Derived cones to reachable sets of semilinear differential inclusions

Aurelian Cernea

***Abstract*— We consider a semilinear differential inclusion and we prove that the reachable set of a certain variational inclusion is a derived cone in the sense of Hestenes to the reachable set of the semilinear differential inclusion. This result allows to obtain a sufficient condition for local controllability along a reference trajectory.**

I. INTRODUCTION

The concept of derived cone to an arbitrary subset of a normed space has been introduced by M.Hestenes in [13] and successfully used to obtain necessary optimality conditions in Control Theory. Afterwards, this concept has been largely ignored in favor of other concepts of tangents cones, that may intrinsically be associated to a point of a given set: the cone of interior directions, the contingent, the quasitangent and, above all, Clarke's tangent cone (e.g., [1]). Mirică ([14,15]) obtained "an intersection property" of derived cones that allowed a conceptually simple proof and significant extensions of the maximum principle in optimal control; moreover, other properties of derived cones may be used to obtain controllability and other results in the qualitative theory of control systems. In our previous papers [4,6,7,9] we identified derived cones to the reachable sets of certain classes of discrete and differential inclusions in terms of a variational inclusion associated to the initial discrete or differential inclusion.

In the present paper we consider semilinear differential inclusions of the form

$$x' \in Ax + F(t,x), \quad x(0) \in X_0 \tag{1.1}$$

where $F : [0,T] \times X \to \mathcal{P}(X)$ is a set valued map, A is the infinitesimal generator of a C_0-semigroup $\{G(t)\}_{t\geq 0}$ on a separable Banach space X and $X_0 \subset X$. Our aim is to prove that the reachable set of a certain variational inclusion is a derived cone in the sense of Hestenes to the reachable set of the semilinear differential inclusion. In order to obtain the continuity property in the definition of a derived cone we shall use a continuous version of Filippov's theorem for mild solutions of semilinear differential inclusions obtained in [2]. As an application, when X is finite dimensional, we obtain a sufficient condition for local controllability along a reference trajectory.

We note that existence results and qualitative properties of the mild solutions of problem (1.1) may be found in [2,3,5,8,10-12,16] etc.

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A. Cernea is with Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, 010014 Bucharest, Romania acernea@fmi.unibuc.ro

The paper is organized as follows: in Section 2 we present the notations and the preliminary results to be used in the sequel and in Section 3 we provide our main results.

II. PRELIMINARIES

Since the reachable set to a control system is, generally, neither a differentiable manifold, nor a convex set, its infinitesimal properties may be characterized only by tangent cones in a generalized sense, extending the classical concepts of tangent cones in Differential Geometry and Convex Analysis, respectively.

From the rather large number of "convex approximations", "tents", "regular tangents cones", etc. in the literature, we choose the concept of derived cone introduced by M.Hestenes in [13].

Let $(X, ||.||)$ be a normed space.

Definition 2.1 ([13]). A subset $M \subset X$ is said to be a *derived set to* $E \subset X$ at $x \in E$ if for any finite subset $\{v_1, ..., v_k\} \subset M$, there exist $s_0 > 0$ and a continuous mapping $a(.) : [0, s_0]^k \to E$ such that $a(0) = x$ and $a(.)$ is (conically) differentiable at $s = 0$ with the derivative col$[v_1, ..., v_k]$ in the sense that

$$\lim_{\mathbf{R}_+^k \ni \theta \to 0} \frac{||a(\theta) - a(0) - \sum_{i=1}^k \theta_i v_i||}{||\theta||} = 0.$$

We shall write in this case that the derivative of the map $a(.)$ at $s = 0$ is given by

$$Da(0)\theta = \sum_{i=1}^k \theta_j v_j \quad \forall \theta = (\theta_1, ..., \theta_k) \in \mathbf{R}_+^k := [0, \infty)^k.$$

A subset $C \subset X$ is said to be a *derived cone* of E at x if it is a derived set and also a convex cone.

For the basic properties of derived sets and cones we refer to M.Hestenes [13]; we recall that if M is a derived set then $M \bigcup \{0\}$ as well as the convex cone generated by M, defined by

$$cco(M) = \{\sum_{i=1}^k \lambda_j v_j; \lambda_j \geq 0,\, k \in N,\, v_j \in M,\, j = \overline{1,k}\}$$

is also a derived set, hence a derived cone.

