Algorithm to Compute Minimal Nullspace Basis of a Polynomial Matrix

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***Abstract*— In this paper we propose a numerical algorithm to compute the minimal nullspace basis of a univariate polynomial matrix of arbitrary size. In order to do so a sequence of structured matrices is obtained from the given polynomial matrix. The nullspace of the polynomial matrix can be computed from the nullspaces of these structured matrices.**

***Index Terms*— Singular Value Decomposition (SVD), polynomial matrix, nullspace of a polynomial matrix, minimal polynomial basis**

I. INTRODUCTION

Computation of the nullspace of a polynomial matrix is a very important problem. We give an algorithm to compute the minimal polynomial basis for a given polynomial matrix. Before stating the problem, we introduce some notation and preliminaries required for this paper.

Let $\mathbb{R}^{p\times q}$ denote the set of all matrices of size $p \times q$ with entries from the field of real numbers. For $A \in \mathbb{R}^{p\times q}$, let $A = U\Sigma V^T$ be the Singular Value Decomposition (SVD) of A. Rank of A is then equal to the number nonzero singular values of A. Let r be the rank of A. Then $A = U_0\Sigma_0V_0^T$ is called the condensed SVD (see [1]) of A where Σ_0 is a square diagonal matrix of size $r \times r$ with nonzero singular values on its diagonal, U_0 is an isometry that is obtained by taking the first r columns of U which correspond to the nonzero singular values and V_0 is obtained from V in similar way. In standard MATLAB notation $U_0 = U(:, 1 : r)$.

Let $\mathbb{R}[s]$ denote the ring of polynomials in a single variable s with coefficients from the real field. A polynomial vector $v \in \mathbb{R}^{\mathtt{w}}[s]$ is a vector of size $\mathtt{w}$ with each entry being a polynomial. The degree of a vector $v \in \mathbb{R}^{\mathtt{w}}[s]$ is the maximum amongst degrees of its polynomial components. Alternatively we can write v as $v = v_0+v_1s+v_2s^2+\cdots+v_ns^n$ where n is the degree of v and $v_i \in \mathbb{R}^{\mathtt{w}}$ for $i = 0, 1, \ldots, n$ and $v_n \neq 0$. Thus v is a polynomial of degree n with the coefficients being the vectors from $\mathbb{R}^{\mathtt{w}}$. Similarly a polynomial matrix $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[s]$ is a matrix of size $\mathtt{g} \times \mathtt{w}$ with the entries from $\mathbb{R}[s]$. The degree of a polynomial matrix is the maximum of the degrees amongst its polynomial entries. As in the case of vectors, a polynomial matrix can be written in polynomial form as

$$R = R_0 + R_1s + R_2s^2 + \cdots + R_ns^n \tag{1}$$

where n is the degree of R and $R_i \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}$ for $i = 0, 1, \ldots, n$ are the coefficient matrices and $R_n \neq 0$. The matrix polynomial representation of polynomial matrices plays a key role in the discussion that follows.

A set of vectors $\{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^{\mathtt{w}}[s]$ is called a polynomially independent set if the polynomial combination $a_1(s)v_1+a_2(s)v_2+\cdots+a_n(s)v_n = 0$ implies that $a_i(s) = 0$ identically for all $i = 1, 2, \ldots, n$. If a set of polynomial vectors is not polynomially independent, then it is called polynomially dependent.

Rank (or normal rank) of a polynomial matrix $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[s]$ is defined as the number $\max_{\lambda\in\mathbb{R}}$ rank $R(\lambda)$. It can be shown that this number is same as the number of polynomially independent rows in R. Further if the matrix R is considered to be an element of $\mathbb{R}^{\mathtt{g}\times\mathtt{w}}(s)$ (the set of matrices with entries from $\mathbb{R}(s)$ which is the quotient field of ring of polynomials $R[s]$), then the above notion of rank matches the notion of rank on this quotient field.

The nullspace of R is defined as $\mathcal{N} = \{v \in \mathbb{R}^{\mathtt{w}}[s] \mid Rv = 0\}$. We now define minimal polynomial basis for the nullspace $\mathcal{N}$.

Definition 1.1: Let $B = \{m_1(s), m_2(s), \ldots, m_k(s)\} \subset \mathbb{R}^{\mathtt{w}}[s]$ be a generating set for $\mathcal{N}$ with degrees $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_k$. This generating set is called minimal basis (see [2]) if for any other basis of $\mathcal{N}$ with degrees $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_k$, it turns out that $\delta_i \leq \gamma_i$ for $i = 1, 2, \ldots, k$.

