Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems – MTNS 2010 • 5–9 July, 2010 • Budapest, Hungary

Interconnection of Dirac Structures and Lagrange-Dirac Dynamical Systems

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***Abstract*— In the paper, we develop an idea of interconnection of Dirac structures and their associated Lagrange-Dirac dynamical systems. First, we briefly review the Lagrange-Dirac dynamical systems (namely, implicit Lagrangian systems) associated to induced Dirac structures. Second, we describe an idea of interconnection of Dirac structures; namely, we show how two distinct Lagrange-Dirac systems can be interconnected through a Dirac structure on the product of configuration spaces. Third, we also show the variational structure of the interconnected Lagrange-Dirac dynamical system in the context of the Hamilton-Pontryagin-d'Alembert principle. Finally, we demonstrate our theory by an example of mass-spring mechanical systems.**

I. INTRODUCTION

A large class of physical and engineering problems can be described in terms of Lagrangian variational mechanics, including mechanical systems, externally forced and dissipative systems, collisions, electrical circuits, problems with holonomic and nonholonomic constraints, stochastic systems, and field theories such as electromagnetism, and elasticity. Dirac structures provide a natural geometric framework for describing the *interconnection* of such diverse Lagrangian systems on the product of configuration spaces by introducing constraints that model interactions between subsystems.

The idea of interconnection was originally developed by [11], which has been known as nonenergic multiports throughout Tellegen's theorem in electrical network theory (see, for instance, [6], [14]). It was shown by [13] that the interconnection can be represented by Dirac structures, which are a generalization of both presymplectic and Poisson structures and also that nonholonomic systems and L-C circuits can be represented in the context of associated *implicit Hamiltonian systems* or *implicit port-Hamiltonian systems* (see also [4], [2], [3]). On the Lagrangian side, it was shown by the authors ([16], [17]) that nonholonomic mechanical systems and L-C circuits as degenerate Lagrangian systems can be formulated in the context of induced Dirac structures and associated *implicit Lagrangian systems*. Further, a notion of *implicit port-Lagrangian systems* for lossless transmission lines was constructed by [18] from the viewpoint of interconnecting L-C circuits.

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In this paper, we develop interconnection of two distinct Dirac structures and associated Dirac dynamical systems. First, we briefly review an induced Dirac structure and an associated Lagrange-Dirac dynamical system (namely, implicit Lagrangian systems). Second, we show how *implicit Lagrange-d'Alembert equations* can be developed by the *Hamilton-Pontryagin-d'Alembert principle* (see [17]). Then, we show how two distinct Lagrange-Dirac systems are energetically interconnected through a Dirac structure on the product of configuration spaces. Finally, we show that equations of motion for the interconnected Lagrange-Dirac dynamical system can be formulated in the context of the Hamilton-Pontryagin-d'Alembert principle.

II. DIRAC STRUCTURES

A. Induced Dirac Structures

Let us review an induced Dirac structure by following [16].

Let Q be an n-dimensional configuration manifold. Given a constraint distribution $\Delta_Q \subset TQ$, define a distribution Δ_{T^*Q} on T^*Q by $\Delta_{T^*Q} = (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q$, where $T\pi_Q : TT^*Q \to TQ$ is the tangent map of the cotangent bundle projection $\pi_Q : T^*Q \to Q$. Let Ω be the canonical symplectic structure on T^*Q and $\Omega^\flat : TT^*Q \to T^*T^*Q$ be the associated bundle map. Then, a Dirac structure D_{Δ_Q} on T^*Q induced from Δ_Q can be defined by, for each $(q,p) \in T^*Q$,

$$\begin{aligned} D_{\Delta_Q}(q,p) = \{ & (w,\alpha) \in T_{(q,p)}T^*Q \times T^*_{(q,p)}T^*Q \mid \\ & w \in \Delta_{T^*Q}(q,p) \text{ and } \alpha - \Omega^\flat(q,p)\cdot w \in \Delta^\circ_{T^*Q}(q,p) \}, \end{aligned} \tag{1}$$

where $\Delta^\circ_{T^*Q}$ is the annihilator of Δ_{T^*Q}.

B. Local Expressions

Let us choose local coordinates q^i on Q so that Q is locally represented by an open set $U \subset \mathbb{R}^n$. The constraint set Δ_Q defines a subspace of TQ, which we denote by $\Delta_Q(q) \subset \mathbb{R}^n$ at each point $q \in U$. If the dimension of $\Delta_Q(q)$ is $n-m$, then we can choose a basis $e_{m+1}(q), e_{m+2}(q), \ldots, e_n(q)$ of $\Delta_Q(q)$.

Recall that the constraint sets can be also represented by the annihilator of $\Delta_Q(q)$, which is denoted by $\Delta^\circ_Q(q)$ spanned by such one-forms that we write as $\omega^1, \omega^2, \ldots, \omega^m$. Using $\pi_Q : T^*Q \to Q$ locally denoted by $(q,p) \mapsto q$ and $T\pi_Q : (q,p,\dot{q},\dot{p}) \mapsto (q,\dot{q})$, it follows that

$$\Delta_{T^*Q} \cong \{ w = (q,p,\dot{q},\dot{p}) \mid q \in U, \dot{q} \in \Delta_Q(q) \}.$$

Then, the annihilator of Δ_{T^*Q} is locally represented as

$$\Delta^\circ_{T^*Q} \cong \left\{ \alpha = (q,p,\beta,v) \mid q \in U,\, \beta \in \Delta^\circ_Q(q) \text{ and } v = 0 \right\}.$$

