $m\times p$ rational function $Q(z)$, which satisfies the conditions below:

- Left tangential interpolation conditions:

$$x_\ell^* Q(z_\ell) = y_\ell^*, \quad \ell = 1,\dots,n_2$$

$$Q(\infty) = D$$

- $Q(z)$ is rational of McMillan degree n_2.
- $Q(z)$ is analytic in $\rho(A_2)$, where

$$A_2 := \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & z_{n_2} \end{bmatrix}. \quad (5)$$

Let us define

$$\begin{aligned} X &:= \begin{bmatrix} x_1 & x_2 & \dots & x_{n_2} \end{bmatrix} \\ Y &:= \begin{bmatrix} y_1 & y_2 & \dots & y_{n_2} \end{bmatrix}. \end{aligned} \quad (6)$$

The state-space realization of the left tangential interpolant is given.

Theorem 2.2: The rational function $Q(z)$ is a solution to the left tangential interpolation problem if and only if it has the realization

$$Q(z) = \left[\begin{array}{c|c} A_2 + X^*C & DX^* - Y^* \\ \hline C & D \end{array}\right] \quad (7)$$

for some matrix C. The rational function $Q(z)$ of the realization (7) is also given by the left coprime factorization

$$Q(z) = F_\ell(z)^{-1}G_\ell(z), \quad (8)$$

where

$$F_\ell(z) = \left[\begin{array}{c|c} A_2 & -X^* \\ \hline C & I \end{array}\right]$$

$$G_\ell(z) = \left[\begin{array}{c|c} A_2 & -Y^* \\ \hline C & D \end{array}\right].$$

Proof: See Appendix. ■

III. Bi-tangential Nevanlinna-Pick Interpolation by LMI

A. Problem Formulation:

The bi-tangential Nevanlinna-Pick interpolation problem of continuous-time or discrete-time systems is that we want to find a strictly bounded real function, which satisfies the conditions below:

- Right and left tangential interpolation conditions:

$$Q(p_r)u_r = v_r, \quad r = 1, \ldots, n_1$$

$$x_\ell^* Q(z_\ell) = y_\ell^*, \quad \ell = 1, \ldots, n_2 \qquad (9)$$

$$Q(\infty) = D,$$

  where the distinct points lie in $\mathbb{C}_+$ for continuous-time system and in $\bar{\mathbb{D}}^c$ for discrete-time system.
- Q is rational of McMillan degree exactly $n_1 + n_2$.
- Q is strictly bounded real in $\mathbb{C}_+$ for continuous-time system or in $\bar{\mathbb{D}}^c$ for discrete-time system.

A parameterization of the solutions to the bi-tangential Nevanlinna-Pick interpolation problem is derived in terms of the state-space realization in $Theorems$ 2.1. We mention that the similar approach by $Theorem$ 2.2 also holds.

B. Two-sided Interpolation by A One-sided Interpolation

We shall prove that the bi-tangential Nevanlinna-Pick interpolation problem is equivalently transformed to a tangential Nevanlinna-Pick interpolation problem. We assume that the $m \times p$ interpolant is tall, i.e., $m > p$.

Theorem 3.1: An $m \times p$ strictly bounded real function $Q(s)$ of $m > p$ satisfies

$$x_\ell^* Q(z_\ell) = y_\ell^*, \quad \ell = 1, \ldots, n_2$$

if $Q(s)$ satisfies

$$Q(-\bar{z}_\ell)y_\ell = x_\ell, \quad \ell = 1, \ldots, n_2.$$

Proof: See Appendix. ■

The similar result holds for the discrete-time systems.