The numbers $\delta_1, \delta_2, \ldots, \delta_k$ are called right minimal indices of $\mathcal{N}$ (see [3]). Given a minimal polynomial basis for $\mathcal{N}$, degree of $\mathcal{N}$ is defined as the maximum amongst the degrees of the minimal polynomial basis vectors, that is, δ_k. We now state the problem.

Problem Statement 1.2: Given a polynomial matrix $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[s]$ with rank r compute a minimal polynomial basis for the nullspace $\mathcal{N}$ of R.

The paper is organized as follows: the rest of this section is devoted to discussing the related work in the literature. In Section II we discuss the main results of the paper. In Section III we propose an algorithm to compute the minimal nullspace basis of the given polynomial matrix. Further we illustrate the algorithm with some numerical examples in Section IV. Finally we conclude in Section V.

Computation of the minimal nullspace basis has a lot of applications in systems theory where the systems are described using the polynomial matrices. See for instance [4], [5]. Recently an application of nullspace basis in fault detection methods was reported in [6]. In literature there are several ways to compute the minimal polynomial nullspace basis of a given polynomial matrix.

One of the approaches to deal with polynomial matrices is to convert the given polynomial matrix into matrix pencil (this is also known in literature as linearization). In [7] and

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[8] the given polynomial matrix is converted to a matrix pencil and then the problem of computing nullspace basis is solved using this matrix pencil. After computing a nullspace basis for the matrix pencil, this result is then converted back into the polynomial domain. In [9] two vector spaces of matrix pencils that generalize the first and the second companion form were introduced. In [3], it was shown that these linearizations can be used to obtain the left and right minimal indices and minimal bases.

As opposed to pencil algorithms, in nonpencil algorithms (see for instance[10], [11]), the problem is converted to the problem of structured matrices obtained from the polynomial matrix. The manipulations are done on these structured matrices in order to compute a minimal nullspace basis.

Another approach to compute the nullspace basis is using randomized algorithm as proposed in [12], [13]). The problem of computing the nullspace of a polynomial matrix is reduced to polynomial matrix multiplication. The randomized algorithm is used to compute the minimal or small degree basis elements in the nullspace of the given polynomial matrix.

II. Main Results

The algorithm we develop to compute the minimal nullspace basis is a non pencil algorithm. The main idea of computing the nullspace of a polynomial matrix involves constructing a sequence of structured real matrices from the given polynomial matrix. The nullspace of the given polynomial matrix is related to the nullspaces of these real structured matrices. Let $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[s]$ of degree n be represented in matrix polynomial form as in the equation (1). We construct matrix $A_0 \in \mathbb{R}^{(n+1)\mathtt{g}\times\mathtt{w}}$ from R as $A_0 = \begin{bmatrix} R_0 \\ R_1 \\ \vdots \\ R_n \end{bmatrix}$.
A sequence of the structured matrices is obtained as follows:

$$A_1 = \left[\begin{array}{c|c} A_0 & 0 \\ \hline 0 & A_0 \end{array}\right], \quad A_2 = \left[\begin{array}{c|cc} A_0 & 0 & 0 \\ \hline 0 & & \\ 0 & & A_1 \end{array}\right], \cdots \tag{2}$$

where 0's in the above equation are all zero matrices of size $\mathtt{g} \times \mathtt{w}$. In general for some $i \geq 0$, $A_i \in \mathbb{R}^{(n+i+1)\mathtt{g}\times(i+1)\mathtt{w}}$. Let $\mathcal{K}_i = \ker(A_i)$ and $d_i = \dim(\mathcal{K}_i)$. A result about the sequence $\{d_i\}_{i=0,1,\ldots}$ is now proved which is essential in the computation of the basis of nullspace of R.