ISBN 978-963-311-370-7

Since we have the local formula $\Omega^\flat(q,p)\cdot w=(q,p,-\dot p,\dot q)$, the condition $\alpha-\Omega^\flat(q,p)\cdot w\in\Delta^\circ_{T^*Q}$ reads $\beta+\dot p\in\Delta^\circ_Q(q)$ and $v-\dot q=0$. Hence, the induced Dirac structure is locally represented by

$$D_{\Delta_Q}(q,p)=\{((\dot q,\dot p),(\beta,v))\mid \dot q\in\Delta_Q(q),\ v=\dot q,\ \beta+\dot p\in\Delta^\circ_Q(q)\}.$$

III. Lagrange-Dirac Dynamical Systems

Let us define Lagrange-Dirac dynamical systems with external force fields.

A. Lagrange-Dirac Dynamical Systems

Let $\tau_Q:TQ\to Q$ be the tangent bundle projection. Given an external force field $F:TQ\to T^*Q$, it induces a horizontal one-form τ_Q^*F on TQ such that, for $v_q\in TQ$,

$$\tau_Q^*F(v_q)\cdot w=\left\langle F(v_q),T\tau_Q(w)\right\rangle,$$

where $\tau_Q^*F=(q,v,F(v_q),0)\in T^*_{v_q}TQ$ and $w=(q,v,\delta q,\delta v)\in T_{v_q}T^*Q$.

Using the *symplectomorphism* between T^*TQ, TT^*Q and T^*T^*Q (refer to [16]), we can define an induced map by, for $v_q\in TQ$,

$$\tilde F=\gamma_Q\circ\tau_Q^*F:TQ\to T^*T^*Q;\ \ (q,v)\mapsto(q,p,-F(v_q),0),$$

where $\gamma_Q:T^*TQ\to T^*T^*Q$ is the symplectomorphism that is locally given by $(q,v,\alpha,p)\to(q,p,-\alpha,v)$.

Given a Lagrangian $L:TQ\to\mathbb{R}$ (possibly degenerate), a ***Lagrange-Dirac dynamical system with an external force field*** (or an *implicit port-Lagrangian system with an external force field*) is defined by a *quadruple* (E,D_{Δ_Q},X,F), which satisfies, in coordinates $(q,v,p)\in TQ\oplus T^*Q$,

$$(X(q,v,p),\mathbf{d}E(q,v,p)|_{TP}+\tilde F(q,v))\in D_{\Delta_Q}(q,p),\qquad(2)$$

where $E(q,v,p):=\langle p,v\rangle-L(q,v)$ denotes a generalized energy, $X:TQ\oplus T^*Q\to TT^*Q$ a partial vector field and $(q,p=\partial L/\partial v)\in P=\mathbb{F}L(\Delta_Q)$. Let us write $X(q,v,p)=(q,p,\dot q,\dot p)$, so that $\dot q$ and $\dot p$ are functions of (q,v,p) and it follows that the local expression of a Lagrange-Dirac system in equation (2) may be given by

$$\dot q=v,\quad \dot p-\frac{\partial L}{\partial q}-F(q,v)\in\Delta^\circ_Q(q),\quad \dot q\in\Delta_Q(q)\qquad(3)$$

and with

$$p=\frac{\partial L}{\partial v}.$$

The curve $(q(t),v(t),p(t))$, $t_1\le t\le t_2$ in $TQ\oplus T^*Q$ that satisfies the condition (2) is a solution curve of (E,D_{Δ_Q},X,F).

B. Power Balance Law

The time derivative of $E(q,v,p)=\langle p,v\rangle-L(q,v)$ restricted to the solution curve $(q(t),v(t),p(t))$ reads

$$\frac{d}{dt}E(q(t),v(t),p(t))=\langle F(q(t),v(t)),\dot q(t)\rangle,$$

where we employed $p(t)=(\partial L/\partial v)(t)$ and $\dot q(t)=v(t)$.

C. Hamilton-Pontryagin-d'Alembert Principle

Let us see how Lagrange-Dirac systems can be developed by the Hamilton-Pontryagin-d'Alembert principle:

$$\delta\int_a^b\{L(q,v)+\langle p,\dot q-v\rangle\}\,dt+\int_a^b F(q,v)\cdot\delta q\,dt=0$$

for the chosen variations $\delta q\in\Delta_Q(q)$, for all δv and δp, and with $v\in\Delta_Q(q)$. Hence, we have

$$\int_{t_!}^{t_2}\left\{\left(\frac{\partial L}{\partial q}-\dot p\right)\delta q+\left(\frac{\partial L}{\partial v}-p\right)\delta v+(\dot q-v)\delta p\right\}\,dt+\int_a^b\langle F(q,v),\delta q\rangle\,dt=0.$$

Then one can obtain ***implicit Lagrange-d'Alembert equations***, which are equivalent with equation (3).

IV. Interconnection of Dirac Structures

Let us consider an idea of *interconnection of Dirac structures*, which is relevant with *composition* of Dirac structures developed by [7].

A. Product Dirac Structures

Let Q_1 and Q_2 be distinct configuration spaces. Given constraint distributions $\Delta_{Q_1}\subset TQ_1$ and $\Delta_{Q_2}\subset TQ_2$, we can define the induced Dirac structures $D_{\Delta_{Q_1}}$ and $D_{\Delta_{Q_2}}$, where we assume that $\Delta_{Q_1}\neq\Delta_{Q_2}$ and $\Delta_{Q_1}\cap\Delta_{Q_2}=\phi$.