The fact that the derived cone is a proper generalization of the classical concepts in Differential Geometry and Convex Analysis is illustrated by the following results ([13]): if $E \subset \mathbf{R}^n$ is a differentiable manifold and $T_x E$ is the tangent space in the sense of Differential Geometry to E at x

$$T_x E = \{v \in \mathbf{R}^n; \quad \exists c(.) : (-s,s) \to X, \quad \text{of class } C^1, \quad c(0) = x, \quad c'(0) = v\},$$

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then T_xE is a derived cone; also, if $E \subset \mathbf{R}^n$ is a convex subset then the tangent cone in the sense of Convex Analysis defined by

$$TC_xE = cl\{t(y-x); \quad t \geq 0,\ y \in E\}$$

is also a derived cone. Since any convex subcone of a derived cone is also a derived cone, such an object may not be uniquely associated to a point $x \in E$; moreover, simple examples show that even a maximal with respect to set-inclusion derived cone may not be uniquely defined: if the set $E \subset \mathbf{R}^2$ is defined by

$$E = C_1 \bigcup C_2,$$
$$C_1 = \{(x,x); x \geq 0\}, \quad C_2 = \{(x,-x), x \leq 0\},$$

then C_1 and C_2 are both maximal derived cones of E at the point $(0,0) \in E$.

On the other hand, the up-to-date experience in Nonsmooth Analysis shows that for some problems, the use of one of the intrinsic tangent cones may be preferable. From the multitude of the intrinsic tangent cones in the literature (e.g. [1]), the contingent, the quasitangent (intermediate) and Clarke's tangent cones, defined, respectively, by

$$K_xE = \{v \in X; \quad \exists\, s_m \to 0+, \quad \exists\, x_m \to x,\ x_m \in E : \tfrac{x_m - x}{s_m} \to v\},$$
$$Q_xE = \{v \in X; \quad \forall s_m \to 0+, \quad \exists\, x_m \to x,\ x_m \in E : \tfrac{x_m - x}{s_m} \to v\},$$
$$C_xE = \{v \in X; \quad \forall\, (x_m, s_m) \to (x, 0+), \quad x_m \in E, \exists\, y_m \in E : \tfrac{y_m - x_m}{s_m} \to v\}$$

seem to be among the most oftenly used in the study of different problems involving nonsmooth sets and mappings.

An outstanding property of derived cone, obtained by Hestenes ([13], Theorem 4.7.4) is stated in the next lemma.

Lemma 2.2 ([13]). *Let $X = \mathbf{R}^n$. Then $x \in int(E)$ if and only if $C = \mathbf{R}^n$ is a derived cone at $x \in E$ to E.*

Corresponding to each type of tangent cone, say τ_xE one may introduce (e.g. [1]) a *set-valued directional derivative* of a multifunction $G(.) : E \subset X \to \mathcal{P}(X)$ (in particular of a single-valued mapping) at a point $(x,y) \in Graph(G)$ as follows

$$\tau_yG(x;v) = \{w \in X;\ (v,w) \in \tau_{(x,y)}Graph(G)\},\ v \in \tau_xE.$$

We recall that a set-valued map, $A(.) : X \to \mathcal{P}(X)$ is said to be a *convex* (respectively, closed convex) *process* if $Graph(A(.)) \subset X \times X$ is a convex (respectively, closed convex) cone. For the basic properties of convex processes we refer to [1], but we shall use here only the above definition.

Let denote by I the interval $[0,T]$ and let X be a real separable Banach space with the norm $||.||$ and with the corresponding metric $d(.,.)$. Denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\},$$
$$d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y \in B} d(x,y)$.

As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $||x(.)||_C = \sup_{t \in I} ||x(t)||$ and by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $||x(.)||_1 = \int_I ||x(t)||dt$.

