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Convergent Rational Interpolation to Cauchy Integrals on an arc

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***Abstract*— We design convergent rational interpolation schemes to functions defined as Cauchy integrals of complex densities over open analytic arcs, under mild smoothness assumptions on the density. The interpolation points must be chosen according to the geometry of the arc, and the convergence is locally uniform outside of the arc. The result essentially settles the convergence issue of multipoint Padé approximants to functions with connected singular set of dimension 1.**

I. INTRODUCTION

Multipoint Padé approximants are the oldest and simplest candidate rational-approximants to a holomorphic function of one complex variable. They were introduced as early as [23], and they are simply those rational functions of type (m, n) that interpolate the function in $m+n+1$ points of the domain of analyticity, counting multiplicity. Here, a rational function is said to be of type (m, n) if it can be written as the ratio of a polynomial of degree at most m and a polynomial of degree at most n. *Classical* Padé approximants refer to the case where interpolation takes place in a single point with multiplicity $m+n+1$.

Besides their importance in number theory [23], [32], [33] and the natural appeal of their linear and elementary character, Padé approximants are of common use in modeling and numerical analysis of various phenomena, from boundary value problems and convergence acceleration, to system and circuits theory, mechanics, quantum mechanics, fluid mechanics, condensed matter physics and even page ranking the *Web*. [5], [39], [14], [22], [20], [17], [12], [3], [38], [31], [29], [15]. From a system-theoretic viewpoint, the computation of classical Padé approximants at infinity is tantamount to solving the so-called partial realization problem [11] from the moments of a transfer function, while multipoint Padé approximants arise naturally in frequency identification [30]. Still their convergence is far from being understood. In this connection, the main interest lies with *diagonal* approximants, *i.e.* interpolants of type (m, m), as they treat poles and zeros on equal footing. Below, we only consider the diagonal case.

Although classical Padé approximants have been proven to converge locally uniformly on the domain of analyticity for certain classes of functions, [27], [40], [4], [25], such a nice property cannot hold in general due to the occurence of "spurious poles" that may wander about the domain of analyticity. We refer the reader to the monograph [6] for a detailed account of the subject. Altogether, the recent disproof of the Padé conjecture [26] and of the Stahl conjecture [13] on the existence of a convergent subsequence have added to the picture that classical Padé approximants are not seen best through the spectacles of uniform convergence. The case of multipoint Padé approximants is somewhat different, since choosing the interpolation points provides one with extra parameters to help the convergence. The multipoint generalization [19] of the classical Markov theorem [27], dealing with the convergence of Padé approximants to Cauchy integrals of a positive measure on a segment (the so-called Markov functions), makes full use of a connection with orthogonal polynomials: the denominator of the n-th diagonal multipoint Padé approximant to a function described as a Cauchy integral is the n-th orthogonal polynomial with respect to the density of the integral, weighted by the inverse of the polynomial whose zeros are the interpolation points (this polynomial is identically 1 for classical Padé approximants). When the interpolation points are conjugate-symmetric, the weight becomes positive and the theory of orthogonal polynomials with varying weight can be used to produce asymptotics from which the convergence follows easily [37].

This inspiring example has attracted much attention towards approximating functions described by Cauchy integrals over arcs, which roughly correspond to those analytic functions having a singular set of dimension 1. However, when dealing with Cauchy integrals of signed or even complex densities over more general arcs than a segment, one has to describe the asymptotic behaviour of non-Hermitian orthogonal polynomials for which the L^2 theory is no longer available, and for a while it was unclear what could be expected.

In a series of pathbreaking papers [34], [35], [36],

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devoted to the convergence *in capacity* of classical Padé approximants to functions with branchpoints, the notion of minimal system of arcs for the logarithmic potential was put to the fore as being contours over which non-Hermitian orthogonal polynomials could be analyzed. This occurence of logarithmic potential theory was decisive and shortly after, in [21], the convergence in capacity of multipoint Padé approximants to Cauchy integrals of continuous non-vanishing densities over arcs of minimal *weighted logarithmic capacity* was established, provided that the interpolation points asymptotically distribute like a measure whose potential is (minus) the logarithm of the weight. Such an extremal system of arcs is called nowadays an S*-contour* and is characterized by a symmetry property of the (two-sided) normal derivatives of its equilibrium potential. This peculiar symmetry for the distribution of the interpolation points may be viewed as a far-reaching generalization of the conjugate-symmetry with respect to the real line that was found necessary to interpolate Markov functions in a convergent way.