Lemma 2.1: Let $\mathcal{K}_i = \ker(A_i)$ and $d_i = \dim(\mathcal{K}_i)$. Let A_j be as in equation (2) where j is the smallest index for which $d_j > 0$. Then

$$d_{j+k} \geq (k+1)d_j \quad \text{for } k \geq 1.$$

Proof: Let $V = \{v_1, v_2, \ldots, v_{d_j}\}$ be a basis for $\mathcal{K}_j$, where $v_i \in \mathbb{R}^{(j+1)\mathtt{w}}$ for $i = 1, \ldots, d_j$. Then $p_1^1 = \begin{bmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and $p_2^1 = \begin{bmatrix} 0 \\ v_1 \\ \vdots \\ 0 \end{bmatrix}$, and so on up to $p_k^1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ v_1 \end{bmatrix}$ are all in the kernel of A_{k+j}, where 0 is a zero vector of size $\mathtt{w}$ and each of $p_i^1 \in \mathbb{R}^{(j+k+1)\mathtt{w}}$ for $i = 1, 2, \ldots, k$. Also note that $\{p_1^1, p_2^1, \ldots, p_k^1\}$ forms a linearly independent set. Since this is true for all v_i's in the kernel of A_j, and V is a linearly independent set, the set $\{p_m^n\}_{m=1,\ldots,k,n=1,\ldots,d_j}$ is a linearly independent set. Hence kernel of A_{j+k} contains at least $(k+1)d_j$ linearly independent vectors. ■

We now illustrate the relation of the vectors in the nullspace $\mathcal{N}$ with the vectors in the kernel $\mathcal{K}_i$. Let for some $i \geq 0$, $v \in \mathbb{R}^{(i+1)\mathtt{w}}$ be such that $v \in \mathcal{K}_i$. Partition this vector v as $v = [v_0 \;\; v_1 \;\; \cdots \;\; v_i]^T$ where $v_j = v(j\mathtt{w}+1 : (j+1)\mathtt{w})$ for $j = 0, 1, \ldots, i$. Then it is easy to see that the polynomial vector $x(s) \in \mathbb{R}^{\mathtt{w}}[s]$ defined as $x(s) = \sum_{j=0}^{i} v_j s^j$ satisfies $Rx(s) = 0$. That is $x(s) \in \mathcal{N}$. Thus a vector $v \in \mathcal{K}_j$ is identified with a polynomial vector $x(s) \in \mathcal{N}$. From Lemma 2.1 it is clear that $\begin{bmatrix} v \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v \end{bmatrix} \in \mathcal{K}_{j+1}$. These vectors present in $\mathcal{K}_{j+1}$ can now be identified with the polynomial vectors $x(s)$ and $sx(s)$ in $\mathcal{N}$ respectively. Thus Lemma 2.1 gives a limit based on all the polynomially dependent vectors present in $\mathcal{K}_{j+1}$ that arise from vectors in $\mathcal{K}_j$.

In the following theorem we give the minimum stage i for which all the polynomially independent vectors in the nullspace $\mathcal{N}$ can be computed from the kernel $\mathcal{K}_i$. Before proceeding to the next result we define what we mean by a steadily increasing sequence of real numbers.

Definition 2.2: Let $\{a_n\}_{n=0,1,\ldots}$ be a non-decreasing sequence of real numbers. Then we say that the sequence $\{a_n\}_{n=0,1,\ldots}$ a steadily-increasing sequence if there exists a non-negative integer $n_0 \in \mathbb{N}$ and $c > 0$, a positive constant, such that $a_{m+1} - a_m = c$, for all $m \geq n_0$.

Theorem 2.3: Let $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[s]$ and construct A_i as in the equation (2) where $A_i \in \mathbb{R}^{(n+i+1)\mathtt{g}\times(i+1)\mathtt{w}}$. Let $\mathcal{K}_i = \ker A_i$ and $d_i = \dim(\mathcal{K}_i)$. Then sequence $\{d_i\}_{i=0,1,\ldots}$ a steadily-increasing sequence and there exists $n_0 \in \mathbb{N}$ such that $d_{i+1} - d_i = k$ for $i \geq n_0$, where k is the number of polynomially independent generators for the nullspace of the polynomial matrix R.