Theorem 3.2: An $m \times p$ strictly bounded real function $Q(z)$ of $m > p$ satisfies

$$x_\ell^* Q(z_\ell) = y_\ell^*, \quad \ell = 1, \ldots, n_2$$

if $Q(z)$ satisfies

$$Q(-\frac{1}{\bar{z}_\ell})y_\ell = x_\ell, \quad \ell = 1, \ldots, n_2.$$

Proof: It is omitted since it is similar to the proof of $Theorem$ 3.1. ■

$Theorem$ 3.1 implies that the left tangential interpolation condition on the strictly bounded real function $Q(s)$ is transformed to the right tangential interpolation condition with the expense of the analyticity of $Q(s)$ at $\rho(-A_2^*)$. By using $Theorem$ 2.1, the state-space realization of a rational function $Q(s)$, which satisfies the right tangential interpolation conditions

$$\begin{cases} Q(p_r)u_r = v_r, & r = 1, \ldots, n_1 \\ Q(-\bar{z}_\ell)y_\ell = x_\ell, & \ell = 1, \ldots, n_2 \end{cases}$$

is given by

$$Q(s) = \left[\begin{array}{c|c} A + BL & B \\ \hline DL - W & D \end{array}\right], \qquad (10)$$

where

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & -A_2^* \end{bmatrix}, \quad L = \begin{bmatrix} U & X \end{bmatrix}, \quad W = \begin{bmatrix} V & Y \end{bmatrix}$$

since we can simply replace A_1, U and V by A, L and W in $Theorem$ 2.1. The rational function $Q(s)$ of (10) does not satisfy the left tangential interpolation conditions (9). However, it satisfies the left interpolation conditions (9) if the free parameter B is chosen such that $Q(s)$ is strictly bounded real by $Theorem$ 3.1.

The similar result holds for the discrete-time systems. By $Theorem$ 2.1, the state-space realization of a rational function $Q(z)$, which satisfies the right interpolation conditions

$$\begin{cases} Q(z_r)u_r = v_r, & r = 1, \ldots, n_1 \\ Q(-\frac{1}{\bar{z}_\ell})y_\ell = x_\ell, & \ell = 1, \ldots, n_2 \end{cases}$$

is also given by (10), where A matrix is replaced by

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2^{-*} \end{bmatrix}. \qquad (11)$$

Then, we obtain a solution to the interpolation problem if the free parameter is chosen such that $Q(z)$ is strictly bounded real by $Theorem$ 3.2.

C. A Convex Characterization of Strictly Bounded Realness for Continuous-time Systems

A characterization of the strictly bounded realness of the rational function $Q(s)$ of (10) is given in terms of an LMI.

Theorem 3.3: The rational function $Q(s)$ of the minimal state-space realization (10) is strictly bounded real if and only if there exist $P > 0$ and K such that

$$\begin{bmatrix} PA + A^*P - \gamma L^*L & K + \gamma L^* & -W^* \\ K^* + \gamma L & -\gamma I & D^* \\ -W & D & -\gamma I \end{bmatrix} < 0. \qquad (12)$$

For P and K of the solutions to the LMI,

$$B = P^{-1}K.$$

Proof: See Appendix. ■

We discuss the existence of the free parameter B of (10) such that $Q(s)$ is strictly bounded real in terms of (12), i.e., it is the solvability condition of the interpolation problem.

Theorem 3.4: There exists the free parameter B of (10) such that $Q(s)$ is strictly bounded real if and only if

$$\begin{bmatrix} -\gamma I & D^* \\ D & -\gamma I \end{bmatrix} < 0 \tag{13}$$

and there exists $P > 0$ such that

$$\begin{bmatrix} PA + A^*P - \gamma L^*L & -W^* \\ -W & -\gamma I \end{bmatrix} < 0. \tag{14}$$

Proof: See Appendix. ■

D. A Convex Characterization of Strictly Bounded Realness for Discrete-time Systems

For the discrete-time system with the state-space realization

$$Q(z) = \left[\begin{array}{c|c} A + BL & B \\ \hline DL - W & D \end{array}\right], \tag{15}$$

a characterization of the strictly bounded realness of the rational function $Q(z)$ of (15) is given in terms of an LMI.