For the reader unfamiliar with potential theory, we hasten to say that no knowledge of the latter is necessary to read the present paper nor to establish the theorems stated in it. However, potential theory is often lurking in the intuition of the mechanisms governing rational interpolation.

In recent years [2], asymptotics of non-Hermitian orthogonal polynomials with respect to analytic nonvanishing densities on S-contours with close-to-power weights were established by generalising the matrix Riemann-Hilbert approach to asymptotics of classical orthogonal orthogonal polynomials [16], [24]. From the sharp asymptotics provided by this method, locally uniform convergence of Padé approximants to Cauchy integrals of analytic nonvanishing densities on S-contours follows easily under quantitative assumptions on the interpolation scheme. However, the question as to which arcs are S contour for *some* weight is a non-trivial geometric inverse problem that remained untouched. In addition, it was unclear if and how the quantitative assumptions on the interpolation scheme required in [2] could be met. Also, analyticity of the density was required in order to apply the saddle point technique involved with the Riemann-Hilbert approach. For these reasons, the class of functions to which [2] can be applied remains somewhat *ad hoc*.

New ground was recently broken in [9] where it is shown that a rectifiable Jordan arc F which is Ahlfors regular at its endpoints is an S-contour if and only if it is analytic. Hence the S-property essentially reduces to strong smoothness (namely analyticity) for Jordan arcs. The proof recasts the S-property for Jordan arcs as the existence of a sequence of “pseudo-rational” functions (*cf.* (1)), holomorphic and tending to zero off the arc, whose boundary values from each side of the latter remain bounded, and whose zeros remain at positive distance from the arc. There are in fact many such sequences that can be computed explicitly from an analytic parameterization of the arc. Subsequently, translating the non-Hermitian orthogonality equation for the denominator into an integral equation involving Hankel operators and using compactness properties of the latter, the reference just quoted establishes that multipoint Padé approximants to Cauchy transforms of Dini-continuous (essentially) non-vanishing densities with respect to the equilibrium distribution of the arc converge locally uniformly in its complement when the interpolation points are the zeros of these pseudo-rational functions. Convergence in L^p-norm also holds on the arc.

The hypotheses of the preceding result entail that the density with respect to arclength in the integral goes to infinity towards the endpoints of the arc, since so does the equilibrium distribution. In particular, the theorem fails to capture ultra-smooth situations like those of Cauchy integrals of smooth functions over analytic arcs, which is of course unsatisfactory. Below, we report on a more recent result that handles *any* non-vanishing integrable Jacobi-type density under the integral, upon making only slightly stronger smoothness assumptions. This settles more or less the issue of convergence in multipoint Padé interpolation to functions defined as Cauchy integrals over Jordan arcs, because it can further be proved that the Cauchy integral of an analytic density over a Jordan arc admits a convergent diagonal interpolation scheme (meaning: uniformly convergent in the complement of the arc) whose interpolation points remain at positive distance from the arc only if the arc is analytic.

II. PADÉ APPROXIMANTS

Let f be analytic on a domain $\mathrm{Dom}f \subset \mathbb{C}$, and $\mathcal{E} = \{E_n\}$ be a triangular scheme of interpolation points, meaning that for each n the set $E_n = \{\xi_{n,1}, \cdots, \xi_{n,2n}\}$ consists of $2n$ points in $\mathrm{Dom}f$. The scheme is said to be nested, or a Newton scheme if $E_n \subset E_{n+1}$. Composing with a Möbius transform if necessary, it is customary to place at least one interpolation point at infinity. Then, if v_n is the monic polynomial with zeros at the finite points of E_n, we make the following definition.

Definition 2.1 (Padé Approximant): The n-th diagonal Padé approximant to f associated with $\mathcal{E}$ is the unique rational function $\Pi_n = p_n/q_n$ satisfying:

- $\deg p_n \le n$, $\deg q_n \le n$, and $q_n \not\equiv 0$;
- $(q_n(z)f(z) - p_n(z))/v_n(z)$ is analytic in $\mathrm{Dom} f$;
- $(q_n(z)f(z) - p_n(z))/v_n(z) = O\left(1/z^{n+1}\right)$ as $z \to \infty$.