Proof: We first prove that the sequence $\{d_i\}_{i=0,1,\ldots}$ is a nondecreasing sequence. Let j be the smallest index such that $d_j > 0$. Let $d_{j+1} = 2d_j + \gamma_{j+1}$. This implies $d_{j+1} \geq d_j$. If $\gamma_{j+1} > 0$, then new polynomially independent vector is added in the basis of $\mathcal{N}$. If $\gamma_{j+1} = 0$ then no new polynomially independent vectors are added in the basis of $\mathcal{N}$. Further $d_{j+2} = 3d_j + 2\gamma_{j+1} + \gamma_{j+2}$. This implies $d_{j+2} \geq d_{j+1}$. Depending on whether γ_{j+2} is zero or not, we can conclude if polynomially independent vectors are added in the basis of $\mathcal{N}$. Continuing this way it can be shown that the sequence $\{d_i\}_{i=0,1,\ldots}$ is a nondecreasing sequence. In general for $\ell \geq 1$

$$d_{j+\ell} = (\ell+1)d_j + \sum_{i=1}^{\ell} (\ell - i + 1)\gamma_{j+i}. \tag{3}$$

Note that at the stage $j+\ell$, the number of polynomially independent vectors added in the basis of $\mathcal{N}$ are given by $d_j+\sum_{i=1}^{\ell}\gamma_{j+i}$ for $\ell \geq 1$. Let $n_0 \in \mathbb{N}$ be such that $k = d_j+\sum_{i=1}^{n_0-j}\gamma_{j+i}$. Then at this stage n_0, all the polynomially independent vectors are added in the the basis of $\mathcal{N}$. Then from equation (3) we get, $d_{n_0+1}-d_{n_0} = d_j+\sum_{i=1}^{n_0-j}\gamma_{j+i} = k$. Since $\gamma_{n_0+1}=0$. In fact $d_{n_0+\ell}-d_{n_0+\ell-1}=k$ as $\gamma_{n_0+\ell}=0$ for $\ell = 1, 2, \ldots$. This proves the theorem. ■

Remark 2.4: Let j be the smallest non-negative integer so that $d_j > 0$. Then $d_0 = d_1 = \cdots = d_{j-1} = 0$. In fact the sequence $\{d_i\}_{i=0,1,2}$ is a strictly increasing sequence after j^{th} term.

Remark 2.5: From Theorem 2.3 it is clear that we pick the polynomially independent vectors in the nullspace of R in an increasing order of degrees without missing any polynomial vector of smaller degree. Thus the basis that is obtained using the theorem is a minimal basis for the nullspace of the polynomial matrix R. Further the degree of the minimal polynomial basis is also obtained from the above theorem.

III. An Algorithm to Compute Minimal Nullspace Basis

In this section we discuss an algorithm to compute the minimal nullspace basis of a given polynomial matrix $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[s]$ with rank r. We extract a minimal polynomial basis of $\mathcal{N}$ from the kernels $\mathcal{K}_i$. In order to achieve greater numerical efficiency we use the matrices obtained from the SVD of A_0 for the computations instead of using matrices A_i themselves. We now describe the construction of a sequence of matrices J_i which are useful in further discussion.

Construction 3.1: For a given polynomial matrix $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[s]$ with degree n, form $A_0 \in \mathbb{R}^{(n+1)\mathtt{g}\times\mathtt{w}}$ as in equation (2). Let $A_0 = U_0\Sigma_0V_0^T$ be the condensed SVD of A_0. Let $J_0 = U_0$ and rank $(U_0) = r_0$. Further construct $J_1 = \left[\begin{array}{c|c} U_0 & 0 \\ \hline 0 & U_0 \end{array}\right] \in \mathbb{R}^{(n+2)\mathtt{g}\times 2r_0}$ where 0's are zero matrices of size $\mathtt{g}\times r_0$. Let $J_1 = U_1\Sigma_1V_1^T$ be the condensed SVD of J_1. Let $r_1 = \text{rank}(J_1)$. Now J_2 is constructed as $J_2 = \left[\begin{array}{c|c} U_0 & 0 \\ \hline 0 & U_1 \\ 0 & \end{array}\right] \in \mathbb{R}^{(n+3)\mathtt{g}\times(r_0+r_1)}$ where 0's below U_0 are zero matrices of size $\mathtt{g}\times r_0$ and 0 above U_1 is the zero matrix of size $\mathtt{g}\times r_1$. Let $J_2 = U_2\Sigma_2V_2^T$ be the condensed SVD and $r_2 = \text{rank}(J_2)$. In general, let

$$J_i = \left[\begin{array}{c|c} U_0 & 0 \\ \hline 0 & \\ \vdots & U_{i-1} \\ 0 & \end{array}\right] \in \mathbb{R}^{(n+i+1)\mathtt{g}\times(r_0+r_{i-1})} \qquad (4)$$

where 0's below U_0 are zero matrices of size $\mathtt{g}\times r_0$ and 0 above U_{i-1} is the zero matrix of size $\mathtt{g}\times r_{i-1}$. Let $r_i = \text{rank}(J_i)$.