We consider $\{G(t)\}_{t \geq 0} \subset L(X,X)$ a strongly continuous semigroup of bounded linear operators from X to X having the infinitesimal generator A and a set valued map $F(.,.)$ defined on $I \times X$ with nonempty closed subsets of X, which define the following differential inclusion:

$$x'(t) \in Ax(t) + F(t, x(t)) \quad a.e.\,(I) \quad x(0) = x_0 \quad (2.1)$$

It is well known that, in general, the Cauchy problem

$$x' = Ax + f(t,x),\ f(t,x) \in F(t,x), \quad x(0) = x_0 \quad (2.2)$$

may not have a classical solution and that a way to overcome this difficulty is to look for continuous solutions of the integral equation

$$x(t) = G(t)x_0 + \int_0^t G(t-u) f(u, x(u)) du.$$

This is why the concept of the mild solution is convenient for solving (2.1)

A mapping $x(.) \in C(I,X)$ is called a *mild solution* of (2.1) if there exists a (Bochner) integrable function $f(.) \in L^1(I,X)$ such that

$$f(t) \in F(t, x(t)) \quad a.e.\,(I), \quad (2.3)$$

$$x(t) = G(t)x_0 + \int_0^t G(t-u) f(u) du \quad \forall t \in I, \quad (2.4)$$

i.e., $f(.)$ is a locally (Bochner) integrable selection of the set-valued map $F(., x(.))$ and $x(.)$ is the mild solution of the initial value problem

$$x'(t) = Ax(t) + f(t), \quad x(0) = x_0. \quad (2.5)$$

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (2.1) if $f(.)$ verifies (2.3) and $x(.)$ is a mild solution of (2.5).

Hypothesis 2.3. i) *$F(.,.) : I \times X \to \mathcal{P}(X)$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(X)$ measurable.*

ii) *There exists $L(.) \in L^1(I, \mathbf{R}_+)$ such that, for any $t \in I, F(t,.)$ is $L(t)$-Lipschitz in the sense that*

$$d_H(F(t,x_1), F(t,x_2)) \leq L(t)||x_1 - x_2|| \quad \forall\, x_1, x_2 \in X.$$

The main tool in characterizing derived cones to reachable sets of semilinear differential inclusions is a certain version of Filippov's theorem for semilinear differential inclusions. In the theorem to follow, S is a separable metric space $X_0 \subset X$, $a_0(.) : S \to X_0$ and $c(.) : S \to [0, \infty)$ are given continuous mappings.

Hypothesis 2.4. *Let $(z(.), f(.)) \in C(I,X) \times L^1(I,X)$ be a trajectory-selection pair of (2.1).*

The continuous mappings $g(.) : S \to L^1(I,X)$, $y(.) : S \to C(I,X)$ are given such that

$$(y(s))'(t) = Ay(s)(t) + g(s)(t), \quad t \in I, \quad y(s)(0) \in X_0.$$

There exist $s_0 \in S$ and a continuous function $p(.) : S \to L^1(I, \mathbf{R}_+)$ such that

$$c(s_0) = 0, \quad c(s) \neq 0, \quad \forall s \in S,\ s \neq s_0,$$

$$a_0(s_0) = z(0),\ y(s_0)(t) = z(t),\ g(s_0)(t) = f(t),\ \forall t \in I,$$

$$d(g(s)(t), F(t, y(s)(t))) \leq p(s)(t) \quad a.e.\,(I),\ \forall\, s \in S.$$

Theorem 2.5 ([2]). *Assume that Hypotheses 2.3 and 2.4 are satisfied.*

Then there exist $M > 0$ and the continuous functions $x(.) : S \to L^1(I,X)$, $h(.) : S \to C(I,X)$ such that for any $s \in S$ $(x(s)(.), h(s)(.))$ is a trajectory-selection of (1.1) satisfying for any $(t,s) \in I \times S$

$$x(s)(0) = a_0(s), \tag{2.6}$$

$$\begin{array}{l} ||x(s)(t) - y(s)(t)|| \leq \\ M[c(s) + ||a_0(s) - y(s)(0)|| + \int_0^t p(s)(u)du]. \end{array} \tag{2.7}$$

III. THE MAIN RESULT

Our object of study is the reachable set of (1.1) defined by

$$R_F(T, X_0) := \{x(T); \quad x(.) \text{ is a mild solution of (1.1)}\}.$$

We consider a certain variational semilinear differential inclusion and we shall prove that the reachable set of this variational inclusion from a derived cone $C_0 \subset X$ to X_0 at time T is a derived cone to the reachable set $R_F(T, X_0)$.