Theorem 3.5: The rational function $Q(z)$ of the minimal state-space realization (15) is strictly bounded real if and only if there exist $P > 0$ and K such that

$$\begin{bmatrix} P + \gamma^2 L^*L - W^*W & -\gamma^2 L^* + W^*D & A^*P \\ -\gamma^2 L + D^*W & \gamma^2 I & K^* \\ PA & K & P \end{bmatrix} > 0. \tag{16}$$

For P and K of the solutions to the LMI,

$$B = P^{-1}K.$$

Proof: See Appendix. ■

We discuss the existence of the free parameter B of (15) such that $Q(z)$ is strictly bounded real in terms of (16), i.e., it is the solvability condition of the interpolation problem.

Theorem 3.6: There exists the free parameter B of (15) such that $Q(z)$ is strictly bounded real if and only if there exists $P > 0$ such that

$$\begin{bmatrix} P + \gamma^2 L^*L - W^*W & -\gamma^2 L^* + W^*D \\ -\gamma^2 L^* + D^*W & \gamma^2 I \end{bmatrix} > 0. \tag{17}$$

$$P - A^*PA + \gamma^2 L^*L - W^*W > 0. \tag{18}$$

Proof: See Appendix. ■

IV. A Convex Relaxation of Additional Interpolations for Discrete-time Systems

A parameterization of the interpolants $Q(z)$ of McMillan degree $n_1 + n_2$ is given in terms of the LMI by $Theorem$ 3.5. Similar to (4) in $Theorem$ 2.1, the rational function $Q(z)$ of (15) is also given by the right coprime factorization

$$Q(z) = G_r(z)F_r(z)^{-1}, \tag{19}$$

where

$$F_r(z) = \left[\begin{array}{c|c} A & B \\ \hline -L & I \end{array}\right]$$
$$G_r(z) = \left[\begin{array}{c|c} A & B \\ \hline -W & D \end{array}\right].$$

We also give a condition on the free parameter B such that $Q(z)$ satisfies the additional interpolation conditions

$$r_k^* Q(q_k) = w_k^*, \quad k = 1, \ldots, n_3 \tag{20}$$

with keeping the strictly bounded realness. We assume that n_3 is strictly less than $n_1 + n_2$. In terms of (4), (20) implies

$$r_k^* G_r(q_k) F_r(q_k)^{-1} = w_k^*$$
$$\iff r_k^* G_r(q_k) = w_k^* F_r(q_k), \quad k = 1, \ldots n_3$$

if $F(q_k)$, $k = 1, \ldots n_3$ is invertible. Thus, we obtain the linear equation

$$\Lambda = \Xi B,$$

where

$$\Lambda := \begin{bmatrix} w_1^* - r_1^* D \\ \vdots \\ w_{n_3}^* - r_{n_3}^* D \end{bmatrix}$$
$$\Xi := \begin{bmatrix} (w_1^* L - r_1^* W)(q_1 I - A)^{-1} \\ \vdots \\ (w_{n_3}^* L - r_{n_3}^* W)(q_{n_3} I - A)^{-1} \end{bmatrix}.$$

Note that $n_3 \times (n_1 + n_2)$ matrix Ξ is flat. Under the assumption that Ξ has the full row rank, the general solution for B is given by

$$B = B_0 + \Xi_\perp \alpha, \tag{21}$$

where

$$B_0 := \Xi^*(\Xi\Xi^*)^{-1}\Lambda.$$

$\Xi_\perp$ is an $(n_1+n_2)\times(n_1+n_2-n_3)$ matrix with full column rank such that

$$\Xi\Xi_\perp = 0,$$

and α is an $(n_1 + n_2 - n_3) \times p$ matrix. By substituting (21) into (15), we obtain

$$Q(z) = \left[\begin{array}{c|c} A + (B_0 + \Xi_\perp\alpha)L & B_0 + \Xi_\perp\alpha \\ \hline DL - W & D \end{array}\right]. \tag{22}$$

The rational function $Q(z)$ satisfies all interpolation conditions if the free parameter α is chosen so that $Q(z)$ is strictly bounded real by $Theorem$ 3.1. The existence of such α is given by a nonlinear matrix inequality.