A Padé approximant always exists since the conditions for p_n and q_n amount to solving a system of $2n+1$ homogeneous linear equations with $2n+2$ unknown coefficients, no solution of which can be such that $q_n \equiv 0$ (we may thus assume that q_n is monic); the fact that p_n/q_n is uniquely defined even if p_n and q_n may not is a classical fact which is easy to check [6]. Note that the required interpolation at infinity is entailed by the last condition and therefore Π_n is, in fact, of type $(n-1, n)$.

III. PSEUDO RATIONAL FUNCTIONS AND ANALYTIC JORDAN ARCS

A Jordan arc $F \subset \mathbb{C}$ is said to be analytic if there is a univalent function p, holomorphic in some neighborhood of $[-1,1]$ such that $F = p([-1,1])$. We normalize F throughout so that its endpoints are ± 1 and we orient it arbitrarily. We put for $z \in D := \overline{\mathbb{C}} \setminus F$:

$$\varphi(z) := z + w(z), \quad \text{where } w(z) = \sqrt{z^2 - 1}$$

where the branch of the square root is chosen so that

$$w(z)/z \to 1 \quad \text{as} \quad z \to \infty.$$

Note that φ has continuous extensions on F from each side, denoted by $\varphi^{\pm}(\xi)$, $\xi \in F$, where $\pm$ is used according whether the limit of $\varphi(z)$ is computed for z tending to ξ from the positive or the negative side of F. The next theorem may be viewed as a new, and rather unusual characterization of analytic Jordan arcs as limits of level sets of sequences of particular analytic functions that we call *pseudo-rational*. This characterization is crucial for the convergence of rational interpolants stated in the next section, as the interpolation points will be taken to be the zeros of these pseudo-rational functions.

Theorem 3.1: Let F be an analytic Jordan arc. Then there exists a triangular scheme $\mathcal{E} = \{E_n\}$ in $D := \overline{\mathbb{C}} \setminus F$ such that the *pseudo-rational* functions

$$r_n(z) := \prod_{\xi \in E_n} \frac{\varphi(z) - \varphi(\xi)}{1 - \varphi(\xi)\varphi(z)} \qquad (1)$$

converge to zero locally uniformly in D when $n \to +\infty$ and have uniformly bounded one-sided limits $r_n^{\pm}$ on F. Moreover, the scheme $\mathcal{E}$ can be chosen to be nested.

There are in fact many such schemes $\mathcal{E}$ that can be computed explicitly from the knowledge of a parameterization p of F. The above theorem is part of [9, Thm. 1] except for the fact that $\mathcal{E}$ can be chosen to be nested. The proof below sketches the necessary modifications to obtain this sharper conclusion. *Proof:* Put $J(z) := (z + 1/z)/2$ for the familiar Joukovski transformation, which is inverse to φ in $D = \overline{\mathbb{C}} \setminus F$. Note that

$$\Gamma := \Gamma^+ \cup \Gamma^-, \quad \Gamma^{\pm} := \varphi^{\pm}(F).$$

Since J maps z and $1/z$ into the same point and $\varphi^{\pm}(z) = \pm 1$ if and only if $z = \pm 1$, it follows that $\Gamma^+ \cap \Gamma^- = \{-1, 1\}$ and therefore Γ is a Jordan curve. Let us show that it is an analytic Jordan curve, namely that there exists a holomorphic univalent function in some neighborhood of the unit circle $\mathbb{T}$ that maps $\mathbb{T}$ onto Γ. For this, let U be a neighborhood of $\mathbb{T}$ such that $J(U)$ lies in the domain of p, where p is a holomorphic univalent parametrization of F, $F = p([-1,1])$. Such a neighborhood exists because J conformally maps the unit disk $\mathbb{D}$ as well as its complement $\overline{\mathbb{C}} \setminus \mathbb{D}$ onto $\overline{\mathbb{C}} \setminus [-1,1]$, with $J(\pm 1) = \pm 1$, and $J(\zeta)$ covers $[-1,1]$ twice as ζ ranges over $\mathbb{T}$. Define

$$\Phi(z) := \begin{cases} 1/\varphi(p(J(z))), & z \in U^+ := U \cap \mathbb{D}, \\ \varphi(p(J(z))), & z \in U^- := U \setminus \overline{\mathbb{D}}. \end{cases}$$