Let $\bar Q=Q_1\times Q_2$ and $\Delta_{\bar Q}=\Delta_{Q_1}\times\Delta_{Q_2}\subset T\bar Q=TQ_1\times TQ_2$. As before, define the induced distribution $\Delta_{T^*\bar Q}$ on $T^*\bar Q$ by

$$\Delta_{T^*\bar Q}=(T\pi_{\bar Q})^{-1}(\Delta_{\bar Q})\subset TT^*\bar Q,$$

where $T\pi_{\bar Q}:TT^*\bar Q\to T\bar Q$ is the tangent map of the cotangent bundle projection $\pi_{\bar Q}:T^*\bar Q\to\bar Q$, while the annihilator of $\Delta_{T^*\bar Q}$ is defined by, for each $(q_e,p_e)\in T^*\bar Q$,

$$\Delta^\circ_{T^*\bar Q}(q_e,p_e)=\{\alpha_{p_e}\in T^*_{p_e}T^*\bar Q\mid\langle\alpha_{p_e},w_{p_e}\rangle=0\ \text{for all}\ w_{p_e}\in\Delta_{T^*\bar Q}(p_e)\}.$$

Let $\bar\Omega$ be the canonical symplectic structure on $T^*\bar Q$ and $\bar\Omega^\flat:TT^*\bar Q\to T^*T^*\bar Q$ be the associated bundle map. Then, we can define a ***product Dirac structure*** $D_{\Delta_{Q_1}}\times D_{\Delta_{Q_2}}$ on $T^*\bar Q$ induced from $\Delta_{\bar Q}$ by taking the product of the Dirac structures, which is given by, for each $(q_e,p_e)\in T^*\bar Q$,

$$\begin{aligned}&(D_{\Delta_{Q_1}}\times D_{\Delta_{Q_2}})(q_e,p_e)\\&=\{(w,\alpha)\in T_{(q_e,p_e)}T^*\bar Q\times T^*_{(q_e,p_e)}T^*\bar Q\mid w\in\Delta_{T^*\bar Q}(q_e,p_e)\\&\text{and}\ \alpha-\bar\Omega^\flat(q_e,p_e)\cdot w\in\Delta^\circ_{T^*\bar Q}(q_e,p_e)\}.\end{aligned}\qquad(4)$$

B. Interconnection of Dirac Structures

Consider ***interconnection*** of the Dirac structures $D_{\Delta_{Q_1}}$ and $D_{\Delta_{Q_2}}$. Suppose there exists a constraint distribution $\Delta_{12}\subset T\bar Q$ due to the interconnection, which may be given by, for each $q_e\in\bar Q$,

$$\Delta_{12}(q_e)=\{v_e\in T_{q_e}\bar Q\mid\langle\omega_{\bar Q}(q_e),v_e\rangle=0\},$$

such that

$$\Delta_{\bar Q}\neq\Delta_{12}\quad\text{and}\quad\Delta_{\bar Q}\cap\Delta_{12}\neq\phi,$$

where $\omega_{\bar{Q}}$ is a one-form on $\bar{Q}$ associated with the interconnection. The annihilator $\Delta^{\circ}_{\bar{Q}} \subset T^*\bar{Q}$ is defined by

$$\Delta^{\circ}_{12}(q_e) = \{ f_e \in T^*_{q_e}\bar{Q} \mid \langle f_e, v_e \rangle = 0 \mid v_e \in \Delta_{12}(q_e) \}$$

and we recall from [16] that $\Delta_{12} \times \Delta^{\circ}_{12}$ can be understood as a Dirac structure on $\bar{Q}$. Obviously, we can define the lifted distribution $\Delta_{\rm int}$ on $T^*\bar{Q}$ by

$$\Delta_{\rm int} = (T\pi_{\bar{Q}})^{-1}(\Delta_{12}) \subset TT^*\bar{Q},$$

where $T\pi_{\bar{Q}} : TT^*\bar{Q} \to T\bar{Q}$ is the tangent map of the cotangent bundle projection $\pi_{\bar{Q}} : T^*\bar{Q} \to \bar{Q}$, while the annihilator of $\Delta_{\rm int}$ is given by, for each $p_e \in T^*_{x_e}\bar{Q}$,

$$\Delta^{\circ}_{\rm int}(p_e) = \{ \alpha \in T^*_{p_e}T^*\bar{Q} \mid \langle \alpha, w \rangle = 0 \\ \text{for all } w \in \Delta_{\rm int}(p_e) \}.$$

Then, the Dirac structure associated to the interconnection $\Delta_{\rm int}$ on $T^*\bar{Q}$ can be defined by, for each $(x_e, p_e) \in T^*\bar{Q}$,

$$D_{\Delta_{\rm int}}(x_e, p_e) = \{ (w, \alpha) \in T_{(x_e,p_e)}T^*\bar{Q} \times T^*_{(x_e,p_e)}T^*\bar{Q} \mid \\ w \in \Delta_{\rm int}(x_e, p_e) \text{ and } \alpha - \bar{\Omega}^{\flat}(x_e, p_e) \cdot w \in \Delta^{\circ}_{\rm int}(x_e, p_e) \}. \tag{5}$$

Notice that the *interconnection* itself can be represented by an induced Dirac structure.