We now describe the relation between matrices J_i and matrices A_i. As in Construction 3.1 let $A_0 = U_0\Sigma_0V_0^T$ be the condensed SVD of A_0. Then we can write A_1 as

$$A_1 = J_1\left[\begin{array}{c|c} \Sigma_0 & \\ \hline & \Sigma_0 \end{array}\right]\left[\begin{array}{c|c} V_0^T & \\ \hline & V_0^T \end{array}\right]. \qquad (5)$$

Clearly the above relation is not the SVD of the matrix A_1 as J_1 is not an isometry. Further since the condensed SVD of A_0 is considered, the second and third matrices in the product on the RHS of equation (5) are always full rank. Thus the rank drop in A_1 is reflected in the rank drop in J_1. Stated otherwise A_1 loses rank if and only if J_1 loses rank. Thus the information of the rank drop in A_1 can be obtained just by studying the rank of J_1. Further, we do similar analysis for A_2.

$$A_2 = \left[\begin{array}{c|cc} U_0 & 0 & 0 \\ \hline & & \\ & & J_1 \\ 0 & & \\ 0 & & \end{array}\right]\left[\begin{array}{c|c|c} \Sigma_0 & & \\ \hline & \Sigma_0 & \\ \hline & & \Sigma_0 \end{array}\right]\left[\begin{array}{c|c|c} V_0^T & & \\ \hline & V_0^T & \\ \hline & & V_0^T \end{array}\right]. \qquad (6)$$

The above equation clearly is not an SVD of A_2. The first matrix in the product on the RHS of equation (6) can be written as

$$\left[\begin{array}{c|cc} U_0 & 0 & 0 \\ \hline & & \\ & & J_1 \\ 0 & & \\ 0 & & \end{array}\right] = J_2\left[\begin{array}{c|cc} I & 0 & 0 \\ \hline 0 & & \\ 0 & & \Sigma_1 \end{array}\right]\left[\begin{array}{c|cc} I & 0 & 0 \\ \hline 0 & & \\ 0 & & V_1^T \end{array}\right]. \qquad (7)$$

From equations (6) and (7) it is clear that the rank drop in A_2 is reflected in the rank drop in J_2 as all the other matrices in the product are always full rank. We now illustrate the relationship between $\ker J_2$ and $\mathcal{K}_2$. Let $v \in \mathbb{R}^{r_0+r_1}$ be such that $v \in \ker J_2$. Then we first compute the corresponding nullspace element, say $y \in \mathbb{R}^{3r_0}$, of the matrix on the LHS of the equation (7). For y to be in the nullspace of that matrix, y has to satisfy the following equation:

$$\left[\begin{array}{c|cc} I & 0 & 0 \\ \hline 0 & & \\ 0 & & \Sigma_1 \end{array}\right]\left[\begin{array}{c|cc} I & 0 & 0 \\ \hline 0 & & \\ 0 & & V_1^T \end{array}\right] y = v. \qquad (8)$$

Since Σ_1 is a diagonal matrix and V_1 is an isometry, we can compute the vector y very easily from the above equation as

$$y = \left[\begin{array}{c|cc} I & 0 & 0 \\ \hline 0 & & \\ 0 & & V_1^T \end{array}\right]^T\left[\begin{array}{c|cc} I & 0 & 0 \\ \hline 0 & & \\ 0 & & \Sigma_1^{-1} \end{array}\right] v. \qquad (9)$$

Since Σ_1 is a diagonal matrix, the number of flops that are required in the above equation are just equal to the flops required to multiply a matrix to a vector. Further both the identity matrices in equation (8) are of the size $r_0 \times r_0$. Hence first r_0 entries in the vector v are not required in the computation of the vector y. Thus matrix vector multiplication in equation (8) is that of a matrix of size $r_1 \times r_1$ (the size of V^T) and a vector of size r_1 (the size of truncated v required in the computation of y). Thus the system of equations in (8) can be solved in a numerically efficient way. From y we now compute $x \in \mathbb{R}^{3\mathtt{w}}$, the corresponding element