Theorem 4.1: There exists α such that (22) is strictly bounded real if and only if there exists $P > 0$ such that

$$P - A^*PA + \gamma^2 L^*L - W^*W > 0 \tag{23}$$

$$\begin{bmatrix} P + \gamma^2 L^*L - W^*W & -\gamma^2 L^* + W^*D \\ -\gamma^2 L + D^*W & \gamma^2 I \\ A & B_0 \end{bmatrix}$$

$$\cdots \quad \begin{bmatrix} A^* \\ B_0^* \\ \Xi P^{-1}\Xi^* \end{bmatrix} > 0. \tag{24}$$

Proof: See Appendix. ■

For a given $P > 0$ of the solution to (24), the free parameter α, which gives the strictly bounded realness, is characterized by a linear matrix inequality. $Theorem$ 4.1 gives the necessary and sufficient condition for the solvability of the bi-tangential Nevanlinna-Pick interpolation problem with the additional interpolations. The matrix inequality (24) is not linear, i.e., (24) gives a nonconvex set of P. We discuss the convex relaxation of (24), i.e., we want to find a convex subset of the nonconvex set of feasible P. The similar relaxation scheme appears for the scalar interpolation problem [6].

Theorem 4.2: The bi-tangential Nevanlinna-Pick interpolation with the additional interpolations is solvable if there exists $P > 0$ such that (23) and

$$\begin{bmatrix} P + \gamma^2 L^*L - W^*W & -\gamma^2 L^* + W^*D \\ -\gamma^2 L + D^*W & \gamma^2 I \end{bmatrix} - \begin{bmatrix} A^*\Xi^\dagger\Xi \\ B_0^*\Xi^\dagger\Xi \end{bmatrix} P \begin{bmatrix} \Xi^\dagger\Xi A & \Xi^\dagger\Xi B_0 \end{bmatrix} > 0. \tag{25}$$

Proof: See Appendix. ■

V. Discussion

We discussed the interpolation problem with the additional interpolation conditions for discrete-time systems in $Theorems$ 4.1 and 4.2. It turns out that the similar scheme of the convex relaxation for the continuous time analogue does not hold. It seems that this is due to the interpolation condition at ∞, i.e, $Q(\infty) = D$ since the interpolation condition is the boundary interpolation in the continuous time setting [14].

Some advantages of the LMI approach in this paper are that our results are easily extended for the bounded real case, i.e., the norm constraint on the interpolant f is $||f||_\infty \le \gamma$ in stead of $||f||_\infty < \gamma$, by replacing the positive definite constraints of the LMIs in $Theorems$ by the positive semidefinite constraints. We mention that the similar LMI approach works for the positive real interpolation problem. Our approach may be applied to the interpolation problem such that interpolant has poles on the boundary since it can handle the case that the matrices P in $Theorems$ are positive semidefinite, which might happen if the interpolant has poles on the boundary.

Appendix

A. Proof of Theorem 2.2

We use the lemma below.

Lemma 1.1: Let $Q(z)$ be a rational function, which is analytic in $\rho(A_2)$ and admits a minimal realization

$$Q(z) = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D \end{array}\right]. \tag{26}$$

Then $(zI - A_2)^{-1}(X^*Q(z) - Y^*)$ is analytic in $\rho(A_2)$ if and only if there exists a matrix S such that

$$X^*D - Y^* = SB_Q \tag{27}$$

and

$$X^*C_Q = SA_Q - A_2S \tag{28}$$

Proof: Suppose that $(zI - A_2)^{-1}(X^*Q(z) - Y^*)$ is analytic in $\rho(A_2)$. Then, it only has poles at the eigenvlaues of A_Q, and it vanishes at ∞. The reachability of (A_Q, B_Q) implies the existence of the matrix S in

$$\begin{aligned} &(zI - A_2)^{-1}(X^*Q(z) - Y^*) = S(zI - A_Q)^{-1}B_Q. \\ \iff &X^*Q(z) - Y^* \\ &\quad = (zI - A_2)S(zI - A_Q)^{-1}B_Q, \end{aligned}$$

which implies

$$\begin{aligned} X^*Q(z) - Y^* &= X^*D - Y^* + X^*C_Q(zI - A_Q)^{-1}B_Q \\ &= (zI - A_2)S(zI - A_Q)^{-1}B_Q. \end{aligned} \tag{29}$$