Then Φ is holomorphic on $U \setminus \mathbb{T}$. Denote by $\Phi^{\pm}$ the traces of Φ from $U^{\pm}$ on $\mathbb{T}$; it is easily checked that these traces exist continuously. Then for $\tau \in \mathbb{T}$, we have

$$\begin{aligned} \Phi^-(\tau) &= \begin{cases} 1/\varphi^+(p(J(\tau))), & \mathrm{Im}(\tau) \ge 0 \\ 1/\varphi^-(p(J(\tau))), & \mathrm{Im}(\tau) < 0 \end{cases} \\ &= \begin{cases} \varphi^-(p(J(\tau))), & \mathrm{Im}(\tau) \ge 0 \\ \varphi^+(p(J(\tau))), & \mathrm{Im}(\tau) < 0 \end{cases} = \Phi^+(\tau). \end{aligned}$$

Thus the traces $\Phi^{\pm}$ from each side do agree, therefore Φ is a holomorphic injective function throughout U that maps $\mathbb{T}$ into Γ, as announced.

Put $\Omega = \Phi(U)$ and $\Psi := \Phi^{-1} : \Omega \to U$. Observe that

$$\Psi(1/z) = 1/\Psi(z), \quad z, 1/z \in \Omega. \qquad (2)$$

We adopt the following notation for circles of center z_0 and radius x:

$$C_x(z_0) := \{z \in \mathbb{C} : \; |z - z_0| = x\}, \quad x > 0,$$

and we let $\rho \in (0,1)$ be such that

$$C_\rho(0), C_{1/\rho}(0) \subset \Psi(\Omega).$$

Set $\Gamma_\rho := \Phi(C_\rho(0))$ and $\Gamma_{1/\rho} := \Phi(C_{1/\rho}(0))$. It immediately follows from (2) that

$$\Gamma_{1/\rho} = \Gamma_\rho^{-1} = \{\tau \in \mathbb{C} : \; 1/\tau \in \Gamma_\rho\}. \qquad (3)$$

Denote by Ω_ρ the annular domain bounded by Γ_ρ and $\Gamma_{1/\rho}$ and define

$$u(z) := \log|\Psi(z)|, \quad z \in \Omega.$$

Then u is a harmonic function in some neighborhood of $\overline{\Omega}_\rho$ such that $u \equiv 0$ on Γ, $u \equiv \log\rho$ on Γ_ρ, and $u \equiv -\log\rho$ on $\Gamma_{1/\rho}$. It takes a straightforward but somewhat lengthy computation [9, Thm. 1] to verify the representation formula:

$$u(z) = \int_{\Gamma_\rho} \log\left|\frac{z-\tau}{1-z\tau}\right| |\Psi'(\tau)| \frac{ds}{2\pi\rho}, \quad z \in \Omega_\rho, \quad (4)$$

where ds is the arclength differential.

We can now construct the desired scheme $\mathcal{E}$. For this, it is enough to find a nested sequence of sets $\{\hat{E}_n\}$, with $\hat{E}_n = \{e_{j,n}\}_{j=1}^{2n} \subset D^+$, where D^+ is the interior of Γ, such that $|\hat{r}_n| = O(1)$ on Γ and $|\hat{r}_n| = o(1)$ locally uniformly in D^+, where

$$\hat{r}_n(z) = \prod_{e \in \hat{E}_n} \frac{z-e}{1-ez}.$$

Indeed, once we have such a sequence, all we have to do is to set $E_n = J(\hat{E}_n)$ and $r_n = \hat{r}_n \circ \varphi$.

Set $\varrho :=$ const.$|\Psi'|$, where const. is adjusted so that $\int_{\Gamma_\rho} \varrho ds = 1$. Observe that ϱ is locally uniformly bounded in Ω, in particular it is bounded on $\Gamma_\rho, \Gamma_{1/\rho}$. For each $n \in \mathbb{N}$, let $\{\Gamma_\rho^{j,n}\}_{j=1}^{2n}$ be a partition of Γ_ρ into $2n$ simple arcs that are pairwise disjoint except for possible common endpoints, in such a way that

$$\int_{\Gamma_\rho^{j,n}} \varrho ds \leq \frac{1}{n}, \quad j = 1, \ldots, 2n.$$