Definition. *The **interconnection of the distinct Dirac structures** $D_{\Delta_{Q_1}}$ and $D_{\Delta_{Q_2}}$ through $D_{\Delta_{\rm int}}$ is defined by*

$$D_{\Delta_C} := (D_{\Delta_{Q_1}} \times D_{\Delta_{Q_2}}) \bowtie D_{\Delta_{\rm int}}, \tag{6}$$

which is given by

$$D_{\Delta_C}(x_e, p_e) = \{ (w, \alpha) \in T_{(x_e,p_e)}T^*\bar{Q} \times T^*_{(x_e,p_e)}T^*\bar{Q} \mid \\ w \in \Delta_C(x_e, p_e) \text{ and } \alpha - \bar{\Omega}^{\flat}(x_e, p_e) \cdot w \in \Delta^{\circ}_C(x_e, p_e) \},$$

where $\Delta_C := \Delta_{T^*\bar{Q}} \cap \Delta_{\rm int} = (T\pi_{\bar{Q}})^{-1}(\Delta_{\bar{Q}} \cap \Delta_{12}) \subset TT^*\bar{Q}$ is the induced distribution on $T^*\bar{Q}$ and Δ°_C is its annihilator. In the above, the symbol $\bowtie$ indicates an operation of interconnection between Dirac structures.

V. Interconnection of Lagrange-Dirac Systems

A. Primitive Lagrange-Dirac Systems

Let $(E_1, D_{\Delta_{Q_1}}, X_1, F_1)$ and $(E_2, D_{\Delta_{Q_2}}, X_2, F_2)$ be two distinct Lagrange-Dirac systems, where L_i indicate Lagrangians on TQ_i, $X_i : TQ_i \oplus T^*Q_i \to TT^*Q_i$ partial vector fields and $F_i : TQ_i \to T^*Q_i$ external force fields (i=1,2).

Let us show how each Lagrange-Dirac dynamical system can be developed by the Hamilton-Pontryagin-d'Alembert principle. For the Lagrange-Dirac dynamical system $(E_1, D_{\Delta_{Q_1}}, X_1, F_1)$ on $TQ_1 \oplus T^*Q_1$, one has

$$\delta \int_a^b \{ L_1(q_1, v_1) + \langle p_1, \dot{q}_1 - v_1 \rangle \} \, dt + \int_a^b \langle F_1, \delta q_1 \rangle = 0$$

for $\delta q_1 \in \Delta_{Q_1}(q_1)$ and with $\dot{q}_1 \in \Delta_{Q_1}(q_1)$. Hence, it follows

$$\dot{q}_1 = v_1 \in \Delta_{Q_1}(q_1),\ \dot{p}_1 - \frac{\partial L_1}{\partial q_1} - F_1 \in \Delta^{\circ}_{Q_1}(q_1),\ p_1 = \frac{\partial L_1}{\partial v_1}.$$

Similarly, for the Lagrange-Dirac dynamical system $(E_2, D_{\Delta_{Q_2}}, X_2, F_2)$ on $TQ_2 \oplus T^*Q_2$, it follows

$$\delta \int_a^b \{ L_2(q_2, v_2) + \langle p_2, \dot{q}_2 - v_2 \rangle \} \, dt + \int_a^b \langle F_2, \delta q_2 \rangle = 0$$

for $\delta q_2 \in \Delta_{Q_2}(q_2)$ and with $\dot{q}_2 \in \Delta_{Q_2}(q_2)$. It follows

$$\dot{q}_2 = v_2 \in \Delta_{Q_2}(q_2),\ \dot{p}_2 - \frac{\partial L_2}{\partial q_2} - F_2 \in \Delta^{\circ}_{Q_2}(q_2),\ p_2 = \frac{\partial L_2}{\partial v_2}.$$

Note that each Lagrange-Dirac dynamical system has no energetic interaction with the other.

B. Interconnected Lagrange-Dirac Dynamical Systems

Define a Lagrangian on $T\bar{Q}$ by $L := L_1 + L_2$ and define a partial vector field on $T\bar{Q} \oplus T^*\bar{Q}$ by

$$X = X_1 \oplus X_2 : T\bar{Q} \oplus T^*\bar{Q} \to TT^*\bar{Q}.$$

Define also an external force field

$$F = F_1 \oplus F_2 : T\bar{Q} \to T^*\bar{Q}.$$

Recall from (5) that the interconnection of the Dirac structures $D_{\Delta_{Q_1}}$ and $D_{\Delta_{Q_2}}$ through $D_{\Delta_{\rm int}}$ is given by

$$D_{\Delta_C} = (D_{\Delta_{Q_1}} \times D_{\Delta_{Q_2}}) \bowtie D_{\Delta_{\rm int}},$$

and the ***interconnected Lagrange-Dirac dynamical system*** (E, D_{Δ_C}, X, F) is given by

$$(X, \mathbf{d}E|_{TP} + \tilde{F}) \in D_{\Delta_C},$$

where $\tilde{F} := \gamma_{\bar{Q}} \circ \tau^*_{\bar{Q}} F : T\bar{Q} \to T^*T^*\bar{Q}$ is the map associated to $F : T\bar{Q} \to T^*\bar{Q}$.