By evaluating the identity at ∞, we obtain

$$X^*D - Y^* = SB_Q. \tag{30}$$

The identity (29) is written as

$$\begin{aligned} &X^*D - Y^* + X^*C_Q(zI - A_Q)^{-1}B_Q \\ &= (zI - A_2)S(zI - A_Q)^{-1}B_Q. \end{aligned} \tag{31}$$

By substituting (30) into (31), we obtain

$$\begin{aligned} &SB_Q + X^*C_Q(zI - A_Q)^{-1}B_Q \\ &= (zI - A_2)S(zI - A_Q)^{-1}B_Q \\ &\iff X^*C_Q(zI - A_Q)^{-1}B_Q \\ &= -SB_Q + (zI - A_2)S(zI - A_Q)^{-1}B_Q \\ &= [-S(zI - A_Q) + (zI - A_2)S](zI - A_Q)^{-1}B_Q \\ &= [SA_Q - A_2S](zI - A_Q)^{-1}B_Q. \end{aligned}$$

The controlability of (A_Q, B_Q) implies

$$X^*C_Q = SA_Q - A_2S.$$

Conversely, if a realization of $Q(z)$ satisfies (27) and (28), by backtracking the above argument, we see that $(zI - A_2)^{-1}(X^*Q(z) - Y^*) = S(zI - A_Q)^{-1}B_Q$ is analytic in $\rho(A_2)$. ■

Assume that $Q(z)$ is a minimal interpolant of McMillan degree n_2. Then, any realization

$$Q(z) = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right] \tag{32}$$

satisfies (27) and (28) by $Lemma$ 1.1, which are written by

$$X^*D - Y^* = SB_Q$$

and

$$SA_QS^{-1} = A_1 + X^*C_QS^{-1}.$$

Thus,

$$\begin{aligned} Q(z) &= \left[\begin{array}{c|c} SA_QS^{-1} & SB_Q \\ \hline C_QS^{-1} & D \end{array}\right] \\ &= \left[\begin{array}{c|c} A_2 + X^*CS^{-1} & X^*D - Y^* \\ \hline C_QS^{-1} & D \end{array}\right]. \end{aligned}$$

By setting $C := C_QS^{-1}$, we obtain (7).

Conversely, if $Q(z)$ has the minimal realization (7), then, it satisfies (27) and (28) by setting $Y = I$ in $Lemma$ 1.1. Moreover, we can verify that

$$\begin{aligned} F_\ell(z)^{-1}G_\ell(z) &= \left[\begin{array}{c|c} A_2 + X^*C & X^* \\ \hline C & I \end{array}\right]\left[\begin{array}{c|c} A_2 & -Y^* \\ \hline C & D \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_2 + X^*C & 0 & X^*D - Y^* \\ 0 & A_2 & -Y^* \\ \hline C & C & D \end{array}\right] \\ &= \left[\begin{array}{c|c} A_2 + X^*C & X^*D - Y^* \\ \hline C & D \end{array}\right], \end{aligned}$$

where we changed the coordinates of the state by multiplying

$$T = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$$

from the left and T^{-1} from the right.