Clearly, we can arrange things so that this partition is nested, *i.e.* each arc in $\{\Gamma_\rho^{j,n}\}$ is an arc in $\{\Gamma_\rho^{j,n+1}\}$. Now, we have that

$$\operatorname{diam}(\Gamma_\rho^{j,n}) \leq \int_{\Gamma_\rho^{j,n}} ds \leq \frac{\text{const.}}{n}, \quad j = 1, \ldots, 2n, \quad (5)$$

since $\inf_{\Gamma_\rho} \varrho > 0$, where const. is an absolute constant. By (4) and the properties of u listed above, it also holds that

$$0 = \int_{\Gamma_\rho} k_z(\tau) \varrho ds, \quad k_z(\tau) := \log\left|\frac{z-\tau}{1-z\tau}\right|, \quad (6)$$

for any $z \in \Gamma$. Moreover, for $z \in \Gamma$ and $\tau_1, \tau_2 \in \Gamma_\rho$, it follows from (3) and the disjointness of $\Gamma_\rho, \Gamma_{1/\rho}$ from Γ that $1 - z\tau_1$ and $z - \tau_2$ are bounded away from 0, therefore

$$|k_z(\tau_1) - k_z(\tau_2)| = \left|\log\left|1 + \frac{(\tau_2-\tau_1)(1-z^2)}{(z-\tau_2)(1-z\tau_1)}\right|\right| \quad (7)$$
$$\leq O(|\tau_2 - \tau_1|),$$

where the last bound does not depend on z.

Now, let $e_{j,n}$ be an arbitrary point belonging to $\Gamma_\rho^{j,n}$, $j = 1, \ldots, 2n$, $n \in \mathbb{N}$. Then it follows from (5) and (7) that for any $z \in \Gamma$ and $\tau \in \Gamma_\rho^{j,n}$, $j = 1, \ldots, 2n$, we get

$$|k_z(\tau) - k_z(e_{j,n})| \leq O(\operatorname{diam}(\Gamma_\rho^{j,n})) \leq \frac{\text{const.}}{n}, \quad (8)$$

where const. is, again, a constant independent of z, j, and n. Therefore, we see that the functions

$$u_n(z) := \frac{1}{2n}\sum_{j=1}^{2n} k_z(e_{j,n}) = \sum_{j=1}^{2n} \int_{\Gamma_\rho^{j,n}} k_z(e_{j,n}) \varrho ds$$

are such that

$$\|u_n\|_\Gamma = \left\| \sum \int_{\Gamma_\rho^{j,n}} (k_z(\tau) - k_z(e_{j,n})) \varrho ds \right\|_{z \in \Gamma} \leq \sum \int_{\Gamma_\rho^{j,n}} \frac{\text{const.}}{n} \varrho ds = \frac{\text{const.}}{n}$$

by (6) and (8). Consequently $|\hat{r}_n| = \exp\{2nu_n\} = O(1)$ on Γ. Since $\hat{E}_n \subset \Gamma_\rho \subset D^+$ contains increasingly many zeros of $|\hat{r}_n|$, it follows from the maximum principle and a normal family argument that $|\hat{r}_n| = o(1)$ locally uniformly in D^+. This achieves the proof of the theorem because the points $e_{j,n}$ can be chosen in a nested fashion if the $\{\Gamma_\rho^{j,n}\}$ are. ■

IV. CONVERGENT INTERPOLANTS TO CAUCHY INTEGRALS

For F an analytic Jordan arc, we consider a function on $D = \overline{\mathbb{C}} \setminus F$ of the form

$$f_h(z) := \int_F \frac{h(t)d(t)}{z-t}, \quad z \in D. \quad (9)$$

It is shown in [9] that the multipoint Padé approximants to f_h associated with $\mathcal{E}$, where $\mathcal{E}$ is as in Theorem 3.1, converge locally uniformly in D if h does not vanish on F and $h(w^+)^\alpha$ is Dini-continuous there for some $\alpha > 0$. In fact, as explained in [9], it is possible for h to vanish at finitely many points $\xi_j \in F$ F like a sufficiently small power $(z-\xi)^\gamma$, but the admissible value for γ is related to the geometry of F in a rather intricate manner that we do not want here to go into.