In local coordinates $q_e = (q_1, q_2) \in \bar{Q}$, $v_e = (v_1, v_2) \in T\bar{Q}$, and $p_e = (p_1, p_2) \in T^*\bar{Q}$, one has

$$\dot{q}_e = v_e \in \Delta_{\bar{Q}}(q_e),\ \dot{p}_e - \frac{\partial L}{\partial q_e} - F \in \Delta^{\circ}_{\bar{Q}}(q_e),\ p_e = \frac{\partial L}{\partial v_e}. \tag{7}$$

Alternatively, the equations of motion for the interconnected Lagrange-Dirac system can be developed by the Hamilton-Pontryagin-d'Alembert principle:

$$\delta \int_a^b \{ L(q_e, v_e) + \langle p_e, \dot{q}_e - v_e \rangle \} \, dt + \int_a^b \langle F(q_e, v_e), \delta q_e \rangle = 0,$$

for the chosen variation $\delta q_e \in (\Delta_{\bar{Q}} \cap \Delta_{12})(q_e) \subset T_{q_e}\bar{Q}$, for all δv_e and δp_e, and with $\dot{q}_e \in (\Delta_{\bar{Q}} \cap \Delta_{12})(q_e) \subset T_{q_e}\bar{Q}$.

In the above, the Hamilton–Pontryagin–d'Alembert principle for the interconnected Lagrange-Dirac dynamical systems is naturally recovered from the Hamilton-Pontryagin-d'Alembert principles for systems 1 and 2, together with the distribution $\Delta_{\bar{Q}} \cap \Delta_{12}$ associated with the interconnection.

VI. Example

A. Mass-Spring Mechanical Systems

Let us consider an illustrative example of a mass-spring system in the context of interconnected Dirac structures and their associated Lagrange-Dirac systems. In this example, there exists *no external force*. Let m_i and k_i be the i-th mass and spring $(i = 1, 2, 3)$.

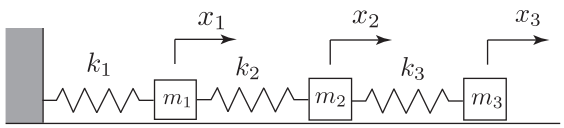

Fig. 1. A Mass-Spring System

B. Tearing and Interconnection

Inspiring by the concept of ***tearing and interconnection*** originally developed by [11], the mechanical system can be torn apart into two subsystems as in Fig 2. Note that *external energy ports* associated with the tearing inevitably appear at the boundaries of the two disconnected subsystems, where it induces the ***continuity conditions:***

$$f_2 + \bar{f}_2 = 0, \qquad \dot{x}_2 = \dot{\bar{x}}_2. \tag{8}$$

Without the above continuity conditions, each disconnected system has no energetic interaction with the other. In other words, *the original mechanical system can be reconstructed by interconnecting the subsystems throughout the continuity conditions* ([15]).

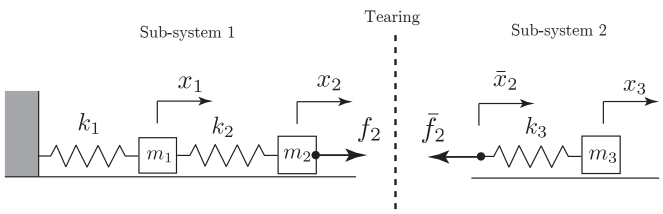

Fig. 2. Torn-apart Systems

C. Primitive Lagrangian Systems

Let us consider how the disconnected subsystems can be formulated in the context of Lagrange-Dirac dynamical systems, each of which can be regarded as a modular unit of the whole system. Let us call each disconnected system a "***primitive system***" by following [11].

The configuration space of the subsystem 1 may be given by $Q_1 = \mathbb{R} \times \mathbb{R}$ with local coordinates (x_1, x_2) for Q_1, while the configuration space of the subsystem 2 is $Q_2 = \mathbb{R} \times \mathbb{R}$ with local coordinates $(\bar{x}_2, x_3)$ for Q_2.

We can naturally define the induced Dirac structures $D_{\Delta_{Q_1}}$ and $D_{\Delta_{Q_2}}$, where $\Delta_{Q_1} = TQ_1$ and $\Delta_{Q_2} = TQ_2$ in this example.

As to the primitive system 1, the Lagrangian $L_1 : TQ_1 \to \mathbb{R}$ is given by

$$\begin{aligned} L_1(x_1, x_2, v_1, v_2) &= T(x_1, x_2, v_1, v_2) - V(x_1, x_2) \\ &= \frac{1}{2}\left\{m_1(v_1)^2 + m_2(v_2)^2\right\} - \frac{1}{2}\left\{k_1(x_1)^2 + k_2(x_2 - x_1)^2\right\}, \end{aligned}$$

and as to the primitive system 2, the Lagrangian $L_2 : TQ_2 \to \mathbb{R}$ is given by

$$\begin{aligned} L_2(\bar{x}_2, x_3, v_3) &= T(\bar{x}_2, x_3, v_3) - V(\bar{x}_2, x_3) \\ &= \frac{1}{2}m_3(v_3)^2 - \frac{1}{2}k_3(x_3 - \bar{x}_2)^2. \end{aligned}$$

Then, we can define the generalized energy E_1 on $TQ_1 \oplus T^*Q_1$ by

$$\begin{aligned} E_1(x_1, x_2, v_1, v_2, p_1, p_2) &= p_1 v_1 + p_2 v_2 - L_1(x_1, x_2, v_1, v_2) \\ &= p_1 v_1 + p_2 v_2 - \frac{1}{2}\left\{m_1(v_1)^2 + m_2(v_2)^2\right\} \\ &\quad + \frac{1}{2}\left\{k_1(x_1)^2 + k_2(x_2 - x_1)^2\right\}, \end{aligned}$$

and we can also define the generalized energy E_2 on $TQ_2 \oplus T^*Q_2$ by

$$\begin{aligned} E_2(\bar{x}_2, x_3, \bar{v}_2, v_3, \bar{p}_2, p_3) &= \bar{p}_2 \bar{v}_2 + p_3 v_3 - L_2(\bar{x}_2, x_3, v_3) \\ &= \bar{p}_2 \bar{v}_2 + p_3 v_3 - \frac{1}{2}m_3(v_3)^2 + \frac{1}{2}k_3(x_3 - \bar{x}_2)^2. \end{aligned}$$