Throughout in this section we assume

Hypothesis 3.1. i) *Hypothesis 2.3 is satisfied and $X_0 \subset X$ is a closed set.*

ii) *$(z(.), f(.)) \in C(I,X) \times L^1(I,X)$ is a trajectory-selection pair of (1.1) and a family $P(t,.) : X \to \mathcal{P}(X)$, $t \in I$ of convex processes satisfying for almost all $t \in I$ the condition*

$$P(t,u) \subset Q_{f(t)}F(t,.)(z(t);u) \quad \forall\, u \in dom(P(t,.)), \tag{3.1}$$

is assumed to be given and defines the variational inclusion

$$v' \in Av + P(t,v). \tag{3.2}$$

Remark 3.2. We note that for any set-valued map $F(.,.)$, one may find an infinite number of families of convex process $P(t,.)$, $t \in I$, satisfying condition (3.1); in fact any family of closed convex subcones of the quasitangent cones, $\overline{P}(t) \subset Q_{(z(t),f(t))}graph(F(t,.))$, defines the family of closed convex process

$$P(t,u) = \{v \in X;\ (u,v) \in \overline{P}(t)\}, \quad u, v \in X,\ t \in I$$

that satisfy condition (3.1). One is tempted, of course, to take as an "intrinsic" family of such closed convex process, for example Clarke's convex-valued directional derivatives $C_{f(t)}F(t,.)(z(t);.)$.

We recall (e.g. [1]) that, since $F(t,.)$ is assumed to be Lipschitz a.e. on I, the quasitangent directional derivative is given by

$$\begin{array}{l} Q_{f(t)}F(t,.)((z(t);u)) = \\ \{w \in X;\ \lim_{\theta \to 0+} \frac{1}{\theta} d(f(t) + \theta w, F(t, z(t) + \theta u)) = 0\}. \end{array} \tag{3.3}$$

We are able now to prove the main result of this paper.

Theorem 3.3. *Assume that Hypothesis 3.1 is satisfied and let $C_0 \subset X$ be a derived cone to X_0 at $z(0)$. Then the reachable set $R_P(T, C_0)$ of (3.2) is a derived cone to $R_F(T, X_0)$ at $z(T)$.*

Proof. In view of Definition 2.1, let $\{v_1, ..., v_m\} \subset R_P(T, C_0)$, hence such that there exist the trajectory-selection pairs $(u_1(.), g_1(.)), ..., (u_m(.), g_m(.))$ of the variational inclusion (3.2) such that

$$u_j(T) = v_j, \quad u_j(0) \in C_0, \quad j = 1, 2, ..., m \tag{3.4}$$

Since $C_0 \subset X$ is a derived cone to X_0 at $z(0)$, there exists a continuous mapping $a_0 : S = [0, \theta_0]^m \to X_0$ such that

$$a_0(0) = z(0), \quad Da_0(0)s = \sum_{j=1}^{m} s_j u_j(0) \quad \forall\, s \in \mathbf{R}_+^m. \tag{3.5}$$

Further on, for any $s = (s_1, ..., s_m) \in S$ and $t \in I$ we denote

$$\begin{array}{c} y(s)(t) = z(t) + \sum_{j=1}^m s_j u_j(t), \\ g(s)(t) = f(t) + \sum_{j=1}^m s_j g_j(t), \\ p(s)(t) = d(g(s)(t), F(t, y(s)(t))) \end{array} \tag{3.6}$$

and prove that $y(.)$, $p(.)$ satisfy the hypothesis of Theorem 2.5.