B. Proof of Theorem 3.1

Denote the state-space realization of $Q(s)$ by

$$Q(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then, there exists matrices B_e and D_e such that

$$Q_e(s) = \left[\begin{array}{c|cc} A & B & B_e \\ \hline C & D & D_e \end{array}\right]$$

is square and inner [13]. Clearly,

$$Q_e(-\bar{z}_\ell)\begin{bmatrix} y_\ell \\ 0 \end{bmatrix} = x_\ell, \quad \ell = 1, \ldots, n_2,$$

holds. Moreover,

$$\begin{aligned} Q_e(-\bar{z}_\ell)\begin{bmatrix} y_\ell \\ 0 \end{bmatrix} = x_\ell &\iff \begin{bmatrix} y_\ell^* & 0 \end{bmatrix} Q_e(-\bar{z}_\ell)^* = x_\ell^* \\ &\iff \begin{bmatrix} y_\ell^* & 0 \end{bmatrix} = x_\ell^* Q_e(z_\ell) \\ &\implies y_\ell^* = x_\ell^* Q(z_\ell) \end{aligned}$$

since

$$\begin{aligned} Q_e(-\bar{z}_\ell)^* &= \begin{bmatrix} B^* \\ B_e^* \end{bmatrix}(-z_\ell I - A^*)^{-1}C^* + \begin{bmatrix} D^* \\ D_e^* \end{bmatrix} \\ &= Q_e(z_\ell)^\sim \end{aligned}$$

and

$$Q_e(z_\ell)^\sim Q_e(z_\ell) = I.$$

C. Proof of Theorem 3.3

By the KYP lemma [5], $Q(s)$ of (10) is strictly bounded real if and only if there exists $P > 0$ such that

$$\begin{bmatrix} P(A+BL) + (A+BL)^*P & PB & (DL-W)^* \\ B^*P & -\gamma I & D^* \\ DL - W & D & -\gamma I \end{bmatrix} < 0.$$

Moreover,

$$\begin{aligned} &T_1^* \begin{bmatrix} P(A+BL) + (A+BL)^*P & PB & (DL-W)^* \\ B^*P & -\gamma I & D^* \\ DL - W & D & -\gamma I \end{bmatrix} T_1 = \\ &\begin{bmatrix} PA + A^*P - \gamma L^*L & PB + \gamma L^* & -W^* \\ B^*P + \gamma L & -\gamma I & D^* \\ -W & D & -\gamma I \end{bmatrix} < 0, \end{aligned} \tag{33}$$

where

$$T_1 = \begin{bmatrix} I & 0 & 0 \\ -L & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

By chaning the variable

$$K = PB$$

of (33), we obtain the LMI (12).

D. Proof of Theorem 3.4

The LMI (33) is written by

$$\begin{aligned} &\begin{bmatrix} PA + A^*P - \gamma L^*L & \gamma L^* & -W^* \\ \gamma L & -\gamma I & D^* \\ -W & D & -\gamma I \end{bmatrix} \\ &+ \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} B \begin{bmatrix} 0 & I & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} B^* \begin{bmatrix} P & 0 & 0 \end{bmatrix} \\ &< 0. \end{aligned}$$

Denote by

$$M_s = \begin{bmatrix} PA + A^*P - \gamma L^*L & \gamma L^* & -W^* \\ \gamma L & -\gamma I & D^* \\ -W & D & -\gamma I \end{bmatrix}.$$

By the elimination lemma [5], the LMI (12) has a solution B if and only if

$$T_2^* M_s T_2 < 0$$
$$T_3^* M_s T_3 < 0,$$

where

$$T_2 = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}, \quad T_3 = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}.$$

Thus, we can verify that $T_2^* M_s T_2 < 0$ is (13) and $T_3^* M_s T_3 < 0$ is (14).

E. Proof of Theorem 3.5

By the KYP lemma [5], $Q(z)$ is strictly bounded real if and only if there exists $P > 0$ such that

$$\begin{bmatrix} A+BL & B \\ DL-W & D \end{bmatrix}^* \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A+BL & B \\ DL-W & D \end{bmatrix} < \begin{bmatrix} P & 0 \\ 0 & \gamma^2 I \end{bmatrix}.$$