Though the original system has no external force, *each disconnected primitive system has an interconnection constraint force as if it plays a role of an external force*. This is because the *tearing always yields constraint forces at the boundaries associated with the disconnected primitive systems*, as shown in Fig. 2

The ***interconnection constraint force fields*** at the boundaries $F_{c1} : T^*Q_1 \to T^*T^*Q_1$ and $F_{c2} : T^*Q_2 \to T^*T^*Q_2$ are given by

$$\begin{aligned} F_{c1} &= (x_1, x_2, p_1, p_2, 0, -f_2, 0, 0), \\ F_{c2} &= (\bar{x}_2, x_3, \bar{p}_2, p_3, -\bar{f}_2, 0, 0, 0). \end{aligned}$$

Further, let $X_1 : TQ_1 \oplus T^*Q_1 \to TT^*Q_1$ be the partial vector field, which is defined at points $(p_1, p_2) \in P_1 = \mathbb{F}L(\Delta_{Q_1})$ as

$$X_1(x_1, x_2, v_1, v_2, p_1, p_2) = (x_1, x_2, p_1, p_2, \dot{x}_1, \dot{x}_2, \dot{p}_1, \dot{p}_2),$$

where $(p_1 = m_1 v_1, p_2 = m_2 v_2) \in P_1$. Let $X_2 : TQ_2 \oplus T^*Q_2 \to TT^*Q_2$ be the partial vector field, which is defined at points $(\bar{p}_2, p_3) \in P_2 = \mathbb{F}L(\Delta_{Q_2})$ as

$$X_2(\bar{x}_2, x_3, \bar{v}_2, v_3, \bar{p}_2, p_3) = (\bar{x}_2, x_3, \bar{p}_2, p_3, \dot{\bar{x}}_2, \dot{x}_3, \dot{\bar{p}}_2, \dot{p}_3),$$

where $(\bar{p}_2 = 0, p_3 = m_3 v_3) \in P_2$, together with the ***consistency condition*** $\dot{\bar{p}}_2 = 0$.

Primitive System 1: We can formulate dynamics of the primitive system 1 as the Lagrange-Dirac dynamical system $(E_1, D_{\Delta_{Q_1}}, X_1, F_1)$ as

$$(X_1, \mathbf{d}E_1|_{TP_1} + F_{c1}) \in D_{\Delta_{Q_1}},$$

which may be given in coordinates by

$$\begin{aligned} &\dot{x}_1 = v_1, \quad \dot{x}_2 = v_2, \\ &\dot{p}_1 = -k_1 x_1 - k_2(x_1 - x_2), \quad \dot{p}_2 = k_2(x_1 - x_2) + f_2, \end{aligned} \tag{9}$$

and with $p_1 = m_1 v_1$ and $p_2 = m_2 v_2$.

Primitive System 2: Similarly, we can also formulate dynamics of the primitive system 2 as the Lagrange-Dirac dynamical system $(E_2, D_{\Delta_{Q_2}}, X_2, F_2)$ as

$$(X_2, \mathbf{d}E_2|_{TP_2} + F_{c2}) \in D_{\Delta_{Q_2}},$$

which may be given in coordinates by

$$\begin{aligned} &\dot{\bar{x}}_2 = \bar{v}_2, \quad \dot{x}_3 = v_3, \\ &\dot{\bar{p}}_2 = k_3(x_3 - \bar{x}_2) + \bar{f}_2, \quad \dot{p}_3 = -k_3(x_3 - \bar{x}_2), \end{aligned} \tag{10}$$

and with $\bar{p}_2 = 0$ and $p_3 = m_3 v_3$ as well as $\dot{\bar{p}}_2 = 0$.

Recall that each primitive systems is physically independent with the other, which means that there exists no energetic interaction between them. Next, let us see how the disconnected primitive systems can be interconnected through a Dirac structure.

D. Interconnection of Dirac Structures

Let $\bar{Q} = Q_1 \times Q_2 = \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ be an ***extended configuration space*** with local coordinates $x_e = (x_1, x_2, \bar{x}_2, x_3)$. In the illustrative example, $\Delta_{\bar{Q}} = \Delta_{Q_1} \times \Delta_{Q_2} = T\bar{Q} = TQ_1 \times TQ_2$ and the product Dirac structure $D_{\Delta_{Q_1}} \times D_{\Delta_{Q_2}}$ on $T^*\bar{Q}$ is defined as in equation (4).

Now, the constraint distribution due to the interconnection is given by

$$\Delta_{12}(x_e) = \{v_e \in T_{q_e}\bar{Q} \mid \langle \omega_{\bar{Q}}(x_e), v_e\rangle = 0\},$$

where $\omega_{\bar{Q}}$ is a one-form on $\bar{Q}$ given by $\omega_{\bar{Q}} = d\bar{x}_1 - d\bar{x}_2$. On the other hand, the annihilator $\Delta_{12}^\circ \subset T^*\bar{Q}$ is defined by

$$\Delta_{12}^\circ(x_e) = \{f_e = (f_1, f_2, \bar{f}_2, f_3) \in T^*_{x_e}\bar{Q} \mid \langle f_e, v_e\rangle = 0 \text{ and } v_e \in \Delta_{12}(x_e)\}.$$

It follows from this codistribution that $f_2 = -\bar{f}_2$.