$$A_i = \left[\begin{array}{c|c|c|c} U_0 & 0 & & 0 \\ & U_0 & & 0 \\ & & \ddots & \vdots \\ 0 & & & \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ & \vdots & & U_0 \\ 0 & 0 & & \end{array}\right] \left[\begin{array}{c|c|c|c} \Sigma_0 & & & \\ \hline & \Sigma_0 & & \\ \hline & & \ddots & \\ \hline & & & \Sigma_0 \end{array}\right] \left[\begin{array}{c|c|c|c} V_0^T & & & \\ \hline & V_0^T & & \\ \hline & & \ddots & \\ \hline & & & V_0^T \end{array}\right]. \tag{10}$$

in $\mathcal{K}_2$, in a similar way. The nullspace vector $x \in \mathcal{K}_2$ is obtained as

$$x = \left[\begin{array}{c|c|c} V_0^T & & \\ \hline & V_0^T & \\ \hline & & V_0^T \end{array}\right]^T \left[\begin{array}{c|c|c} \Sigma_0^{-1} & & \\ \hline & \Sigma_0^{-1} & \\ \hline & & \Sigma_0^{-1} \end{array}\right] y. \tag{11}$$

Note that Σ_0 is a diagonal matrix and we obtain x from y in a numerically efficient way. Thus in the case when $i = 2$, we obtain the nullspace vector of A_2 from that of J_2 in two steps.

We now generalize the procedure to obtain nullspace element of A_i from that of J_i. In order to do so, we start with the following recurrence relation.

$$J_{i+1} \begin{bmatrix} I & 0 \\ 0 & \Sigma_i \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & V_i^T \end{bmatrix} = \left[\begin{array}{c|c} U_0 & 0 \\ \hline 0 & \\ \vdots & J_i \\ 0 & \end{array}\right] \tag{12}$$

Now for some $i \in \mathbb{N}$, we write A_i as in equation (10). Further using recurrence relation (12) we have,

$$\left[\begin{array}{c|c|c|c} U_0 & 0 & & 0 \\ & U_0 & & 0 \\ & & \ddots & \vdots \\ 0 & & & \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ & \vdots & & U_0 \\ 0 & 0 & & \end{array}\right] = J_i P_{i-1} P_{i-2} \cdots P_1, \tag{13}$$

where matrix P_j, for $j = 1, \ldots, i-1$, is defined as

$$P_j = \begin{bmatrix} \overbrace{\begin{matrix} I & & \\ & \ddots & \\ & & I \end{matrix}}^{i-j \text{ times}} & \\ & \Sigma_j \end{bmatrix} \begin{bmatrix} \overbrace{\begin{matrix} I & & \\ & \ddots & \\ & & I \end{matrix}}^{i-j \text{ times}} & \\ & V_j^T \end{bmatrix}. \tag{14}$$

From equation (10) and (13) it is clear that the rank loss in A_i is reflected in the rank loss of J_i since Σ_j's are diagonal matrices with nonzero entries on the diagonal and V_j's are isometries. We now illustrate the method to compute the corresponding vector in $\mathcal{K}_i$ to a vector in $\ker J_i$. Let $v_i \in \mathbb{R}^{r_0 + r_{i-1}}$ be such that $J_i v_i = 0$. Then we first compute $y \in \mathbb{R}^{(i+1)r_0}$ such that y is in the nullspace of the matrix on the LHS of equation (13). In order to do so, we go through a series of steps as follows. Given $v_i \in \mathbb{R}^{r_0 + r_{i-1}}$, we first compute $v_{i-1} \in \mathbb{R}^{r_0 + r_{i-1}}$ such that v_{i-1} is in the nullspace of the matrix $J_i P_{i-1}$. Since Σ_{i-1} is a diagonal matrix and V_{i-1}^T is an isometry, v_{i-1} can be obtained from v_i as a matrix vector multiplication. Note that since the identity matrices in P_{i-1} are of the size $r_0 \times r_0$, the first r_0 entries in v_i are not required in the computation of v_{i-1}. Further we compute v_{i-2} such that it is in the nullspace of the matrix $J_i P_{i-1} P_{i-2}$. We compute v_{i-2} from v_{i-1} in a similar way. Note here that the identity matrices in P_{i-2} are of the size of $2r_0 \times 2r_0$. Thus first $2r_0$ entries in v_{i-1} are not required in the computation of v_{i-2}. Continuing in this way we compute $y = v_1$ which is in the nullspace of $J_i P_{i-1} P_{i-2} \cdots P_1$. Now we compute $x \in \mathbb{R}^{(i+1)\mathtt{w}}$ from y such that $A_i x = 0$. In order to do so we use relation in equation (10). Thus we have

$$x = \left[\begin{array}{c|c|c|c} V_0^T & & & \\ \hline & V_0^T & & \\ \hline & & \ddots & \\ \hline & & & V_0^T \end{array}\right]^T \left[\begin{array}{c|c|c|c} \Sigma_0^{-1} & & & \\ \hline & \Sigma_0^{-1} & & \\ \hline & & \ddots & \\ \hline & & & \Sigma_0^{-1} \end{array}\right] y. \tag{15}$$