In any case, the previous assumptions entail that h is unbounded at the endpoints of F, which is not satisfactory as it falls short of handling smooth densities h. We shall present a generalisation of this result that removes this restriction when h lies in the fractional Sobolev space $W^{1-1/p,p}(F)$ for some $p > 2$. Recall $W^{1-1/p,p}(F)$ is comprised of those those $g : F \to \mathbb{C}$ for which

$$\int_{t,t' \in F} \left|\frac{g(t) - g(t')}{t - t'}\right|^p ds(t)ds(t') < \infty; \quad (10)$$

it coincides with the so-called Besov space $B_p^{1-1/p,p}(I)$. If we indicate by $H_\alpha(F)$ the familiar space of Hölder-continuous functions on F of exponent $\alpha \in (0,1)$, it is clear that $H_\alpha(I) \subset W^{1-1/p,p}(I)$ if $1 < p < 1/(1-\alpha)$. Also, by the Sobolev embedding theorem [1, thm. 4.12], it holds as a weak converse that $W^{1-1/p,p}(I) \subset H_{1-2/p}(I)$ if $p > 2$. These estimates should help the reader unfamiliar with Sobolev spaces to figure out the level of generality of the assumptions in the next theorem.

More generally,

Theorem 4.1: Let F be an analytic Jordan arc connecting ± 1, $\mathcal{E}$ an associated interpolation scheme as in Theorem 3.1, and r_n the pseudo-rational functions (1). For $p \in (2,\infty)$, put $s = 1 - 1/p$, and assume $\psi \in W^{s,p}(F)$ does not vanish on F. Moreover, let $\alpha, \beta \in (-s,s) \cap (-1,\infty)$ and define the Jacobi weight $W(z) := (1-z)^\alpha(1+z)^\beta$. Then, with $h := W\psi$ and f_h as in (9), it holds for the multipoint Padé approximants Π_n associated to $\mathcal{E}$ that

$$(f_h - \Pi_n)w = [2G_h + O(1/n)]\, S_h^2 r_n, \qquad (11)$$

locally uniformly in D, where

$$G_h := \exp\{\int_F \log h(t)\, d\omega(t)\}, \text{ with } d\omega(t) := \frac{idt}{\pi w^+(t)}$$

is the geometric mean of h and, for $z \in D$,

$$S_h(z) := \exp\left\{\frac{w(z)}{2}\int \frac{\log h(t)}{z-t} d\omega(t) - \frac{1}{2}\int \log h d\omega\right\}$$

is the Szegö function of h.

The proof of Theorem 4.1 is rather technical and can be found in [10]. We shall be content here with some remarks.

First, our slightly stronger conclusion that the interpolation scheme can be chosen to be nested requires no modification with respect to [10] since only the fact that $r_n = o(1)$ locally uniformly in D and that it has uniformly bounded one-sided limits $r_n^\pm$ on F is used.

Second, Theorem 4.1 extends to general $\alpha, \beta > -1$ under additional smoothness assumptions. Roughly speaking, the higher the Jacobi exponents the smoother the density should be. We refer the reader to [10] for a precise statement to this effect and further generalizations. When the Jacobi exponents are negative, which is the case we presented, only a fraction of a derivative is needed, which compares not too badly with the Dini-continuity assumption in [9]. In the present setting, however, the density cannot vanish whereas some weak vanishing is still allowed in [9]. We are of course rewarded here with more general Jacobi weights and stronger asymptotics

Thirdly, let us stress that the proof of Theorem 4.1 dwells on the Riemann-Hilbert aproach to asymptotics of (non-Hermitian) orthogonal polynomials [24] and to the $\bar{\partial}$-variation thereof [28] to relax the analyticity requirement. This provides us with strong (*i.e.* Plancherel-Rotach type) asymptotics for the denominator polynomials of the multipoint Padé interpolants we construct and for their associated functions of the second kind, from which (11) follows easily. It is interesting to note that the Riemann-Hilbert approach, which is typically a tool to obtain sharp quantitative asymptotics, is here used as a means to solve a qualitative question namely, the convergence of the interpolants.