Denote by

$$T_4 = \begin{bmatrix} I & 0 \\ -L & I \end{bmatrix}.$$

Then,

$$\begin{aligned}
& T_4^* \begin{bmatrix} P & 0 \\ 0 & \gamma^2 I \end{bmatrix} T_4 > \\
& T_4^* \begin{bmatrix} A+BL & B \\ DL-W & D \end{bmatrix}^* \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \\
& \qquad \times \begin{bmatrix} A+BL & B \\ DL-W & D \end{bmatrix} T_4 \\
& \Longleftrightarrow \begin{bmatrix} P+\gamma^2 L^*L & -\gamma^2 L^* \\ -\gamma^2 L & \gamma^2 I \end{bmatrix} > \\
& \begin{bmatrix} A & B \\ -W & D \end{bmatrix}^* \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ -W & D \end{bmatrix} \\
& \Longleftrightarrow \begin{bmatrix} P+\gamma^2 L^*L & -\gamma^2 L^* \\ -\gamma^2 L & \gamma^2 I \end{bmatrix} \\
& - \begin{bmatrix} A & B \\ -W & D \end{bmatrix}^* \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ -W & D \end{bmatrix} > 0 \\
& \Longleftrightarrow \begin{bmatrix} P+\gamma^2 L^*L & -\gamma^2 L^* \\ -\gamma^2 L & \gamma^2 I \end{bmatrix} \\
& \qquad - \begin{bmatrix} -W^* \\ D^* \end{bmatrix} \begin{bmatrix} -W & D \end{bmatrix} \\
& \qquad - \begin{bmatrix} A^* \\ B^* \end{bmatrix} P \begin{bmatrix} A & B \end{bmatrix} > 0
\end{aligned}$$

$$\begin{aligned}
& \Longleftrightarrow \begin{bmatrix} P+\gamma^2 L^*L - W^*W & -\gamma^2 L^* + W^*D \\ -\gamma^2 L + D^*W & \gamma^2 I \end{bmatrix} \\
& - \begin{bmatrix} A^* \\ B^* \end{bmatrix} P \begin{bmatrix} A & B \end{bmatrix} \\
& \Longleftrightarrow \begin{bmatrix} P+\gamma^2 L^*L - W^*W & -\gamma^2 L^* + W^*D & A^* \\ -\gamma^2 L + D^*W & \gamma^2 I & B^* \\ A & B & P^{-1} \end{bmatrix} > 0, \qquad (34)
\end{aligned}$$

where we used the Schur complement at the last step. Moreover,

$$\begin{aligned}
& T_5^* \begin{bmatrix} P+\gamma^2 L^*L - W^*W & -\gamma^2 L^* + W^*D & A^* \\ -\gamma^2 L + D^*W & \gamma^2 I & B^* \\ A & B & P^{-1} \end{bmatrix} T_5 \\
& = \begin{bmatrix} P+\gamma^2 L^*L - W^*W & -\gamma^2 L^* + W^*D & A^*P \\ -\gamma^2 L + D^*W & \gamma^2 I & B^*P \\ PA & PB & P \end{bmatrix}, \qquad (35)
\end{aligned}$$

where

$$T_5 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & P \end{bmatrix}.$$

By chaning the variable

$$K = PB$$

of (35), we obtain the LMI (16).

F. Proof of Theorem 3.6

The LMI (34) is written by

$$\begin{bmatrix} P+\gamma^2 L^*L - W^*W & -\gamma^2 L^* + W^*D & A^* \\ -\gamma^2 L + D^*W & \gamma^2 I & 0 \\ A & 0 & P^{-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} B \begin{bmatrix} 0 & I & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} B^* \begin{bmatrix} 0 & 0 & I \end{bmatrix} > 0.$$

Denote by

$$M_d = \begin{bmatrix} P+\gamma^2 L^*L - W^*W & -\gamma^2 L^* + W^*D & A^* \\ -\gamma^2 L + D^*W & \gamma^2 I & 0 \\ A & 0 & P^{-1} \end{bmatrix}.$$

By the elimination lemma [5], the LMI (16) has a solution B if and only if

$$\begin{aligned} T_6^* M_d T_6 > 0 \\ T_7^* M_d T_7 > 0, \end{aligned}$$

where

$$T_6 = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, \quad T_7 = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}.$$

Thus, we can verify that $T_6^* M_d T_6 > 0$ is (17) and $T_7^* M_d T_7 > 0$ is (18) by taking the Schur complement.