Hence, we obtain the conditions for the interconnection as in equation (8); namely,

$$f_2 + \bar{f}_2 = 0, \qquad v_2 = \bar{v}_2,$$

Recall from [16] that $\Delta_{12} \times \Delta_{12}^\circ$ can be understood as a Dirac structure on $\bar{Q}$. As before, since we can define the distribution Δ_{int} on $T^*\bar{Q}$ by $\Delta_{\mathrm{int}} = (T\pi_{\bar{Q}})^{-1}(\Delta_{12}) \subset TT^*\bar{Q}$, the Dirac structure $D_{\Delta_{\mathrm{int}}}$ can be defined as in (5). Further, recall that the interconnected Dirac structure Δ_{int} on $T^*\bar{Q}$ is given by

$$D_{\Delta_C} = (D_{\Delta_{Q_1}} \times D_{\Delta_{Q_2}}) \bowtie D_{\Delta_{\mathrm{int}}}.$$

E. Interconnected Lagrange-Dirac Systems

Now, let us see how two primitive Lagrange-Dirac systems, namely, $(E_1, D_{\Delta_{Q_1}}, X_1, F_1)$ and $(E_2, D_{\Delta_{Q_2}}, X_2, F_2)$ can be interconnected to be a Lagrange-Dirac dynamical system.

Define the Lagrangian $\bar{L} : T\bar{Q} \to \mathbb{R}$ for the interconnected system by $\bar{L} = L_1 + L_2$, and hence the generalized energy $\bar{E} = E_1 + E_2 : T\bar{Q} \oplus T^*\bar{Q} \to \mathbb{R}$ is given in coordinates $(x_e, v_e, p_e) = (x_1, x_2, \bar{x}_2, x_3,\ v_1, v_2, \bar{v}_2, v_3, p_1, p_2, \bar{p}_2, p_3)$ as

$$\begin{aligned}
&\bar{E}(x_1, x_2, \bar{x}_2, x_3, v_1, v_2, \bar{v}_2, v_3, p_1, p_2, \bar{p}_2, p_3)\\
&\quad = E_1(x_1, x_2, v_1, v_2, p_1, p_2) + E_2(\bar{x}_2, x_3, \bar{v}_2, v_3, \bar{p}_2, p_3)\\
&\quad = p_1 v_1 + p_2 v_2 + \bar{p}_2 \bar{v}_2 + p_3 v_3\\
&\qquad\quad - \frac{1}{2}\left\{m_1(v_1)^2 + m_2(v_2)^2 + m_3(v_3)^2\right\}\\
&\qquad\quad + \frac{1}{2}\left\{k_1(x_1)^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_3(x_3 - \bar{x}_2)^2\right\}.
\end{aligned}$$

Let us introduce a partial vector field $\bar{X} = X_1 \oplus X_2 : T\bar{Q} \oplus T^*\bar{Q} \to TT^*\bar{Q}$, which is defined at points $(x_1, x_2, \bar{x}_2, x_3, v_1, v_2, \bar{v}_2, v_3, p_1 = m_1 v_1, p_2 = m_2 v_2, \bar{p}_2 = 0, p_3 = m_3 v_3) \in \bar{P} = \mathbb{F}\bar{L}(\Delta_{\bar{Q}})$ as

$$\begin{aligned}
&\bar{X}(x_1, x_2, \bar{x}_2, x_3, v_1, v_2, \bar{v}_2, v_3, p_1, p_2, \bar{p}_2, p_3)\\
&\quad = (x_1, x_2, \bar{x}_2, x_3, , p_1, p_2, \bar{p}_2, p_3, \dot{x}_1, \dot{x}_2, \dot{\bar{x}}_2, \dot{x}_3, \dot{p}_1, \dot{p}_2, \dot{\bar{p}}_2, \dot{p}_3),
\end{aligned}$$

where the consistency condition $\dot{\bar{p}}_2 = 0$ holds.

Now, the ***interconnected Lagrange-Dirac system*** is given by $(\bar{L}, \Delta_{\bar{Q}}, \bar{X}, \bar{F})$, which satisfies, for each (x_e, v_e, p_e),

$$(\bar{X}(x_e, v_e, p_e), \mathbf{d}\bar{E}(x_e, v_e, p_e)|_{T\bar{P}}) \in D_{\Delta_C}(x_e, p_e),$$

and with $(x_e, p_e) \in \bar{P} = \mathbb{F}\bar{L}(\Delta_{\bar{Q}})$. Thus, it follows that

$$\bar{\Omega}^\flat(x_e, p_e)\cdot\bar{X}(x_e, v_e, p_e) - \mathbf{d}\bar{E}(x_e, v_e, p_e)|_{T\bar{P}} = \bar{F}_c(x_e, v_e),$$

with the interconnection constraint $\omega_{\bar{Q}}(x_e)v_e = 0$. In the above, $\mathbf{d}\bar{E}(x_e, v_e, p_e)|_{T\bar{P}} = (-\partial L/\partial x_e, v_e)$ and the force field $\bar{F}_c = F_{c1} \oplus F_{c2} : T\bar{Q} \to T^*T^*\bar{Q}$ is given, in coordinates $(x_e, v_e) = (x_1, x_2, \bar{x}_2, x_3, v_1, v_2, \bar{v}_2, v_3)$ for $T\bar{Q}$, as $\bar{F}_c(x_1, x_2, \bar{x}_2, x_3, v_1, v_2, \bar{v}_2, v_3) = (0, 0, 0, 0, 0, -f_2, -\bar{f}_2, 0)$, where the force $\bar{F}_c(x_e, v_e)$ is an element in $\Delta^\circ_{T^*\bar{Q}}(x_e, p_e) \subset T^*T^*\bar{Q}$ such that $f_2 = -\bar{f}_2$. This induces