Fourthly, in the proof of Theorem 4.1, when deriving asymptotics for the denominators of the rational interpolants that are non-Hermitian orthogonal polynomials on F, one has to handle a varying weight which is not of power type, nor in general converging sufficiently fast to a weight of power type to take advantage of the results of [2], where the Riemann-Hilbert approach is adapted to non Hermitian orthogonality with analytic weights on smooth S-arcs. Instead, when "opening the lens" in the terminology of the steepest descent method associated to the Riemann-Hilbert approach, one has to set up a *sequence* of Riemann-Hilbert problems with *varying contours* whose solutions converge to the desired one by properties of the pseudo rational functions.

Finally, as the varying part of the weight is analytic, the extension inside the lens with controlled $\bar{\partial}$-estimates, introduced in [28] for power weights, needs only deal with the density defining the Cauchy integral we interpolate. This step is treated using either tools from real analysis, *e.g.* weighted norm inequalities and Sobolev extensions, or else classical Hölder estimates for singular integrals, whichever yields the best results granted the Jacobi exponents.

V. SOME REMARKS ON INTERPOLATION AND IDENTIFICATION

A few observation on the meaning of Theorem 4.1 in modeling and identification are perhaps in order.

- Observe that the convergence stated in the theorem follows immediately from (11), since r_n converges to zero locally uniformly off F. It is easy to see that this convergence is in fact geometric, locally uniformly in D. Indeed, with the notation of the proof of Theorem 3.1, we have that $r_n = \hat{r}_n \circ \varphi$ so it is enough to show that $\hat{r}_n$ converges to 0 geometrically, locally uniformly in D^+. Since $\hat{r}_n$ has uniformly bounded continuous trace on Γ and vanishes at $2n$ points on Γ_ρ which is compactly included in D^+, the question reduces by conformal mapping and the maximum principle to proving that a bounded sequence of H^∞-functions in the unit disk with increasingly many zeros in a smaller disk converges to zero geometrically, locally uniformly. This is immediate upon writing the fonctions in the form $b_{2n}g_n$ where g_n is bounded in H^∞ and b_{2n} is a Blaschke product of degree $2n$ whose zeros remain at positive distance from $\mathbb{T}$ [18, Ch. II].

- The interpolation points in the theorem are the zeros of the pseudo-rational functions r_n so they are chosen in close connection with the arc F, that is, with the singular set of f_h. In [9, Thm. 1], it is shown that the asymptotic distribution of these zeros generate a logarithmic potential whose opposite is the field making F into an S-contour. On the one hand, this means that they must distribute symmetrically with respect to F in some sense. For example if F is a real segment, then the asymptotic distribution of the zeros – thus of the interpolation points– should be conjugate symmetric. A generalization of this example is shown in the numerical examples of the next section. On another hand, the interpolation points *do not depend* on the density hW, which is perhaps counter-intuitive.
- Elaborating on the prvious item, we arrive at the (somewhat negative) conclusion that it Padé approximation is to be used in frequency identification, say we are given on the unit circle a sequence of values of a stable transfer function f holomorphic in $\{z;\ |z|>1-\varepsilon\}$, the interpolation points should be selected according to the geometry of the singular set of f inside the disk. Thus, very strong prior on f is to be implicitly used in such a procedure. Note that if the prior is wrong in the sense that f is indeed a Cauchy integral but on a different arc than expected, the interpolation procedure will diverge, at least generically with respect to analytic densities in the integral. This last fact is a little difficult to explain and requires more potential-theoretic prepartion than we possibly can have here, but it is of great significance from the identification viewpoint.
- Theorem 4.1 only deals with analytic Jordan arcs F. However, this is in some sense the most general convergence result that can be obtained in diagonal rational interpolation to Cauchy integrals over rectifiable Jordan arcs (analytic or not). Indeed, if the Cauchy integral of a Jacobi weight can be interpolated in a convergent way *via* a triangular scheme of interpolation points that stay away from the arc, then it turns out that the arc *must be analytic*.
- The convergence asserted in Theorem 4.1 entails that the poles of the rational interpolants p_n/q_n eventually lie in any neighborhood of F when n is large enough. In other words the singularities of the approximants converge to the singularities of the approximated function and there are no spurious poles. In fact more can be said, namely the asymptotic distribution of these poles is the *weighted equilibrium distribution on* F. This result connects to convergence in capacity and is already established in [19]. The numerical examples in the next section, where the poles of the interpolants are plotted for a couple of values of n, illustrate this fact. A related phenomenon can be used to approach inverse problems for the Laplacian pertaining to nondestructive control, see [8], [7].
- It is natural to ask about an analog to Theorem 4.1 for functions whose singular set has dimension 2, that is functions holomorphic outside a system of *c*losed curves. As for today, this extremely interesting issue is wide open.