Using the lipschitzianity of $F(t,.,.)$ we have that for any $s \in S$, the measurable function $p(s)(.)$ in (3.6) it is also integrable.

$$\begin{array}{l} p(s)(t) = d(g(s)(t), F(t, y(s)(t))) \leq \sum_{j=1}^m s_j ||g_j(t)|| \\ + d_H(F(t, z(t)), F(t, y(s)(t))) \leq \sum_{j=1}^m s_j ||g_j(t)|| + \\ L(t) \sum_{j=1}^m s_j ||u_j(t)||. \end{array}$$

Moreover, the mapping $s \to p(s)(.) \in L^1(I, \mathbf{R}_+)$ is continuous (in fact Lipschitzian) since for any $s, s' \in S$ one may write succesively

$$\begin{array}{l} ||p(s)(.) - p(s')(.)||_1 = \int_0^T ||p(s)(t) - p(s')(t)||dt \leq \\ \int_0^T [||g(s)(t) - g(s')(t)|| + d_H(F(t, y(s)(t)), \\ F(t, y(s')(t))))]dt \leq ||s - s'||(\sum_{j=1}^m \int_0^T [||g_j(t)|| + \\ L(t)||u_j(t)||]dt) \end{array}$$

Let us define $c(.) : S \to [0, \infty)$, $c(s) := ||s||^2$, $s_0 = 0 \in S$. It follows from Theorem 2.5 the existence of a continuous function $x(.) : S \to C(I,X)$ such that for any $s \in S$, $x(s)(.)$ is a mild solution of (1.1) with the properties (2.6)-(2.7).

Finally, we define the function $a(.) : S \to R_F(T, X_0)$ by

$$a(s) = x(s)(T) \quad \forall s \in S.$$

Obviously, $a(.)$ is continuous on S and satisfies $a(0) = z(T)$.

To end the proof we need to show that $a(.)$ is differentiable at $s_0 = 0 \in S$ and its derivative is given by

$$Da(0)(s) = \sum_{j=1}^{m} s_j v_j \quad \forall\, s \in \mathbf{R}_+^m$$

which is equivalent with the fact that:

$$\lim_{s\to 0} \frac{1}{||s||}(||a(s) - a(0) - \sum_{j=1}^{m} s_j v_j||) = 0. \tag{3.7}$$

From (2.7) we obtain

$$\frac{1}{||s||}||a(s) - a(0) - \sum_{j=1}^{m} s_j v_j|| \le \frac{1}{||s||}||x(s)(T) -y(s)(T)|| \le M||s|| + \frac{M}{||s||}||a_0(s) - z(0)- \sum_{j=1}^{m} s_j u_j(0)|| + M\int_0^T \frac{p(s)(u)}{||s||}du$$

and therefore in view of (3.5), relation (3.7) is implied by the following property of the mapping $p(.)$ in (3.6)

$$\lim_{s\to 0} \frac{p(s)(t)}{||s||} = 0 \quad a.e.\,(I). \tag{3.8}$$

In order to prove the last property we note since $P(t,.)$ is a convex process for any $s \in S$ one has

$$\sum_{j=1}^{m} \frac{s_j}{||s||} g_j(t) \in P(t, \sum_{j=1}^{m} \frac{s_j}{||s||} u_j(t)) \subset Q_{f(t)}F(t,.)(z(t); \sum_{j=1}^{m} \frac{s_j}{||s||} u_j(t)) \quad a.e.\,(I).$$

Hence by (3.3) we obtain

$$\lim_{h\to 0+} \frac{1}{h} d(f(t) + h\sum_{j=1}^{m} \frac{s_j}{||s||} g_j(t), F(t, z(t)+ h\sum_{j=1}^{m} \frac{s_j}{||s||} u_j(t))) = 0. \tag{3.9}$$

In order to prove that (3.9) implies (3.8) we consider the compact metric space $S_+^{m-1} = \{\sigma \in \mathbf{R}_+^m; ||\sigma|| = 1\}$ and the real function $\phi_t(.,.) : (0, \theta_0] \times S_+^{m-1} \to \mathbf{R}_+$ defined by

$$\phi_t(h,\sigma) := \frac{1}{h} d(f(t) + h\sum_{j=1}^{m} \sigma_j g_j(t), F(t, z(t) + h\sum_{j=1}^{m} \sigma_j u_j(t))) \tag{3.10}$$

where $\sigma = (\sigma_1, ..., \sigma_m)$ and which according to (3.9) has the property

$$\lim_{\theta\to 0+} \phi_t(\theta,\sigma) = 0 \quad \forall\, \sigma \in S_+^{m-1}\ a.e.\,(I). \tag{3.11}$$

Using the fact that $\phi_t(\theta,.)$ is Lipschitzian and the fact that S_+^{m-1} is a compact metric space, from (3.11) it follows easily (e.g. Proposition 4.4 in [9]) that

$$\lim_{\theta\to 0+} \max_{\sigma\in S_+^{m-1}} \phi_t(\theta,\sigma) = 0$$

which implies the fact that

$$\lim_{s\to 0} \phi_t(||s||, \frac{s}{||s||}) = 0 \quad a.e.\,(I)$$

and the proof is complete.