G. Proof of Theorem 4.1

Following the proof of $Theorem$ 3.5, the rational function $Q(z)$ of (22) is strictly bounded real if and only if there exist $P > 0$ and α such that

$$\begin{bmatrix} P+\gamma^2 L^* L - W^* W & -\gamma^2 L^* + W^* D & A^* \\ -\gamma^2 L + D^* W & \gamma^2 I & (B_0+\Xi_\perp \alpha)^* \\ A & B_0 + \Xi_\perp \alpha & P^{-1} \end{bmatrix} > 0.$$

Moreover, it can be written by

$$\begin{bmatrix} P+\gamma^2 L^* L - W^* W & -\gamma^2 L^* + W^* D & A^* \\ -\gamma^2 L + D^* W & \gamma^2 I & B_0^* \\ A & B_0 & P^{-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \Xi_\perp \end{bmatrix} \alpha \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}^* + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \alpha \begin{bmatrix} 0 \\ 0 \\ \Xi_\perp \end{bmatrix}^* > 0. \quad (36)$$

Denote by

$$M_a = \begin{bmatrix} P+\gamma^2 L^* L - W^* W & -\gamma^2 L^* + W^* D & A^* \\ -\gamma^2 L + D^* W & \gamma^2 I & B_0^* \\ A & B_0 & P^{-1} \end{bmatrix}.$$

By the elimination lemma [5], there exists α for the LMI (36) if and only if there exists $P > 0$ such that

$$T_8^* M_a T_8 = \begin{bmatrix} P+\gamma^2 L^* L - W^* W & A^* \\ A & P^{-1} \end{bmatrix} > 0$$
$$\Longleftrightarrow P+\gamma^2 L^* L - W^* W - A^* P A > 0$$

$$T_9^* M_a T_9 = \begin{bmatrix} P+\gamma^2 L^* L - W^* W & -\gamma^2 L^* + W^* D & A^* \\ -\gamma^2 L + D^* W & \gamma^2 I & B_0^* \\ A & B_0 & \Xi P^{-1} \Xi^* \end{bmatrix} > 0,$$

where

$$T_8 = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}, \quad T_9 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \Xi \end{bmatrix}.$$

Thus, we obtain (23) and (24).

H. Proof of Theorem 4.2

The feasibility of

$$\begin{bmatrix} P+\gamma^2 L^* L - W^* W & -\gamma^2 L^* + W^* D & A^* & 0 \\ -\gamma^2 L + D^* W & \gamma^2 I & B_0^* & 0 \\ A & B_0 & \Xi P^{-1} \Xi^* & \Xi P^{-1} \Xi_\perp \\ 0 & 0 & \Xi_\perp^* P^{-1} \Xi^* & \Xi_\perp^* P^{-1} \Xi_\perp \end{bmatrix} > 0$$

implies the feasibility of (24). Note that

$$\begin{bmatrix} \Xi \\ \Xi_\perp^* \end{bmatrix}^{-1} = \begin{bmatrix} \Xi^\dagger & \Xi_\perp^{\dagger *} \end{bmatrix}. \quad (37)$$

By taking the Schur complement of (24), we obtain

$$\begin{bmatrix} P+\gamma^2 L^* L - W^* W & -\gamma^2 L^* + W^* D \\ -\gamma^2 L + D^* W & \gamma^2 I \end{bmatrix} - \begin{bmatrix} A^* & 0 \\ B_0^* & 0 \end{bmatrix} \begin{bmatrix} \Xi^* & \Xi_\perp \end{bmatrix}^{-1} \times P \begin{bmatrix} \Xi \\ \Xi_\perp^* \end{bmatrix}^{-1} \begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix} > 0,$$

which is equivalent to (25) by (37).

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