VI. NUMERICAL ILLUSTRATION

Denote by F_α, $\alpha\in\mathbb{R}$, the following set

$$F_\alpha := \left\{\frac{i\alpha+x}{1+i\alpha x}:\ x\in[-1,1]\right\},$$
$$F_\alpha^{-1} := \{z:\ 1/z\in F_\alpha\}.$$

Clearly, F_α is an analytic arc joining -1 and 1. Besides, it is easily computed that $|r^\pm(e;t)|\equiv 1$, $t\in F_\alpha$, if $e\in F_\alpha^{-1}\setminus\{\pm1\}$. Hence, if $\mathcal{E}=\{E_n\}$, $E_n=\{e_{j,n}\}_{j=1}^{2n}$, is such that $e_{j,n}\in F_\alpha^{-1}\setminus\{\pm1\}$ for all $n\in\mathbb{N}$ and $j\in\{1,\dots,2n\}$, then $\mathcal{E}$ is adapted to F in the sense of Theorem 3.1. Moreover, it can be easily checked that for any $e\notin F_\alpha$

$$|r^+(e;t)r^+(e^*;t)|\equiv|r^-(e;t)r^-(e^*;t)|\equiv 1,\quad t\in F_\alpha,$$

where we have set

$$e^* := \frac{2i\alpha+(1-\alpha^2)\bar{e}}{(1-\alpha^2)+2i\alpha\bar{e}}.$$

Thus, if $\mathcal{E}=\{E_n\}$ is such that each E_n contains e and e^* simultaneously, then $\mathcal{E}$ is again adapted to F. Observe also that for $\alpha=0$ we get that $F_0=[-1,1]$ and $e^*=\bar{e}$.

In the first example below, we set $\alpha=-1/2$ and we plot the poles of the multipoint Padé approximants to f_μ, $d\mu(t)=ie^tdt/w^+(t)_{|F_{-1/2}}$, that corresponds to the interpolation scheme with half of the points at 0 and another half at $-4i/3$. Finding the denominator of the approximant of degree n amounts to solving a system of linear equations whose coefficients are obtained from the moments of the measures $t^{-n}(t+4i/3)^{-n}d\mu(t)$. The computations were carried with MAPLE 9.5 software using 24 digits precision.

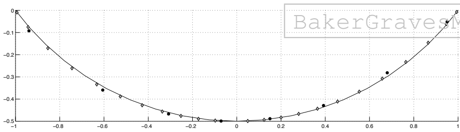

Fig. 1. Zeros of q_8 (disks) and q_{24} (diamonds).

In the second example, the contour F is generated by $e_1 := (i-3)/4$, $e_2 := (87+6i)/104$, and $e_3 := -i/10$, in the sense that

$$\left|(r(e_1;t)r(e_2;t)r(e_3;t))^{\pm}\right| \equiv 1,$$

and this time F is computed numerically. Thus, a scheme adapted to F is obtained by letting E_{3m} consist of e_1, e_2, and e_3 appearing m times each, while E_{3m+1} (E_{3m+2}) is obtained by adding to E_{3m} one (two) arbitrary point (points) from $D = \overline{\mathbb{C}} \setminus F$. Based on the derived discretizations of F, the moments of the measures $[(t-e_1)(t-e_2)(t-e_3)]^{-2m}h(t)dt$, $n = 3m$, are estimated numerically for $n = 24$ and $n = 66$, with $h(t) = t$ if $\mathrm{Im}(t) \geq 0$ and $h(t) = \bar{t}$ otherwise, and the coefficients of the corresponding denominators were found by solving the respective linear systems. The computations were carried with MAPLE 9.5 software using now 52 digits precision.

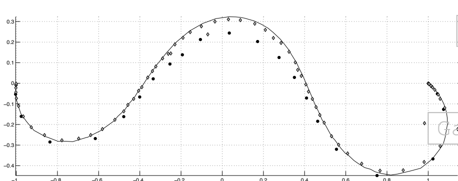

Fig. 2. Zeros of q_{24} (disks) and q_{66} (diamonds).

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