Since Σ_0 is a diagonal matrix, we obtain x from y in a numerically efficient way. We obtain y from v_{i-1} in $i-1$ steps and in the final step we obtain x from y. Thus we require i steps in order to compute the nullspace element in $\mathcal{K}_i$ given a nullspace element of J_i.

We now propose an algorithm to compute a minimal polynomial basis for the nullspace $\mathcal{N}$ of a given polynomial matrix $R \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}[s]$. In order to compute this basis, we use the sequence of matrices J_i obtained from R. Let $\delta_i = \dim(\ker J_i)$. In Theorem 3.2 we show that δ_i denotes the number of polynomially independent vectors upto degree i in $\mathcal{N}$. In Theorem 2.3 we proved that the sequence $\{d_i\}_{i=0,1,\ldots}$ saturates at some stage $n_0 \in \mathbb{N}$ and the degree of minimal polynomial basis is n_0. Thus at the stage n_0, δ_{n_0} denotes the number of all the polynomially independent vectors in $\mathcal{N}$ upto degree n_0.

Theorem 3.2: Let $R \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}[s]$ be a polynomial matrix with degree n. Let $\mathcal{N}$ denote the nullspace of the polyno-

$$N = \begin{bmatrix} -0.153 & -0.327 & -0.548 & -0.262 & 0.286 & 0.034 & -0.378 & 0.343 & 0.290 & 0.260 & -0.009 & -0.059 \\ 0.258 & -0.438 & 0.352 & -0.341 & 0.189 & 0.123 & -0.027 & 0.060 & -0.106 & -0.093 & 0.389 & 0.522 \end{bmatrix}^T \tag{16}$$

mial matrix R. Further construct the sequence of matrices $\{J_i\}_{i=0,1,\dots}$ as in Construction 3.1. Let δ_i = nullity (J_i). Then the following statements hold:

(a) $\delta_{i+1} \geq \delta_i$ for all $i \geq 0$.

(b) δ_i denotes the number of polynomially independent vectors of degree upto i in the minimal polynomial basis of $\mathcal{N}$.

Remark 3.3: From the statement (b) of the above theorem we get the terminating criterion for the algorithm to compute the minimal nullspace basis of $\mathcal{N}$ as follows: let r be the normal rank of $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[s]$ with $\mathtt{g} \leq \mathtt{w}$ without loss of generality. Then the number of polynomially independent vectors in $\mathcal{N}$ is given by $\mathtt{w} - r$. Thus for the smallest i such that $\delta_i = \mathtt{w} - r$, we get all the polynomially independent vectors in $\mathcal{N}$ and we know that the highest degree of minimal nullspace basis vectors is i.

Algorithm 3.4: Computation of a minimal nullspace basis

Input: Polynomial matrix $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[s]$ with rank r
Output: Polynomial matrix $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}-r}[s]$ where columns of M form the minimal polynomial basis of $\mathcal{N}$

Construct A_0 as in equation (2).
Compute the condensed SVD of A_0 as $A_0 = U_0\Sigma_0V_0^T$.
$r_0 \leftarrow$ rank(A_0), $\delta_0 \leftarrow$ nullity(A_0), $J_0 \leftarrow U_0$
$i \leftarrow 0$

while $\delta_i < \dim(\mathcal{N}) = \mathtt{w} - r - \delta_0$, repeat
  $i \leftarrow i + 1$
  $J_i \leftarrow \begin{bmatrix} U_0 & 0 \\ 0 & U_{i-1} \end{bmatrix}$
  Compute the condensed SVD of J_i as $J_i = U_i\Sigma_iV_i^T$
  $r_i \leftarrow$ rank(J_i), $\delta_i \leftarrow$ nullity(J_i)
end while