An application of Theorem 3.3 concerns local controllability of the semilinear differential inclusion in (1.1) along a reference trajectory, $z(.)$ at time T, in the sense that

$$z(T) \in Int(R_F(T, X_0)).$$

We recall that if $X = \mathbf{R}^n$ then (2.5) is a Cauchy problem associated to an affine (linear nonhomogenous) differential equation and its solution (2.4) is obtained with the variation of constants method. In this case $G(t) = \exp(tA)$, $A \in L(\mathbf{R}^n, \mathbf{R}^n)$, $t \in I$.

Theorem 3.4. *Let $X = \mathbf{R}^n$, $z(.), F(.,.)$ and $P(.,.)$ satisfy Hypothesis 3.1 and let $C_0 \subset X$ be a derived cone to X_0 at $z(0)$. If, the variational semilinear differential inclusion in (3.2) is controllable at T in the sense that $R_P(T, C_0) = \mathbf{R}^n$, then the differential inclusion (1.1) is locally controllable along $z(.)$ at time T.*

Proof. The proof is a straightforward application of Lemma 2.2 and of Theorem 3.3.

References

[1] J.P. Aubin and H. Frankowska, *Set-valued Analysis*, Birkhauser, Boston, Berlin; 1990.

[2] A. Cernea, Continuous version of Filippov's theorem for a semilinear differential inclusion, *Stud. Cerc. Mat.*, vol. 49, 1997, pp. 319-330.

[3] A. Cernea, Some qualitative properties of the solution set of an infinite horizon operational differential inclusion, *Revue Roumaine Math. Pures Appl.*, vol. 43, 1998, pp. 317-328.

[4] A. Cernea, Local controllability of hyperbolic differential inclusions via derived cones, *Revue Roumaine Math. Pures Appl.*, vol. 47, 2002, pp. 21-31.

[5] A. Cernea, On the relaxation theorem for semilinear differential inclusions in Banach spaces, *Pure Math. Appl.*, vol. 13, 2002, pp. 441-445.

[6] A. Cernea, Derived cones to reachable sets of differential-difference inclusions, *Nonlinear Anal. Forum*, vol. 11, 2006, pp. 1-13.

[7] A. Cernea, Derived cones to reachable sets of discrete inclusions, *Nonlinear Studies*, vol. 14, 2007. pp. 177-187.

[8] A. Cernea, On the solution set of some classes of nonconvex nonclosed differential inclusions, *Portugaliae Math.*, vol. 65, 2008, pp. 485-496.

[9] A. Cernea and Ş. Mirică, Derived cones to reachable sets of differential inclusions, *Mathematica (Cluj)*, vol. 40(63), 1998, pp. 35-62.

[10] F. S. De Blasi and G. Pianigiani, Evolutions inclusions in non separable Banach spaces, *Comment. Math. Univ. Carolinae*, vol. 40, 1999, pp. 227-250.

[11] F. S. De Blasi, G. Pianigiani and V. Staicu, Topological properties of nonconvex differential inclusions of evolution type, *Nonlinear Anal.*, vol. 24, 1995, pp. 711-720.

[12] H. Frankowska, A priori estimates for operational differential inclusions, *J. Diff. Eqs.*, vol. 84, 1990, pp. 100-128.

[13] M.R. Hestenes, *Calculus of Variations and Optimal Control Theory*. Wiley, New York; 1966.

[14] Ş. Mirică, New proof and some generalizations of the Minimum Principle in optimal control, *J. Optim. Theory Appl.*, vol. 74, 1992, pp. 487-508.

[15] Ş. Mirică, Intersection properties of tangent cones and generalized multiplier rules, *Optimization*, vol. 46, 1999, pp. 135-163.

[16] V. Staicu, Continuous selections of solutions sets to evolution equations, *Proc. Amer. Math. Soc.*, vol. 113, 1991, pp. 403-413.