$$\bar{F}_c(x_1, x_2, \bar{x}_2, x_3, p_1, p_2, \bar{p}_2, p_3) = (0, 0, 0, 0, 0, -f_2, f_2, 0)$$

and hence, it follows

$$\bar{F}_c(x_e, p_e) = \bar{\mathbf{N}}^T(x_e, p_e) f_2,$$

where $\bar{\mathbf{N}} = (0, 0, 0, 0, \mathbf{N}) = (0, 0, 0, 0, 0, -1, 1, 0)$.

In local coordinates, the interconnected Lagrange-Dirac dynamical system is given by

$$\left(\begin{array}{cccc|cccc}
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right)
\left(\begin{array}{c}
\dot{x}_1\\ \dot{x}_2\\ \dot{\bar{x}}_2\\ \dot{x}_3\\ \hline \dot{p}_1\\ \dot{p}_2\\ \dot{\bar{p}}_2\\ \dot{p}_3
\end{array}\right)$$

$$= \left(\begin{array}{c}
k_1 x_1 - k_2(x_2 - x_1)\\
k_2 x_2\\
-k_3(x_3 - \bar{x}_2)\\
k_3(x_3 - \bar{x}_2)\\
\hline
v_1\\ v_2\\ \bar{v}_2\\ v_3
\end{array}\right)
+ \left(\begin{array}{c}
0\\ -1\\ 1\\ 0\\ \hline 0\\ 0\\ 0\\ 0
\end{array}\right) f_2, \tag{11}$$

together with the Legendre transformation

$$\begin{pmatrix} p_1\\ p_2\\ \bar{p}_2\\ p_3 \end{pmatrix} = \begin{pmatrix} m_1 v_1\\ m_2 v_2\\ 0\\ m_3 v_3 \end{pmatrix}, \tag{12}$$

the interconnection constraint

$$\begin{pmatrix} 0 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \bar{v}_2 \\ v_3 \end{pmatrix} = 0 \qquad (13)$$

as well as the consistency condition

$$\dot{\bar{p}}_2 = 0. \qquad (14)$$

F. Hamilton-Pontryagin-d'Alembert Principle

Let us consider how the interconnected Lagrange-Dirac system can be developed by the Hamilton-Pontryagin-d'Alembert principle:

$$\delta \int_a^b \{L(x_e, v_e) + \langle p_e, \dot{x}_e - v_e \rangle\}\; dt = 0$$

for the chosen variation $\delta x_e \in \Delta_C(x_e)$, for all δv_e and δp_e, and with $\dot{x}_e \in \Delta_{\bar{Q}}(x_e)$. Hence, it follows from

$$\int_{t_1}^{t_2} \left\{ \left(\frac{\partial L}{\partial x_e} - \dot{p}_e \right) \delta x_e + \left(\frac{\partial L}{\partial v_e} - p_e \right) \delta v_e + (\dot{x}_e - v_e) \delta p_e \right\}\; dt = 0$$

that one can obtain

$$\dot{x}_e = v_e \in \Delta_C(x_e), \quad \dot{p}_e - \frac{\partial L}{\partial x_e} \in \Delta_C^{\circ}(x_e), \quad \dot{p}_e = \frac{\partial L}{\partial v_e}.$$

In the example, we can set $x_e = (x_1, x_2, \bar{x}_2, x_3)$, $v_e = (v_1, v_2, \bar{v}_2, v_3)$ and $p_e = (p_1, p_2, \bar{p}_2, p_3)$ and hence the Hamilton-Pontryagin-d'Alembert principle for the interconnected Lagrange-Dirac system is denoted by

$$\delta \int_a^b [L_1(x_1, x_2, v_1, v_2) + p_1(\dot{x}_1 - v_1) + p_2(\dot{x}_2 - v_2)] + [L_2(\bar{x}_2, x_3, v_3) + \bar{p}_2(\dot{\bar{x}}_2 - \bar{v}_2) + p_3(\dot{x}_3 - v_3)]\; dt,$$

for all $\delta x_2 = \delta \bar{x}_2$, for all δv_e and δp_e, and with $v_2 = \bar{v}_2$.

Thus, we obtain the system equations of the interconnected Lagrange-Dirac system, which are equivalent with equations of motion in (11), the Legendre transformation in (12), the interconnection constraint in (13) and with the consistency condition in (14).

VII. Summary

In this paper, we showed a notion of interconnection of Dirac structures and associated Lagrange-Dirac dynamical systems. The principal idea lies in the fact that the *interconnection of Dirac structures itself is represented as a Dirac structure* and we showed how distinct Lagrange-Dirac dynamical systems can be interconnected throughout the Dirac structure. We also clarified the variational structure of the interconnected Lagrange-Dirac dynamical system by the Hamilton-Pontryagin-d'Alembert principle. Lastly, we demonstrated our theory with an illustrative examples of mass-spring mechanical systems.

Research of HY is partially supported by JSPS Grant-in-Aid 20560229, JST-CREST and Waseda University Grant for SR 2010A-606; Research of JEM is partially supported by NSF grant DMS-0505711.

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