$m \leftarrow i$
$N \leftarrow [n_1\ n_2\ \dots\ n_{\delta_m}] \in \mathbb{R}^{(r_0+r_{m-1})\times\delta_m}$
  such that $J_mN = 0$
$M_f \leftarrow N(1 : r_0, :)$

for $j = m - 1 : 1$, repeat
  $N \leftarrow V_j\Sigma_j^{-1}N(jr_0 + 1 : jr_0 + r_j, :)$
  $M_f \leftarrow [M_f; N(1 : r_0, :)]$
end for

$\hat{M} \leftarrow [\ ]$

for $j = 0 : m$, repeat
  $\hat{M} \leftarrow [\hat{M}; V_0\Sigma_0^{-1}M_f(jr_0 + 1 : (j + 1)r_0, :)]$
end for

$M \leftarrow \sum_{j=0}^{m} \hat{M}(:, j\mathtt{w} + 1 : (j + 1)\mathtt{w})s^j$

Remark 3.5: The main advantage of the proposed algorithm is that all the polynomially independent vectors in $\mathcal{N}$ are computed at one stage as is proved in Theorem 3.2. Here all the polynomially independent vectors are obtained at the last iteration of the while loop of Algorithm 3.4.

IV. Numerical Examples

In this section we consider the numerical examples to illustrate the algorithm proposed to compute the minimal nullspace basis.

Example 4.1: [11, Example 3]. Let $R = \begin{bmatrix} -1 & s^q & & & \\ & -1 & s^q & & \\ & & \ddots & \ddots & \\ & & & -1 & s^q \end{bmatrix} \in \mathbb{R}^{\mathtt{g}\times\mathtt{g}+1}[s]$ for some $\mathtt{g}, q \in \mathbb{N}$. The nullspace is spanned by $[s^{q\mathtt{g}}\ \ s^{q(\mathtt{g}-1)}\ \ \dots\ \ s^q\ \ 1]^T$. Here $m = q\mathtt{g}$ and for $q = 2$ and $g = 3$, $M = [-0.7071s^6\ \ -0.7071s^4\ \ -0.7071s^2\ \ -0.7071]^T$.

Example 4.2: In this example we illustrate the different steps in the algorithm with a randomly generated polynomial matrix of size 2×4 with degree 2. Consider the polynomial matrix R with rank 2 as

$$R = \begin{bmatrix} 1 & 2s & 2s+1 & 2s^2+2s+1 \\ 2s^2-1 & -s+1 & 2s^2-s & s^2 \end{bmatrix}$$

Here the degree m of the minimal nullspace basis is 2. In the notation of the algorithm N at $i = 2$ is given as in equation (16). Further the matrix M_f is formed by appending the following matrices as $M_f = \begin{bmatrix} M_1^T & M_2^T & M_3^T \end{bmatrix}^T$ where

$$M_1 = \begin{bmatrix} -0.153 & -0.327 & -0.548 & -0.262 \\ 0.258 & -0.438 & 0.352 & -0.341 \end{bmatrix},$$
$$M_2 = \begin{bmatrix} -0.081 & 1.351 & 1.167 & 0.372 \\ -3.641 & -9.413 & -11.380 & -6.229 \end{bmatrix},$$
$$M_3 = \begin{bmatrix} 1.358 & -0.527 & 0.721 & 1.149 \\ -10.992 & 4.245 & -5.714 & -9.611 \end{bmatrix}.$$

Further the matrix M is obtained as follows where the columns of M form the minimal nullspace bas

$$M = \begin{bmatrix} 0.080 & 0.228 \\ -0.319 & 0.205 \\ -0.213 & -0.236 \\ 0.133 & 0.007 \end{bmatrix} + \begin{bmatrix} 0.045 & 1.869 \\ 0.600 & -6.609 \\ 0.476 & -5.347 \\ -0.362 & 2.815 \end{bmatrix} s + \begin{bmatrix} -0.795 & 6.672 \\ 0.627 & -5.040 \\ 0.530 & -4.448 \\ 0.000 & -0.000 \end{bmatrix} s^2 \tag{17}$$

V. Concluding Remarks

In this paper we discussed a numerical algorithm to compute the minimal nullspace basis of a given polynomial matrix. The algorithm involved constructing structured real matrices and working with these matrices to compute the minimal nullspace basis. Advantage of our algorithm is that all the vectors in the nullspace basis are computed at one time and the efforts to track the new polynomially independent vectors that are added in the nullspace at each stage is avoided.

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